On the calculating power of Laplace's demon (Part I)

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Abstract

We discuss several ways of making precise the informal concept of physical determinism, drawing on ideas from mathematical logic and computability theory. We outline a programme of investigating these notions of determinism in detail for specific, precisely articulated physical theories. We make a start on our programme by proposing a general logical framework for describing physical theories, and analysing several possible formulations of a simple Newtonian theory from the point of view of determinism. Our emphasis throughout is on clarifying the precise physical and metaphysical assumptions that typically underlie a claim that some physical theory is 'deterministic'. A sequel paper is planned, in which we shall apply similar methods to the analysis of other physical theories.

Along the way, we discuss some possible repercussions of this kind of investigation for both physics and logic.

1 Introduction

> *Given for one instant an intelligence which could comprehend all the forces by which nature is animated and the respective situations of the beings who compose it — an intelligence sufficiently vast to submit these data to analysis — it would embrace in the same formula the movements of the greatest bodies and those of the lightest atom; for it, nothing would be uncertain and the future, as the past, would be present to its eyes.* [53, Chapter II]

In these famous words, Laplace articulated his vision of an orderly, mechanistic universe whose history unfolds according to fixed deterministic laws. This vision may in essence be traced back at least to Democritus, and it has exerted an enormous influence on scientific thinking down to the present time (see *e.g.* [66] for a modern incarnation). Philosophers still argue over whether or not the issue of physical determinism has any bearing on the question of human free will (see *e.g.* [19]). It is therefore very natural to ask how well Laplace's claim holds up in the light of our present-day understanding of science and mathematics.

Broadly speaking, the answer to this question will depend on two kinds of considerations. Firstly, it obviously depends on what the 'laws of physics' actually are: for example, some proposed interpretations of quantum theory allow for some kind of nondeterminism *e.g.* in connection with the so-called 'collapse of the wave function', interface between quantum and classical levels, whilst others do not (see [64, Chapter 29] for a survey of the main positions). Questions of this kind are clearly a matter for the physicists. Secondly, and less obviously, one can ask how the very concept of 'determinism' is to be made precise. In this paper our concern is primarily with this latter question, though our investigations will also naturally involve a close scrutiny of the precise ways in which physical theories are formulated.

As we shall argue by drawing on ideas from mathematical logic and computability theory, there are in fact a host of different ways in which Laplace's claim might be formulated. Whilst some of the choices involved are rather technical in nature, others touch on much deeper issues of physical and metaphysical ontology. We will show, moreover, that even in for very simple physical theories, such considerations can radically affect whether a theory is deemed to be 'deterministic' or not.

Laplace's imagery of a hypothetical predictive 'intelligence' (nowadays widely referred to as *Laplace's demon*) provides a valuable prop for the imagination, and a convenient metaphor for expressing many of our ideas. Broadly speaking, we will be interested in questions such as the following:

- How exactly is the instantaneous state of the universe (or more modestly, that of some 'closed' physical system) presented to the demon? That is, what kinds of information about this state is the demon supposed to have access to?

- What kinds of 'analysis' is the demon able to perform on this information? For example, what kinds of *infinitary* deductions or computations are permitted?

To elaborate a little further on what lies behind each of these questions:

- Exactly what 'information' is deemed to be present in a physical system at a given instant in time? In particular, which entities or quantities are considered to be 'physically real'? As we shall see, the attempt to make Laplace's claim precise forces us to be very explicit about the ontological assumptions that underlie a physical theory.

- What exactly is meant by saying that certain information can be 'deduced' or 'computed' from certain other information (under the assumption that certain physical laws hold)? To answer this question, one is naturally led to draw on ideas from mathematical logic, and here one finds that many different conceptions of truth, provability and computability have been elaborated. Moreover, some of the choices here turn out to be related to deep metaphysical issues. Again, trying to clarify the idea of determinism forces us to be very explicit about what we are presupposing.

Our main aims in the paper are as follows. Firstly, we wish to examine closely various notions of 'determinism', and show that the 'determinism' or otherwise of a theory is often very sensitive to details of how the question is formulated. In particular, we wish to draw attention to the kinds of physical and metaphysical assumptions that might typically underlie a claim that some physical theory is 'deterministic'. We believe that an awareness of such issues is vital to any attempt to think clearly about physical determinism and its philosophical implications.

Secondly, we hope to explore an area of overlap between physics and mathematical logic which we suspect might repay further study — an area we might broadly characterize as the *metamathematical investigation of physical theories*. Our intention here is to delineate a general field of investigation by means of a somewhat loose assortment of observations and examples. We hope to continue this programme with further examples in a sequel paper [56]. We shall discuss below some of the reasons why we expect this kind of interaction between physics and logic might be beneficial to both parties.

1.1 Outline of paper

The present paper is structured as follows. In Section 2, by way of preparation for our discussion, we clarify the stance we intend to adopt on the crucial issue of the relationship between physical *theories* and the physical universe itself. In Section 3 we then discuss semi-formally three possible approaches to formulating a definition of 'determinism' for a general physical theory, which we call *metaphysical*, *logical* and *computational* determinism. Each of these approaches is itself parameterized by various choices and so may be further subdivided — in Section 4 we review some of these options and (in some cases) the philosophical issues they impinge on. Sections 3 and 4 together constitute a high-level overview of our proposed programme of investigating all these notions in the case of particular physical theories. Section 5 discusses some further motivations and ramifications of this programme in terms of its possible benefits for both physics and logic.

The serious technical work commences in Section 6. Here we propose a general logical framework for the discussion of physical theories, oriented initially towards the investigation of logical determinism. In Section 7 we show how even the rudimentary issue of representing (a single value of) a continuous parameter can be approached in two quite different ways — a *point-based* and an *interval-based* approach — and discuss the significance of the distinction between them. Section 8 is the most technically substantial part of the paper. Here we undertake a detailed investigation of many possible ways of formulating a simple Newtonian situation (that of a falling object in a uniform gravitational field) within our framework. Even for this seemingly simple problem, a surprising number of deep issues arise — we give particular attention to the precise assumptions that are required to support the various approaches. We offer some brief concluding remarks in Section 9.

Our intention in the sequel paper is to carry on our investigation by working up to successively more complex physical situations. As one does so, questions of computational as well as logical determinism come to the fore. In particular, we intend to consider (at least) the gravitational N-body problem; the case of a particle moving in a non-uniform force; and the propagation of electromagnetic waves as described by Maxwell's equations. We shall also briefly consider the outlook for more advanced theories such as quantum field theory.

1.2 Related work

Our work brings together two strands of thought, each of which is represented in the existing literature albeit in somewhat scattered form. The first strand is the consideration of foundational questions from the philosophy of mathematics in relation to physical theories. Typical questions here are: What portions of mathematics are actually needed for physics? Does the apparent indispensability of certain kinds of mathematics constitute an argument in favour of one foundational position over another? Notable contributions in this area have been made by Feferman [28, 29] and Hellman [41, 42, 43]; we believe our present results advance the discussion somewhat further (see especially Section 8 below). Similar questions are considered in a somewhat more speculative vein by Svozil [82, 83, 84], who suggests the possible physical relevance even of ideas from higher set theory; see also [12] for a contribution from a more constructive standpoint.

The second strand, which has received far more attention to date, concerns the issue of computability in connection with physical theories. Here we see our work as continuing a general line of inquiry initiated by Kreisel [50], and later informed by the contributions by Pour-El and Richards [69]. The general issues were further discussed and reviewed by Geroch and Hartle [36] and Shipman [76]. The subject was given new impetus by the ideas of Penrose [62, 63], who has argued that non-computability might be essential to a physical theory that can explain human thought processes. Subsequently, questions of computability in specific physical contexts have attracted attention [13], particularly in relation to the gravitational N-body problem [78, 14, 94]. The papers of Cooper [15, 16, 17] discuss the wider issues at stake in a broad intellectual context, whilst Beggs and Tucker [8] outline a methodological programme similar in spirit to ours.

Whilst our investigations make contact with this body of work at many points, among recent work in the area the most directly relevant to our concerns is that of Weihrauch and his colleagues [90, 91, 92], who discuss physical theories in the light of computability notions appropriate for mathematical spaces involved. Our work makes some further progress in this general direction; in particular, we argue that alternative perspectives offered by other recent work in theoretical computer science (*e.g.* [5, 80]) can make a valuable contribution here (We should mention, however, that most of the *technical* substance of our contribution along these lines will be deferred to the sequel paper.)

Our work also has some affinities with work on 'hypercomputation' (*e.g.* [18, 47, 26]). However, we see our basic standpoint as somewhat different from that taken in these works. Whereas most work in hypercomputation is seemingly driven by the question of whether some physical system might enable us to compute a non-Turing-computable function, our basic concern is with whether some computing agent (or demon) could simulate a physical system. Clearly the two questions are closely related; however, we hope to show that even if the former question has a negative answer, there is plenty to discuss in connection with the latter question.

2 Theories versus physical reality

We wish to stress at the outset that we are going to study the question of determinism as applied to physical *theories*, rather than to physical systems themselves. There are two main reasons for this. Firstly, a theory admits (or ought to admit) of an exact mathematical formulation which makes questions such as determinism amenable to rigorous and precise analysis. By contrast, if we try to study the physical systems themselves, we are forced to make some kinds of theoretical assumptions about them, and all kinds of doubts may then arise concerning the validity of these assumptions. We thus draw a sharp distinction between intrinsic properties of a theory which are open to precise mathematical investigation, and the issue of whether the theory in fact accurately models the behaviour of real physical systems. Our concern here is with the former, the latter being within the province of physics itself. This methodological principle was enunciated by Kreisel in [50]:[1]

> We are here primarily interested in a distinction between classes of theories, not classes of phenomena. The reader should not allow himself to be confused at this stage by doubts about the validity of a theory with regard to the phenomena for which it is intended.

A second, closely related, reason for focusing on theories is that it does not even appear possible to give content to the idea that the physical universe itself behaves deterministically other than by saying that it behaves according to some deterministic *theory.* Even this notion is in danger of trivialization if no limits are placed on the complexity or arbitrariness of the theory, since in an extremal case the 'theory' might be nothing more than a complete record of the actual history of the universe. This was observed by Russell in [72, pp.398–401] (see also the discussion in [15, p.85]). For this reason, it seems more fruitful to focus on the question of determinism for specific (simple) theories.

This does not, of course, mean that we should have no interest whatever in the question of validity for the theories we consider. To continue the above quotation from [50]: 'Naturally, such doubts imply doubts about the relevance (to those phenomena) of any results about [...] the theory.' Thus, detailed metamathematical investigations might appear to be of limited significance in the case of theories that are already known *not* to be precisely valid (such as Newtonian mechanics), but acquire an added dimension of interest in the

[1] A similar attitude has been echoed by more recent workers; see *e.g.* [8, 15].

case of theories that are believed to be candidates for an 'ultimate' description of physical reality. Nevertheless, in view of the complexity and mathematical sophistication of much of modern physics, it seems reasonable to begin such investigations by considering some simple and well-understood physical theories, in order to understand the *kinds* of issues that can arise, and to try to study these issues in a relatively simple context. In this paper and its sequel, we will make a modest start on this programme, drawing our examples from well-known and mainly 'classical' physical theories. We believe that many of the issues we will consider are likely to be representative of issues that will also arise for more sophisticated theories (though it remains to be seen how far this is really the case). Thus, our work could be viewed as a stepping-stone towards an investigation of present-day physical theories (although this latter task would be far beyond the competence of the present author).

2.1 Continua in the physical universe?

Questions of the exact applicability of physical theories are particularly thorny in connection with the use of the mathematical notion of a *continuum* (*e.g.* the real line) in such theories. Most of the questions we will be discussing are in some way bound up with this notion (indeed, the precise nature of the continuum will be our main concern in Section 8 below). Consequently, most of the issues we consider would trivialize if, in fact, it turned out that the universe could be completely modelled by some discrete, finite mathematical structure. A brief discussion of this possibility and our attitude towards it is therefore in order.[2]

On the one hand, the vast majority of successful physical theories in use today rely heavily on the calculus, which makes essential use of the notion of an infinitely subdivisible continuum. On the other hand, we do not yet know whether genuine continua — or indeed genuine infinities of any kind — actually occur anywhere in the physical world. For example, the 'continuous model' of space is believed to hold at least down to the 'Planck scale' of 10^{-35} metres, but it remains a vast leap from this to the idea that that space is a true continuum in the mathematical sense. Physicists have sometimes expressed unease at the seeming ontological extravagance of such a hypothesis.[3]

There have been several interesting attempts to put physics onto a more 'discrete' footing (see [64, Chapter 33] for a survey), but it would appear that this is not so easy to do, and such approaches still have a long way to go in

[2] I am here indebted here particularly to [64, Chapters 3,33], and to [76, II.D]. For a stimulating discussion which tackles various possible objections to the discrete space hypothesis, see [33].

[3] For example, Schrödinger in [74]: 'The idea of a *continuous range*, so familiar to mathematicians in our days, is something quite exorbitant, and enormous extrapolation of what is accessible to us.' Or Feynman in [32]: 'It always bothers me that, according to the laws as we understand them today, it takes a computing machine an infinite number of logical operations to figure out what goes on in no matter how tiny a region of spaces, and no matter how tiny a region of time. How can all that be going on in that tiny space? Why should it take an infinite amount of logic to figure out what one tiny piece of space/time is going to do?'

order to rival the success of their continuous counterparts. (A challenge for any such approach, presumably, will be to account for why continuous theories are so successful for describing the aspects of the universe that we have hitherto been able to understand.) Most leading-edge physical theories, it seems, still take the mathematical continuum for granted.

In the meantime, it seems that we should do what we can with the best theories available. In view of the prevalence of continua in physical theories, it seems to us that it is, at the very least, interesting to explore the implications of the hypothesis that physical continua *do* exist. Moreover, many different conceptions of the nature of the continuum are possible (although this is perhaps not widely appreciated outside mathematical logic), and so it is natural to consider the implications of various alternative viewpoints in the arena of physics.[4] Even if, in the end, such an investigation only served to furnish evidence *against* the plausibility of *e.g.* a genuine spacetime continuum, we would still regard this as an interesting and worthwhile outcome.

3 Some formulations of determinism

In this section we outline, in a semi-formal way, some of the possible ways in which the notion of determinism might be formulated, and indicate the kinds of choices that we are led to consider. Throughout this section the issues are phrased in general terms — we do not as yet concern ourselves with specific physical theories, as we shall do in later sections. Our purpose here is to convey at an intuitive level some of the main themes that will recur throughout the examples that we shall treat more formally later on.

As a first step, we may broadly distinguish between 'ontological' and 'epistemological' versions of Laplace's claim. Informally, given an initial state of some physical system, an ontological notion of determinism would say that there is, *in fact*, only one thing that can happen, while an epistemological notion would claim that there is some way to know or 'predict' what will happen.

Here we shall try to refine these notions by more precisely formulating three possible definition of determinism: one with an ontological flavour, which we shall call *metaphysical* determinism (following [58]), and two with an epistemological flavour, which we shall call *logical* and *computational* determinism respectively. At this stage in the discussion, our point is simply that these give three different approaches to *formulating* a notion of determinism — not that they necessarily give rise to inequivalent notions. Indeed, we shall see later that each of these approaches can itself be further subdivided, and that in many cases there is considerable overlap between the three notions.

Let us now explain each of these notions in turn.

[4] This point is admirably expressed by Feferman at the conclusion of [29]: 'But as long as science takes the real number system for granted, its philosophers must eventually engage the basic foundational question of modern mathematics: "What are the real numbers, really?" '

3.1 Metaphysical determinism

Broadly speaking, an *metaphysical* version of determinism would claim that for any possible 'initial state' of our physical system, there is, *in fact*, only one possible course of history starting from this state in which the laws of physics are upheld — regardless of whether we have any way of knowing what this history is. Suppose, for example, that we have some physical theory which postulates a set *Times* of instants in time, a set *States* of possible instantaneous states of the system, and a set *Histories* of 'possible histories'. Given a history h and a time t, let us write $h(\!|t|\!)$ for the state at time t according to the history h.[5] For simplicity, we assume throughout the following discussion that $t_0 \in \mathit{Times}$ is some 'start time' which we regard as fixed. Then a statement of metaphysical determinism might have the schematic form[6]

$$\forall s_0 \in \mathit{States}.\ \exists! h \in \mathit{Histories}.\ h(\!|t_0|\!) = s_0 \qquad (\mathbf{MD})$$

Intuitively, we are here imagining a predictive demon so powerful that it can magically survey all potential histories and single out the unique one that satisfies the given initial conditions.

An important question here is: what ontological status do the 'possible histories' here really have? It appears that they have to be 'metaphysical' entities rather than strictly 'physical' ones: indeed, the uniqueness assertion in (**MD**) would become vacuous if h ranges only over 'physically real' histories, since there is in any case only one history — the *actual* one[7] — that can reasonably lay claim to any kind of physical reality.[8] Likewise, *States* should be construed as some set of potential states rather than just those that are ever physically realized. The point here is, of course, a well-known and widely discussed one in philosophical logic (see *e.g.* [58]).

In order to sharpen the discussion, let us suppose that, in our theory, we have some *mathematical* description of the set *States*, and *Histories* is defined to be the set of all mathematical *functions* $h : \mathit{States} \rightarrow \mathit{Times}$ satisfying a certain specified predicate $\mathit{Laws\_Of\_Physics}(h)$. (For instance, this predicate might say that h satisfies certain differential equations at all times t; the state s_0 in (**MD**) would then correspond to a boundary condition.) This shifts the metaphysical discussion to a discussion of the status of mathematical entities

[5] For the purpose of this illustration, we presuppose a Galilean conception of time in which the notion of simultaneity is absolute. For relativity theories, one should replace the idea of 'state at time t' by that of a general 'time slice' through the system; the essence of the ensuing discussion can then be interpreted in a relativistic setting *mutatis mutandis*.

[6] The formulae we give in this section are intended as merely illustrative, and are given in order to clarify the *typical* form that some statement might take. They should be interpreted somewhat informally; a more formal logical framework for our investigations will be presented in Section 6.

[7] Even under many-worlds interpretations of quantum theory this point remains valid, since if one wants to formulate a version of (**MD**) in such a setting, one must anyway consider states as pertaining to the entire multiverse.

[8] Even the supposition of 'physical existence' for this one history might be considered problematic. *E.g.* following Augustine [3, Book XI.15], we might ask: '*When* does it exist?'

(such as elements of *States* and functions from *States* to *Times*) and statements involving them. We will return to this issue in Section 4.1 below.

Of course, if (**MD**) is understood as a purely *mathematical* statement about some model of physics, it does not by itself succeed in saying anything about how the actual universe behaves. This, however, is quite consistent with our decision to concentrate on the intrinsic properties of theories and to isolate these from questions about how the theories relate to physical reality.[9] We will henceforth restrict our attention entirely to the theories and models themselves.

3.2 Logical determinism

In epistemological versions of determinism, the idea is that given complete knowledge of the initial state, there is some way in which we may know or 'work out' how the future will unfold. This idea may itself be understood in two ways.

A *logical* version of determinism would claim that given knowledge of all relevant facts about an initial state and of the physical laws, it is possible to 'deduce' facts about future states. The emphasis here is on the idea of logical deduction or inference; here we may informally imagine our demon as engaging in *reasoning* on the basis of some given facts.

Crucial to such a notion of determinism will be the choice of a *system of inference* which we may envisage our demon as using. This in turn must presumably involve a (possibly infinitary) *language* of some kind — perhaps the demon's private language of thought — in which assertions are expressed. Of course, for a given choice of language, it may not be possible to give a complete description of an arbitrary state by means of a single assertion; it might therefore be too much to demand the deducibility of an assertion of the form 'the state at time t_1 is s_1'. In general, therefore, we may have to content ourselves with assertions *about* states which are expressible in the language. Thus, we might ask for which predicates P_0 and P_1 of the language can the entailment

$$\mathit{Laws\_Of\_Physics}(\bar{h}) \ \wedge\ T(\bar{t_0}, \bar{t_1}) \ \wedge\ P_0(\bar{h}(\!|\bar{t_0}|\!)) \quad \vdash \quad P_1(\bar{h}(\!|\bar{t_1}|\!)) \qquad (\dagger)$$

be proved within the given system of inference. Here $\bar{h}, \bar{t_0}, \bar{t_1}$ are formal free variables within the language, as distinct from elements h, t_0, t_1 of the corresponding sets. (We will not bother to distinguish notationally between the predicate *Laws_Of_Physics* and operation $(\!| - |\!)$ and their syntactic counterparts.) The predicate $T(\bar{t_0}, \bar{t_1})$ should be thought of as fixing the times t_0, t_1

[9] One possible way of trying to bridge the gap between the model and the physical universe itself might be to supplement an assertion such as (**MD**) with the following statement, in which T, T' ranges over actual points in time and $S(T)$ is the actual state of the universe at time T.

$$\begin{array}{l} \forall T, T' : \mathit{Times}, t, t' : \mathit{Times}, h : \mathit{Times} \rightarrow \mathit{States}. \\ \quad \mathit{Models}(t, T) \wedge \mathit{Models}(t', T') \wedge \mathit{Models}(h(\!|t|\!), S(T)) \wedge \mathit{Laws\_Of\_Physics}(h) \ \Rightarrow \\ \quad \mathit{Models}(h(\!|t'|\!), S(T')) \end{array}$$

However, it is not clear whether anything interesting is achieved by doing this.

with reference to constants of the language, *e.g.* $\bar{t_0} = 0 \wedge \bar{t_1} = 1$. For the sake of the discussion we shall suppose that t_0, t_1 may be uniquely pinned down by a single predicate T.

A reasonable statement of determinism might then say that there are enough provable assertions of the above form to uniquely determine the state at t_1 from that at t_1. For this, we will still need to appeal to a mathematical notion of possible states. We therefore assume we have a set *States* and a notion of satisfaction for formal predicates P with respect to elements $s \in \mathit{States}$; we will denote this by $\models P(s)$ rather than by the more correct but cumbersome $\models_{\bar{s} \mapsto s} P(\bar{s})$. However, a notion of possible *histories* is no longer required — in fact, we need never speak (even syntactically) of 'histories' other than the actual one.

First, it is natural to suppose that there are enough definable predicates to distinguish between any two different states:

$$(\forall P.\ \models P(s) \ \Leftrightarrow \ \models P(s')) \ \Rightarrow \ s = s'$$

If this is not the case, there is redundancy in our model and we may as well work with the quotient of *States* modulo Leibniz equality. Our formulation of logical determinism now says that for any s_0, there is only one s_1 that satisfies all the properties inferrable from properties of s_0:

$$\forall s_0.\ \exists! s_1.\ \forall P_0, P_1.\ \ (\models P_0(s_0) \ \wedge \ `(\dagger)\text{'}) \ \Rightarrow \ \models P_1(s_1) \qquad (\mathbf{LD})$$

Here '(†)' expresses the provability the entailment (†) given above. If (**LD**) holds for every time-fixing predicate T, then by varying this predicate so as to allow t_1 to vary, then under mild metamathematical assumptions we may define a function $h : \mathit{Times} \to \mathit{States}$ and prove that it is indeed the unique function satisfying the *Laws_Of_Physics* predicate. Thus, under rather mild assumptions about the class of possible histories, (**LD**) implies (**MD**).

If the language is powerful enough that arbitrary states can be specified completely by assertions, we can do better. Suppose that for each $s \in \mathit{States}$ we have a characteristic predicate P_s such that $\models P_s(s')$ if and only if $s = s'$; we shall for readability write $P_s(s')$ as '$s' = s$'. In this case, we may restrict attention to these characteristic predicates:

$$\begin{array}{ll} \forall s_0.\ \exists! s_1.\ `\mathit{Laws\_Of\_Physics}(\bar{h}) \wedge T(\bar{t_0}, \bar{t_1}) \wedge \bar{h}(\!|\bar{t_0}|\!) = s_0 & \\ \qquad\qquad \vdash \ \bar{h}(\!|\bar{t_1}|\!) = s_1\text{'} & (\mathbf{LD}^+) \end{array}$$

In this setting, clearly ($\mathbf{LD}^+$) implies (**LD**). Conversely, (**LD**) will imply ($\mathbf{LD}^+$) if our logic satisfies a suitable *completeness* property, but not in general.

More generally only *some* states are logically specifiable, we may relativize ($\mathbf{LD}^+$) to a statement ($\mathbf{LD}^+_d$) by taking s_0, s_1 to range only over some set $\mathit{States}_d \subseteq \mathit{States}$ of definable states. It would seem that ($\mathbf{LD}^+_r$) should imply (**LD**) if the set of definable states is suitably *dense*; however, ($\mathbf{LD}^+_d$) will not in general be implied by (**LD**) as it imposes the stronger condition that definable starting states give rise to definable finishing states (if the starting and finishing times are definable).

3.3 Computational determinism

By contrast, a *computational* version of determinism would claim that there is some way of 'computing' the future evolution of the system from a given starting state s_0 — that is, some kind of computable operation

$$\Phi : \mathit{States} \times \mathit{Times} \to \mathit{States}$$

Here we wish to say that $\Phi(s_0, t)$ is *in fact* the state the system will be in at time t, whether or not we can prove that it is. In order to express this idea, we again need recourse to a set of possible histories:

$$\begin{array}{l} \forall s_0 \in \mathit{States}.\, \forall h \in \mathit{Histories}. \\ \qquad (h(\!|t_0|\!) = s_0 \;\Leftrightarrow\; \forall t \in \mathit{Times}.\, h(\!|t|\!) = \Phi(s_0, t_0)) \end{array} \qquad (\mathbf{CD})$$

Of course, what exactly we mean by a 'computable operation' Φ in this context requires clarification — we return to this issue below. Clearly (**CD**) implies (**MD**) (assuming Φ is total), but not conversely.

The *a priori* possibility that the evolution of physical systems might be mathematically deterministic but not algorithmically computable in nature was explicitly discussed in [36], and has been highlighted by Penrose [62, 63], who has furthermore suggested that physical laws of this kind might play an essential role in the science of consciousness.[10] The distinction was again taken up in [13], where some candidates for (metaphysically) deterministic but not computationally predictable physical systems were proposed.

Let us comment briefly on the relationships between logical and computational determinism. It is tempting to think that (**LD**) or ($\mathbf{LD}_d^+$) should imply (**CD**) in general, since one way to compute the state at t_1 is by enumerating all possible proofs in the inference system. However, there are several reasons why this might not be the case. Firstly, a complete linguistic description of a starting state will typically give us more information than can be computably extracted from the state itself. (This is analogous to the fact that equality is decidable for the rationals, but not for the reals.) Secondly, the enumeration trick does not carry over readily to *infinitary* inference systems: for instance, in an inference system with the ω-rule the set of proofs is uncountable. Thirdly, in the case of (**LD**), further assumptions on the expressive power of the predicates P_0, P_1 may be necessary. Whether (**LD**) does indeed imply (**CD**) for some family of cases will therefore depend on the details of the inference system and the nature of the computable operation Φ (see Section 4.3).

Conversely, if the logical language is strong enough to specify the operation Φ, and the evident formula $\mathit{Laws\_Of\_Physics}(\bar{\Phi}(\bar{s}))$ is provable in the system, and the system is strong enough to derive the evident formalization of the statement of (**MD**), then (**CD**) will typically imply (**LD**). Again, whether all these conditions are met will depend on the details of the situation at hand.

The ingredients of logical and computational determinism can be combined into a formulation (**LCD**) of *logical-computational* determinism, in which the

[10] See *e.g.* [62, page 220]: 'Computability is a different question from determinism — and the fact that it *is* a different question is something that I am trying to emphasize in this book.'

statement of (**LD**) is supplemented with the condition $s_1 = \Phi(s_0, t_1)$. However, it is easy to see that under mild assumptions, (**LCD**) holds iff both (**LD**) and (**CD**) hold.

Some further relationships between these formulations will be discussed in Section 4.

3.4 'Provable' variants

Each of our statements (**MD**), (**LD**), (**CD**) is itself a mathematical assertion. We may therefore in principle take one step back and ask whether these statements are themselves provable in some particular meta-system for mathematics. We thus obtain 'provable' variants ($\mathbf{MD}_p$), ($\mathbf{LD}_p$), ($\mathbf{CD}_p$) of these formulations, which may be expressed by adding a meta-level syntactic turnstile $\vdash_M$ at the far left of the statements given earlier. From this standpoint, the original versions of these statements may then be distinguished by means of a corresponding semantic turnstile $\models_M$. (In the case of (**LD**) we must distinguish notationally between the turnstiles for the meta-system and those for the object-system, which may in general be different systems.) A version of ($\mathbf{MD}_p$) raised its head briefly in the above discussion of equivalences.

In a sense, it is the provable variants that we naturally find ourselves investigating when we consider particular theories, since *e.g.* if (**MD**) were true but not provably so in any formal system that we ourselves believe in, we would never find this out. However, the point of introducing the provable variants is not really because we care about the difference in the case of strong meta-systems such as ZF set theory — it would be astonishing if there were a plausible physical theory for which the statement of metaphysical determinism furnished an example of Gödelian incompleteness! — but rather because we will wish to consider the provable variants in the case of weaker meta-systems, as we shall be interested in the precise strength of the mathematical principles required to conduct the proofs of the semantic versions.

Let us now consider the lines along which our three main formulations may themselves be further subdivided.

4 Truth, provability and computability

Broadly speaking, throughout the above discussion we have appealed repeatedly to three basic notions, whose precise nature was left somewhat open:

- A notion of *truth* for mathematical statements. Of course, all of our formulations are to some extent contingent on such a notion, simply because they are all mathematical statements; but the dependency is particularly marked for those notions, such as (**MD**), which rely heavily on abstract notions such as potential states and potential histories.

- A notion of *provability*. This clearly features in the formulation of (**LD**), and also in that of the 'provable' variants as discussed in Section 3.4.

- A notion of *computability*, which features in (**CD**) in the requirement that the operation Φ be computable.

And it could be fairly said that the notions of truth, provability and computability (and the relationships between these notions) constitute the very core of modern mathematical logic. Our purpose, then, is to see how ideas and experience from logic can contribute to the clarification and study of our various notions of determinism.

What we find from the study of logic is that for each of these three fundamental notions, many different conceptions and formulations have been developed and studied. In fact (as we shall explain), for none of them is there a single, definitive conception or definition which suffices for all the applications we have in mind. We therefore obtain, for each of our three flavours of determinism, a whole range of interpretations according to which concept of truth, provability or computability is adopted. Let us now examine each of these three basic notions in turn.

4.1 The notion of mathematical truth

Recall that in the case of metaphysical determinism (for example), we are interested in whether (some instance of) the formula (**MD**) is a *true*. Many physicists and mathematicians might feel that, once we have succeeded in bringing the question of determinism for some theory down to a purely mathematical question, we have removed all fuzziness or doubt concerning what the question means, and it only remains to answer it. Implicit in such a view, typically, is a (sometimes tacit) assumption that regardless of whether we are able to know or decide whether a given mathematical statement is true or not, there really is some 'fact of the matter' about its truth or falsity which we are trying to discover. This is often referred to as a *Platonist* view of mathematical truth.[11] It is the 'classical' philosophy of mathematics, and is widely held by many working mathematicians and physicists today.

However, some mathematicians, logicians and philosophers have not been comfortable with this view, and with its apparent metaphysical commitment to a realm of 'truth' beyond human knowledge and beyond empirical investigation. After all, they might ask, what *meaning* does (**MD**), with its implicit quantification over 'potential histories', really have? Naturally, when we make such statements we may be entertaining a certain picture in our minds — probably a vague picture of some 'space of potential histories' — but this picture does not (they would claim) *refer* to anything. Moreover, even if the metaphysical status of 'potential histories' could be clarified, there remains the issue of whether *quantification* over them has any meaning. Our understanding of quantification (they might argue) derives from our experience of being able to check properties

[11] The question of whether *statements* of mathematics have a determinate truth-value independent of our knowledge can in principle be distinguished from the question of whether the *entities* considered by mathematics have an independent 'existence'. Our focus here is mainly on the issue of Platonism with regard to mathematical truths rather than entities.

exhaustively for finite collections of objects, and extrapolations of this idea to infinite collections are simply illusory.

This rejection of the Platonic view by some workers has led to the development of several alternative approaches to mathematics under the broad umbrellas of *constructivism* and *finitism* (see [87, Chapter 1] for a survey). We shall not describe these schools of thought in detail here, but a common feature of them is that mathematical statements are not regarded as meaningful unless they can be endowed with some kind of empirical content. Moreover, the *law of excluded middle* $\vdash \phi \vee \neg\phi$, unproblematic to the classical mathematician, is typically rejected by such schools, and this has far-reaching consequences for the development of constructive or finitist mathematics.

Many intermediate positions between strict constructivism and full-blown 'set-theoretic Platonism' have also been developed: these differ, roughly, in regard to which *kinds* of mathematical entities and truths are accepted as having knowledge-independent existence. For instance, in positions we shall broadly call *semi-constructivist*, one accepts the principle that if some property $P(n)$ of *natural numbers* is meaningful for each natural number n then there is a 'fact of the matter' about whether or not $\forall n.P(n)$ is true; however, one typically rejects principles involving more powerful appeals to the infinite. An important example of a semi-constructivist position, for our purposes, is that of *predicativism* (see *e.g.* [30]).

It is not our purpose here to take a side in all these debates, but merely to point out that one's metaphysical attitude towards mathematical statements will affect how one construes the idea of metaphysical determinism. And since the idea of determinism is itself of such philosophical interest, it is surely natural that one should wish to scrutinize the philosophical presuppositions that one's concept of determinism rests on. The 'natural' reading of metaphysical determinism as we have presented it (that is, the way one would expect it to be understood by a reader not forearmed by an acquaintance with the philosophy of mathematics) is, we would suggest, tacitly dependent on a Platonistic view. However, there is nothing to stop us from interpreting the statement (**MD**) from the standpoint of other mathematical ideologies; we thus obtain a whole spectrum of possible interpretations of (**MD**), each with its concomitant set of philosophical presuppositions. Constructive readings, in which truth is in some degree identified with knowability, will naturally tend to approximate to versions of (**LD**), (**CD**), ($\mathbf{MD}_p$) or perhaps all of these. Similar remarks also apply to computational determinism (which also refers to potential histories), and to a lesser extent to logical determinism, since in (**LD**) we still have the idea of quantification over potential states.

Laplace's appeal to the imagery of a demon is particularly telling at this point, and it is instructive to relate it to some recurring themes in the Platonist-constructivist debate.[12] Presumably for Laplace this was a mere figure of speech, and he would have regarded the question of whether or not such an 'intelligence'

[12]See also [21], to which we are indebted for some of the ingredients of the following discussion.

really exists as irrelevant to the point he was making. However, to remove any suspicion of counterfactuality, one might try to reformulate the essence of Laplace's claim without recourse to the idea of a demon. Suppose a classical mathematician does this along the lines of (**MD**). A constructivist might then challenge him as to the meaning of his quantification over all potential histories, which appears to lack empirical content. The classicist might respond that though we only have direct experience of finite quantifications, we can form *by analogy* a perfectly clear conception of what it would mean to check an infinite set of things for a certain property. If pressed to convey the nature of this analogical conception more clearly, he may find himself led naturally, perhaps even inevitably, to appeal to the idea of what *would* be knowable to some agent or process that *could* 'see all potential histories at once' (or could in some other way perform the requisite search).[13] So we are back to the idea of a demon again! The classicist might protest, of course, that this is only a way of *suggesting* what he means, and that the hypothetical nature of this manner of speaking does not seriously impugn the coherence of the underlying conception. By contrast, the constructivist might maintain that the impossibility of such infinite searches is precisely the point at issue, and that the classicist's inability to express his conception in any fundamentally better terms betrays the fact that the conception partakes *essentially* of this hypothetical character — and therein (he might say) lies its fatal flaw.[14]

In other words, the mere imagery of Laplace's demon tends to smuggle into our conception the idea of the *abstract possibility* of a being with the requisite infinite powers of surveillance — and it is precisely this idea that is rejected as meaningless by the constructivists. We are not thinking here primarily of the hypothesis involved in postulating a being that has complete access to each of the raw *physical* facts that comprise the state of the universe, but of the conception of the demon as being able to perform arbitrary unrestricted infinitary manipulations of these facts. This conception, when scrutinized, appears to be wedded to a Platonistic view of truth for the statements in question.

One might object that potential states, potential histories and the like are not really mathematical objects, but rather metaphysical objects of some other kind (perhaps having some hybrid mathematical-physical status), so that one's philosophy of mathematics had no bearing on one's attitude to statements such as (**MD**). This, however, would seem to be merely a terminological quibble. Of course, one is at liberty to reserve the term 'mathematics' for *e.g.* some autonomous mental activity bearing no relation to the external physical world. But even if one regards potential states and histories as entities of some other kind, exactly the same questions and debates that arise in the context of math-

[13]This idea is expressed with particular clarity by Wittgenstein [93, §352]: ' "In the decimal expansion of π either the group '7777' occurs, or it does not — there is no third possibility." That is to say: "God sees — but we don't know." But what does that mean? — We use a picture; the picture of a visible series which one person sees the whole of and another not. ...Here saying "There is no third possibility" or "But there can't be a third possibility!" — expresses our inability to turn our eyes away from this picture.'

[14]Once again, we are trying not to take sides here!

ematics carry over to these other entities. In this case, one should understand the term ‘Platonism’ in the above discussion as meaning ‘Platonism with regard to the truth of such statements as (**MD**)’.

4.2 Formal systems and provability

As we have noted, logical notions of determinism, as well as the ‘provable’ version of metaphysical determinism, depend on some idea of logical inference, perhaps as presented by a formal system. Here there is less to say, as the existence of many different formal systems (*e.g.* covering different portions of mathematics) is very familiar and hardly requires comment.

Of course, we are here interested primarily in deductions, not in arbitrary formal systems, but in systems expressing principles that it is reasonable to believe in. The spectrum of possible philosophies of mathematics which we outlined in Section 4.1 is thus closely paralleled by an array of formal systems for mathematics which embody (more or less) the principles and rules of inference that are deemed acceptable by these philosophies. Although one should beware of making too close an identification between a philosophical stance and a formal system that embodies it (indeed, many philosophies explicitly resist the idea that the content of mathematics can be *exhaustively* captured by a formal system), there is no doubt that the study of these formal systems has contributed greatly to an elucidation of the various foundational positions and the relationships between them. We recommend [27] for an overview of this area.

One may also consider formal systems with *infinitary* rules, such as the ω-rule for arithmetic. For such systems, the notion of provability will clearly tend to approximate to that of truth.

4.3 Notions of computability

Let us now turn to the question of computability as it features in computational notions of determinism. Here, there is a prevalent assumption that the notion of ‘computability’ is unambiguous and well-understood — more specifically, that if one accepts (some version of) the Church-Turing thesis, the Turing definition gives us all we need by way of a working definition of computability, as all other reasonable definitions turn out to be equivalent to this one. Of course, the Church-Turing thesis can itself be questioned (and has been in recent proposals for ‘hypercomputation’, *e.g.* [18, 47, 26]), but that is not the main issue we wish to discuss here. Our point, rather, is that the situation alluded to above only really gives us a canonical notion for functions acting on *natural numbers*, or on other finite entities that can be effectively coded by them in an essentially canonical way.[15] But what do we mean by computation where *infinite* entities are concerned, such as functions on $\mathbb{N}$, real numbers, continuous functions on the reals, or elements of even larger ‘spaces’? Such questions are crucial to how one formulates the notion of computational determinism, since states of a

[15]Equivalently, we get a canonical notion of a (single) *computable real number*, as in Turing’s famous paper [88].

physical system may typically be represented by real numbers (*e.g.* for positions and velocities of particles), functions on real numbers (*e.g.* for gravitational or electromagnetic fields), or even distributions over spaces of such functions (*e.g.* for quantum field theory).

In fact, as soon as one leaves the realm of natural numbers and other finite objects, the issue of computability becomes much more complex. Broadly speaking, we have to decide how we think of an infinite object as being given to us, and what kinds of manipulations we are allowed to perform on it. Even if we restrict our attention to 'computations' that that manipulate the infinite data only in intuitively 'finitary' ways, the situation is still complex. For computations that take ordinary functions on $\mathbb{N}$ as data, there are already some choices to be made, and the situation becomes rapidly more complex as one passes to higher types (see [54] for a survey and discussion). In other contexts, such as that of computable metric spaces, there is arguably a single canonical computability notion which may be characterized in several ways (see *e.g.* [80]). A fairly clear overall map of much of this territory is now emerging, though the picture is still unclear for some parts of the landscape, such as higher type operations over the reals (see [7, 59]).

Typically, the computability notions we have in mind are based on finitary manipulations of the data; such notions invariably specialize to the familiar Turing notion if we restrict attention to the natural numbers. It is also possible, however, to consider notions in which certain infinitary operations are allowed (such as Kleene's higher type computability in the presence of various quantifiers [48, 73]). Such notions are of interest since they tend to relate closely to semi-constructive or classical notions of truth.

Our point, then, is that the 'states' and 'histories' featuring in a physical theory are likely to be highly infinitary objects, for which there may or may not be a *canonical* notion of computability. In the context of computational determinism, therefore, we will have to think carefully about how we consider the state of the universe to be given to our demon — in particular, it need not be given by symbols on a Turing tape. Even in contexts where only one notion of computability can appropriately be applied, this fact will require justification.

It is interesting to note that workers who have considered issues of computability for physical theories have concentrated, for the most part, on the question of whether the theory can ever give rise to 'first-order' non-computable phenomena, such as a single measurable real number that is not computable from the initial conditions, or a means of computing a 'non-computable' function $\mathbb{N} \to \mathbb{N}$ (see *e.g.* [50, 69, 36, 78]). The more general question of the computability of an entire history from an initial state has received less attention to date.[16] One might surmise that this stems from an uncertainty regarding the right way to discuss computability in this setting, and it is here that an acquaintance with the body of work mentioned above may be of use.

[16] A notable exception is the work of Weihrauch and his colleagues [90, 92, 91].

4.4 Relating notions of determinism

It will be clear by now that there are close connections, and some substantial overlap, between the three formulations of determinism that we have been considering. Some relationships of a 'general' nature — that is, those that could be discussed without reference to specific conceptions of truth, provability and computability — were observed in Section 3.3; the tendency of these observations was that (**LD**) or (**CD**) usually implies (**MD**). We now briefly summarize the kinds of more specific connections between particular formulations that one might expect; some of these have already been alluded to in the course of the above discussion.

On the one hand, the statement ($\mathbf{MD}_p$) understood with reference to a constructive meta-system will usually imply a form of (**CD**), since in virtually all proposed formal systems for constructive mathematics, one can extract from a proof of $\forall s \exists h \cdots$ some kind of computable operation for finding an h given an s. Likewise, it will imply a form of (**LD**), since a proof of $\forall s \exists h. A(s, h)$ provides an operation which, given an s, yields not just a suitable h but a proof of $A(s, h)$ for this particular s, h. In fact, there is already an extensive body of metamathematical work on relating constructive formal systems to computability notions (*e.g.* by means of *realizability* interpretations), and also on investigating notions of constructive provability from a more syntactical point of view. (See *e.g.* [86] for a monumental treatment of much of this material.) Some of this work is clearly relevant to the question of relating notions of determinism in particular cases.

Moving higher up, (**MD**) from non-constructive standpoints can sometimes be related to version of (**CD**) based on infinitary computability notions. For instance, what 'exists' from a semi-constructivist standpoint is closely related to what is 'computable' in the presence of the (second-order) operation of existential quantification over the natural numbers (see [48]) — informally, what would be computable to a demon who could 'see all the natural numbers at once'. In the presence of even more powerful infinitary operations, computability would approach classical notions of truth (see *e.g.* [73]). On the other hand, the proof-theoretic analysis of semi-constructive and more powerful systems sheds important light on how these are related to various conceptions of truth (see *e.g.* [27]). In fact, a full survey of all the work that is potentially relevant to our concerns would rapidly lead us into an review of a huge area of mathematical logic!

One may now roughly imagine the possible formulations of determinism as occupying a two-dimensional space, with three columns corresponding to our three basic flavours, and with a vertical spectrum ranging from the most powerful (*e.g.* Platonistic) notions at the top to the weakest (*e.g.* finitist) notions at the bottom. The import of the above observations is that there are many horizontal connections between formulations in different columns. In view of the multitude richness of these connections, it might perhaps seem by now that it is the vertical dimension that really holds most of the interest, with (**MD**), (**LD**) and (**CD**) merely playing the role of different formulations. However, the three approaches are still of value and interest in their own right in that they offer different *perspectives* and ways of thinking which enrich our understanding. Moreover, in particular cases, most of the interest lies not in the mere existence of horizontal correlations but in the details of which *particular* notions can be correlated, and in what way. When we look closely, of course, we find that the precise relationships are often rather subtle and our formulations do not map neatly onto a simple one-dimensional axis.

What we are proposing in this paper and its sequel, then, is a detailed investigation of possible notions of determinism, their interrelationships, and their applicability to specific, precisely articulated physical theories, using ideas and methods developed in mathematical logic. We will make a start on this programme in both papers, by presenting a selection of examples to show that even for very simple physical theories there is something to explore, in that the 'determinism' of a theory often turns out to be quite sensitive to the precise definition of determinism, the details of how the physical theory is formulated, and the metaphysical assumptions adopted.

5 Some further motivations

Now that we have laid out in informal terms the main ideas of the paper and the kind of interplay between physics and logic that we wish to explore, we are in a position to explain more fully some of the reasons why this kind of interplay seems to us interesting and potentially fruitful, beyond the general philosophical interest of the notion of determinism.

Our discussion so far has been somewhat oriented towards the 'grand question' of whether, and in what sense, the universe as a whole behaves according to a deterministic theory. From this point of view, our present investigation of simple physical theories could be seen as a stepping-stone to a more ambitious study of current theories which could have interesting scientific or philosophical repercussions. However, there is another perspective which is also interesting, particularly with regard to the more 'down-to-earth' theories we will consider. In many of these cases we already know in practice that the theory is deterministic in practice, in the sense that it enables the successful prediction of certain future events (such as eclipses). Thus, here it is not the determinism of the theory that is in question, but rather the adequacy of our formulation of the theory and of the notion of determinism in question. If the formulation of such theories does not allow us to infer some reasonable statement of determinism, we may conclude that there is something deficient about the formulation in the sense that it does not by itself explain the observed phenomena it is designed to explain; we are thus led to consider what other missing principles or covert assumptions need to be explicitly incorporated. In other words, we are subjecting physical theories to a discipline of axiomatization analogous to the axiomatic formulation of mathematics; such a discipline can only serve to rigorize our thinking and clarify our presuppositions.

It will be clear by now that there is no particular need to limit our considerations to the Laplacian problem of determining the future from the present.

Any 'predictive' problem in physics, such as that of determining a field throughout a region from information about its values on the boundary, would do just as well. Indeed, the whole of the above discussion of notions of determinism could be framed more abstractly in terms of the general question of which collections of physical facts abstractly imply [resp. enable us to compute, or allow us to deduce] which other physical facts. We have chosen here to concentrate on questions of Laplacian determinism partly in order to give the discussion a focus (examples relating to determinism being as good as any others), and partly because of the perceived philosophical significance of this topic.

We next mention two possible further motivations for our inquiry — one regarding what physics might have to contribute to logic, and one regarding what logic might have to offer to physics.

5.1 Indispensability arguments

The first of these relates to the issue of whether physics has any light to shed on the debates between competing philosophies of mathematics. On the one hand, the apparent indispensability of certain mathematical entities and principles as ingredients of scientific theories has sometimes been urged as one kind of justification for a realist attitude towards these entities and principles themselves. Such 'indispensability arguments' were first developed by Quine and Putnam; the general position is summarized by Maddy [57] as follows:[17]

> We have good reason to believe our best scientific theories, and mathematical entities are indispensable to those theories, so we have good reason to believe in mathematical entities. Mathematics is thus on an ontological par with natural science. Furthermore, the evidence that confirms scientific theories also confirms the required mathematics, so mathematics and science are on an epistemological par as well.

Early versions of this argument were somewhat coarse-grained, and tended to speak of 'mathematics' as a monolithic structure as if it stood or fell as a whole. However, the discussion has been considerably sharpened by the work of Feferman [29, 28], who asks exactly *which* mathematical entities and principles are in fact indispensable for *which* scientific theories. Much of the value of Feferman's contribution lies in his insistence in bringing the discussion down to specific, detailed questions about the strength of particular formal systems for particular applications, and the precise ontological commitments needed to justify these systems. A useful methodology here is provided by the *Reverse Mathematics* program of Friedman and Simpson (see [77]), which investigates the *necessity* of certain key axioms for certain theorems by showing that the theorem actually implies the axiom (typically, in the presence of less controversial axioms and

[17]We find a similar attitude expressed in Deutsch [20]: 'Mathematical entities are part of the fabric of reality ... We have no choice but to assume that the incomprehensible mathematical entities are real too, because they appear inextricably in our explanations of the comprehensible ones.'

inference rules). Results of this kind allow one to make precise the idea that a certain axiom or principle is indeed 'indispensable' for some desired conclusion.

Feferman's conclusion is that in fact a rather modest system based entirely on the principles of *predicativism* (such as Weyl's system **W**) is sufficient for developing practically all of "scientifically applicable mathematics", with some possible (and disputed) exceptions arising *e.g.* from the use of infinite-dimensional spaces in quantum mechanics. Thus, for Feferman, the success of real analysis (for example) in physical theories does not furnish any justification for impredicative concepts such as that of the completed powerset of the natural numbers, let alone any 'higher' set theory. Although it might be traditional to make use of such concepts in the classical development of the relevant parts of analysis, it is not *necessary* to do so. However, it remains to be seen whether Feferman's contention holds good for the further reaches of modern physics, such as quantum field theory.

In a series of contributions highly relevant to our concerns, Hellman [42, 41, 43] concurs with Feferman on the sufficiency of predicativist principles for large parts of applied mathematics, but argues that a stricter *constructivist* stance is not sufficient for physics, despite of the impressive body of analysis that can be developed constructively as in [9]. Hellman's main counterexamples draw on rather advanced physics, such as the spectral theorem for linear operators in quantum mechanics, or the Hawking-Penrose singularity theorems in general relativity. As we shall see below, some interesting problems in a similar vein in fact arise much earlier on, in connection with even the most rudimentary parts of physics. One possible merit of the more down-to-earth examples we shall consider is that they lie much closer than Hellman's to what can be directly validated by observation (*e.g.* by the predictive success of some classical theory), and the case that such observation serves to confirm the required mathematics would therefore seem to be correspondingly strengthened.

5.2 'Constructivization' of physical theories

We now venture to suggest, even more speculatively, some possible ways in which this area of investigation might provide a fruitful source of inspiration for physics itself.[18]

In the study of fundamental physics, it is natural to try to develop theories on as economical an ontological base as possible. By 'ontology' here one ordinarily understands what *physical* entities are supposed to exist, but one might also seek to cut down on one's *mathematical* ontology in order to keep one's overall metaphysical commitments to a minimum.

As already suggested, difficulties seem to arise if one's mathematical ontology is *too* restricted — at least for physical theories as they are usually formulated. However, as we shall see in later sections, the level of mathematical ontology that is required is often quite sensitive to the finer details of which *physical*

[18] The author is not himself a physicist, and is only too aware of his lack of expertise in the subject. He wishes to offer these ideas humbly as tentative suggestions arising from experience in another discipline.

entities or relations are treated as ontologically real. There is therefore at least the possibility that one might be led by metamathematical considerations to recast one's physical ontology in more 'finitistic' terms, if this seemed to lead to a more economical system overall.[19]

How might one arrive at such an alternative physical ontology? It is here that ideas from logic may be able to help. Indeed, the problem at hand is broadly analogous to that of putting *mathematical* concepts and theories on a more constructive or effective footing, and considerable experience in this arena has been gained by logicians over several decades. A very pertinent case in point is that of topology. The classical 'point-set' development of this area of mathematics makes heavy use of highly infinitistic concepts from set theory; however, it turns out that the essential content of large portions of the theory can be developed on a much more modest 'constructive' base in the form of *locale theory* (see [46, 89]). Here there is, so to speak, a switch in the basic ontology: rather than considering the *points* of a space to be its fundamental constituents, we take its abstract lattice of *open sets* as primary. This simple idea turns out to lead to a rich and compelling theory.

In fact, there is by now a rather extensive body of material, in a broad and loosely defined area spanning aspects of locale theory, domain theory and computability theory, and offering a variety of perspectives on this general idea that a more finitary, 'effective' handle of certain topological spaces can be obtained by giving primacy to some concept of 'region' or 'observable property', and relegating 'points' to the role of a convenient idealization or limit concept. (As representatives of this general area we mention [2, 24, 23, 6, 5, 10, 80, 81, 90].) Most of the more recent work here comes from theoretical computer science, and derives its impetus from the desire for appropriate theories of computability for the spaces typically arising in classical analysis.

Now that these ideas are approaching maturity and some compelling mathematical structures and concepts are starting to emerge, it may be that the time is ripe for seeing whether such ideas can be applied to the 'reconstruction' of portions of physical theory. For instance, in one's treatment of the spacetime manifold, one might ascribe ontological reality not to points in spacetime (whose physical status might in any case seem dubious) but rather to intervals or regions. Indeed, various approaches having something like this general flavour have already been proposed (see *e.g.* [71, 35, 22]), but our suggestion is that the mathematical theory mentioned above may have something to contribute. The hope is that this might ultimately lead to a simpler and more economical formulation of physical theories, and one which makes their computability properties more transparent. The general tendency of such a simplification would be to bring ontology into closer alignment with epistemology.[20]

[19] A similar programme is suggested by Shipman at the end of [76]: 'Feynman's dictum that it shouldn't take an infinite amount of information to describe what is going on in a finite region of spacetime still seems reasonable, and the task before us is to come up with better theories and models, more constructive and finitary in nature but compatible with the quantum world we live in.'

[20] Our proposal seems to be along similar lines to a suggestion in [65, Chapter 3].

Our suggestion, then, is that work in logic and computer science might provide a source of inspiration for alternative ontologies for physical theories. In this paper we will make a few preliminary forays in this direction, and give a rudimentary impression of what such an approach might look like.

6 A logical framework for physical theories

In this section we propose a simple logical framework for describing physical theories, their respective ontologies and their predictive power. The purpose of the framework is to allow us to formalize physical theories to the extent that they become amenable to precise logical analysis. We regard our proposal only as a first attempt at a suitable general framework for the logical discussion of physics, and one which invites further improvement. For the time being, we are content if our framework is helpful for the discussion of the particular examples we shall consider. Our presentation in this section leans rather heavily towards *logical* notions of determinism, which is what will chiefly occupy our attention in the next few sections. Later on, we will see how the same ideas also enable us to discuss computability issues.

Any physical theory requires an *ontology* — intuitively, a stock of concepts corresponding to entities which are deemed by the theory to 'exist'. Moreover, a theory will treat certain properties or attributes of these entities as being *ontologically real*.[21] The totality of ontologically real facts about a system (at a given time t) will constitute a description of the *state* of the system according to the theory. For various theories, we will be investigating the question of whether, and in what sense, the ontologically real facts about a system at a time t_0 determine the ontologically real facts at a time t_1 later than t_0.

Of course, we may take the view that the entities acknowledged by the theory are *all* that exists, in which case the set of ontologically real facts will constitute a *complete* description of the physical system — or we may take the view that there may be other entities but we are simply not choosing to talk about them, in which case our set of facts will simply provide a certain window onto the physical reality. The latter perspective means that we can use the same framework to study more epistemically oriented kinds of determinism, /eg/ we can consider what can be predicted or deduced from certain *observable* facts without committing ourselves to the view that these are all the facts there are. For simplicity, however, we will tend to favour ontological language in the ensuing discussion, as if we are adopting the former perspective.

Rather than get embroiled in concerns over what it means for something to 'exist', we shall take the 'ontology' of a theory to mean the total repertoire of concepts that one requires in order to articulate the theory. In particular,

[21]For example, in a Newtonian theory of celestial mechanics, one treats particles as real entities possessing the real attributes of mass, position and momentum at any point in time, but it is not necessary to treat gravitational fields as real. By contrast, in Maxwell's theory of electromagnetic waves, we are led to treat the electromagnetic fields themselves as ontologically real, since electromagnetic waves may carry energy even in a vacuum.

this might include not only *physical* concepts such as that of a particle or wave, but *mathematical* concepts such as that of a real number or a set of reals, if these turn out to be indispensable for expressing the theory. What is important here is not so much whether one thinks of these entities as 'really existing', as whether certain facts and relations involving them are considered to be ontologically meaningful.[22] One might, of course, hold that mathematical entities enjoy a quite different kind of 'existence' from physical ones, but for our present purposes there is no particular need to draw a sharp distinction between the two. Indeed, we will sometimes wish to postulate entities of a somewhat hybrid character, such as *sets of points in time*.

6.1 Languages for physical theories

A physical theory will thus comprise, among other things, a *language* for representing ontologically real facts. Our notion of a language is reminiscent of that of a signature for a logic. For our purposes, a language $\mathcal{L}$ will consist of the following:

- A collection (typically finite) of *type symbols*, *e.g.* *Particles*, *Positions*, *Times*, ω, $\mathcal{P}\omega$. These correspond to the types of entities we wish to talk about. We do not distinguish formally between physical and mathematical entities.

- For each type name, a class of *constants* of that type.[23] These will be used as names for the particular entities that may feature in ontologically real facts. Intuitively, we will need a constant for each entity that is treated as 'existing'; this often means we will require infinitely many constants of certain types (*e.g.* one for each position in space that a particle might occupy). We will use ordinary identifiers as examples of constants, and checked identifiers (*e.g.* $\check{x}$) for metavariables ranging over constants.

- A collection of *relation symbols*, each with a given *arity* consisting of some argument slots each with an assigned type. For instance, a theory might have a ternary relation symbol $P(- : \mathit{Particles}, - : \mathit{Positions}, - : \mathit{Times})$ to represent the fact that a given particle has a certain position at a certain time. Arities may in principle be allowed to be infinite, though in practice we will consider only finite arities in our examples. The relation

[22]One might therefore argue that it is ultimately only the *facts* that are ontologically real, and that the role of concepts corresponding to *entities* is merely to provide a language for expressing these facts. 'The world is the totality of facts, not of things' (Wittgenstein, *Tractatus* §1.1). This suggests that one should really consider theories modulo some relation of mutual interpretability, two theories being considered equivalent if they express the same facts, regardless of whether there is any correspondence between their respective classes of entities. However, for the purpose of this paper this more sophisticated point of view will not be required.

[23]In the spirit of Gödel-Bernays set theory, we shall say 'a class of Xs' (rather than 'a set of Xs') when we do not mean to imply that the Xs in question can be collected into a completed totality which can be apprehended as a whole (either by the demon or by ourselves). Thus, 'a class of Xs' means roughly 'a notion of an X'.

P should be distinguished from *e.g.* the formula $P(A, x, t)$ obtained by filling its slots with certain variables (see below); nevertheless, by abuse of language we shall sometimes speak of 'a relation $P(A, x, t)$'.

By filling the argument slots in relations with constants of the appropriate type, we obtain the *atomic propositions*, (written *e.g.* as $P(\check{A}, \check{x}, \check{t})$). The intention is that atomic propositions express potential facts, and a particular physical system will be described (according to the theory) by some class of atomic propositions corresponding to the *actual* (*i.e.* true) facts.

- A class of *formulae*, including all atomic propositions, and closed under certain (possibly infinitary) conjunctions and disjunctions. That is, for certain *conjoinable* sets S of formulae we have a formula $\bigwedge S$, and for certain *disjoinable* sets T we have a formula $\bigvee T$. We suppose that all finite sets are both conjoinable and disjoinable, and usually write $\phi_1 \wedge \cdots \wedge \phi_n$ and $\phi_1 \vee \cdots \vee \phi_n$ for $\bigwedge\{\phi_1, \ldots, \phi_n\}$ and $\bigvee\{\phi_1, \ldots, \phi_n\}$ respectively. We also write $\top$ for $\bigwedge \emptyset$ and $\bot$ for $\bigvee \emptyset$.

  Usually, the class of formulae will contain atomic formulae constructed using (typed) *variables* as well as constants, and will also be closed under binary implication and universal and existential quantification (denoted by $\Rightarrow, \forall, \exists$).

The possibly infinite sets S, T here are not themselves treated as entities in the theory, and need not be considered to 'exist' by the demon. Nevertheless, we do require that the demon has *some* way of contemplating the resulting formulae $\bigwedge S, \bigvee T$, whether by knowing about suitable indexing sets for S, T or otherwise, and this, as well as the explicit ontology embodied by types and relations, needs to be taken into account when assessing the metaphysical commitments that a theory depends on.

We do not include *function symbols* in our definition of a language. This is because we see functions as themselves a kind of 'entity', and the spirit of our approach is to make explicit, as types of the language, all the kinds of entities we need to refer to. A particular language could, of course, feature a type ρ of functions from type σ to type τ, together with an ternary relation symbol $AppEq(f : \rho, x : \sigma, y : \tau)$ (corresponding to '$f(x) = y$'), plus *laws* asserting totality and single-valuedness of this relation. As usual, we can enrich our language with function symbols if we wish, on the understanding that formulae involving them are simply shorthand for the corresponding formulae involving relations. In practice, we will do this only rarely below.

Some further remarks to clarify the intentions behind the above machinery are in order. We are thinking of our language not as a finitary medium for communication between agents such as ourselves, but as a framework for expressing all true facts about a physical system, and moreover as an idealized 'language of thought' for a supposed demon who can see all these true facts. This explains the infinitary aspects of the above definition. We intuitively conceive of the demon as looking at the universe 'from outside', and able to observe whatever facts it pleases without affecting the state of the system in any way. Moreover, we suppose the demon to be 'outside physical time' in the sense that it can make as many observations as it likes on a single instantaneous state without any physical time elapsing in between, and any 'thinking' on the demon's part is likewise assumed not to cost any physical time.[24]

Intuitively, the true atomic facts are the things that the demon has immediate access to as it looks at the physical system. For convenience, we treat facts as *affirmable* from the demon's perspective: if a fact is true, the demon will 'eventually' be able to see it; but if a fact is not true, the demon will not in general be able to detect its absence. (Refutable facts, if required, may of course be incorporated into the framework via their negations.) However, there is no requirement that true facts should be affirmable to an observer *within* the physical universe.

By contrast, we may informally think of general formulae as expressions of things such as a demon might know, believe, or entertain as a possibility in the course of a deduction. This allows for a level of logical superstructure on top of the realm of brute physical facts. In contrast to atomic propositions, we do not presuppose a notion of *truth* for general formulae; this leaves us free, if we so wish, to take a purely formalist or instrumentalist view which eschews any notion of 'meaning' for such formulae. An exception is made for conjunctions and disjunctions, which we understand as having their 'usual' meaning — see below.

6.2 Theories and logical entailments

We now describe the remaining ingredients which we shall consider to constitute a physical theory. For our purposes, a *theory* $\mathcal{T}$ will consist of a language $\mathcal{L}$ as above along with the following:

- A class L of distinguished formulae of $\mathcal{L}$ called *laws*. Typically, these may express the 'physical laws' postulated by the theory. However, they may also include any necessary axioms for the 'mathematical' notions that are involved in the theory. Once again, we do not need to distinguish formally between physical and mathematical laws here: we are concerned only with the totality of assumptions that form the basis of the theory. We sometimes need to consider infinite classes of laws since certain laws are given schematically: *e.g.* we might wish to include $\phi \vee \neg\phi$ as a law for every formula ϕ. We may imagine that all the laws are known to and believed by our predictive demon.

- Some sets of general formulae, called *surveyable sets*, such that each surveyable set is conjoinable and disjoinable, and every finite set is surveyable. The intuition is that if S is designated as surveyable and the demon can

[24]Even in this highly idealized scenario, there will be plenty of interesting questions to discuss. In the sequel paper we will briefly touch on alternative scenarios in which the demon behaves somewhat more like an agent within the physical universe itself.

see each formula ϕ of S to be true, then it can see the truth of all of these formulae at once, and can hence see the conjunction $\bigwedge S$ to be true.

- A *consequence relation* $S \vdash \phi$ between sets of propositions and propositions, capturing some notion of logical entailment. We require $\vdash$ to have at least the following closure properties:
  - if $\phi \in S$ then $S \vdash \phi$;
  - if $S \vdash \phi$ and $S \sqsubseteq T$ then $T \vdash \phi$;
  - if $S \vdash \phi$ for each $\phi \in T$ and $T \vdash \psi$, then $S \vdash \psi$;
  - if S is conjoinable [resp. disjoinable] and $\phi \in S$, then $\{\bigwedge S\} \vdash \phi$ [resp. $\{\phi\} \vdash \bigvee S$];
  - if S is surveyable then: $S \vdash \bigwedge S$, and if $T \cup \{\phi\} \vdash \psi$ for all $\phi \in S$ then $T \cup \{\bigvee S\} \vdash \psi$;
  - $\vdash$ is closed under the usual (intuitionistic) inference rules for the connectives $\Rightarrow, \forall, \exists$ (if these are present).

  If A is a *class* of propositions, we will for convenience write $A \vdash \phi$ to mean $S \vdash \phi$ for some $S \subseteq A$.

A further word about the status of sets of formulae. As we have mentioned these are not themselves entities that the demon knows about (except in the weak sense that it must be able to form the corresponding formulae) — rather, they are our way of describing what the demon can do. For us, however, these sets have the status of entities. Thus, in order to discuss what the demon can and cannot see, we may need to suppose that we ourselves can see a little more than the demon can. (Note also the third clause above, where we implicitly assume that T is surveyable to us, though not necessarily to the demon.) This is a rather common situation in metamathematical studies. That said, we usually will not need to suppose we can see *much* more than the demon — in fact, our concept of a set can be left rather vague, and only needs to be strong enough to support whatever we want to use it for. We do not ourselves need to assume a full-blown Platonic universe of sets in order to discuss a demon with only modest powers of surveillance.

We should comment briefly on the difference between the formulae $\forall x : \sigma.\phi(x)$ and $\bigwedge_{\check{x}:\sigma} \phi(\check{x})$. The former will typically be proved in the usual mathematical way, that is, by (finitary) reasoning involving a hypothetical 'arbitrary' element x. This does not in itself involve an ability to 'survey' the type σ, but it does require that we believe in the type σ in order to parameterize our reasoning by an arbitrary element of it. By contrast, a surveyable conjunction is 'proved' by proving all the conjuncts separately — this is typically an infinitary principle, but an advantage is that we are not so strongly committed to the presence of the type σ in the theory. Thus, if each conjunct $\phi(\check{x})$ separately is equivalent to some ψ_x that makes no reference to σ, we could write $\bigwedge_{x:\sigma} \psi_x$, where σ is now treated as a meta-level type. Finally, a *non-surveyable* conjunction can typically not be proved by either means, though the individual conjuncts can be inferred from it.

When considering logical determinism, we will normally take $\vdash$ to be a *syntactically* defined entailment relation, generated *e.g.* by certain inference rules which we are free to specify. However, we may also be interested (*e.g.* for metaphysical determinism) in *semantic* notions of entailment, depending for instance on Platonic notions of truth for the natural numbers or other mathematical structures. Such notions can typically be captured in our framework by postulating sufficiently many surveyable sets (this of course relies on the availability at the meta-level of the notions of truth in question).

The following examples illustrate typical uses of surveyable sets. Suppose $\mathcal{T}$ is a theory containing the type ω of natural numbers, with constants $0, 1, 2, \ldots$. By a *formula context* $\phi[- : \omega]$ we mean, informally, a formula with zero or more occurrences of a 'blank' of type ω; by plugging a constant $\check{n}$ into these blanks we may obtain an ordinary formula $\phi[n]$.[25] We may say T is a *weak ω-theory* if for each formula context $\phi[- : \omega]$, the set $S_\phi = \{\phi[0], \phi[1], \ldots\}$ is surveyable. Thus, for instance, if T contains the language and inference rules of first order arithmetic, a weak ω-demon will be able to see all classically true statements of first order arithmetic.

We may also say $\mathcal{T}$ is a *strong ω-theory* if it satisfies the following 'collection principle': Given any set S and any formula context $\phi[-_1 : \sigma_1, \ldots, -_r : \sigma_r, - : n]$ such that for any constant $\check{n} : \omega$ there is some choice of constants $\check{b}_1, \ldots, \check{b}_r$ such that $S \vdash \phi[\check{b}_1, \ldots, \check{b}_r, \check{n}]$, there is a surveyable set T such that

- each $\psi \in T$ has the form $\phi[\check{b}_1, \ldots, \check{b}_r, \check{n}]$,
- $S \vdash \psi$ for each $\psi \in T$
- for each $\vec{n} : \omega$ we have $\phi[\check{b}_1, \ldots, \check{b}_r, \check{n}] \in T$ for some choice of the $\check{b}_i$.[26]

Intuitively, whereas a weak ω-demon can verify an ω-indexed family of facts given to it by means of a formula, a strong ω-demon has the power to 'notice' many infinite families of facts for itself. It is easy to see that any strong ω-demon is also a weak ω-demon. Analogous survey principles can be defined for other types besides ω.

6.3 Models for theories

The central problem we wish to study (for various theories) is roughly as follows: given a class A of atomic propositions describing the state of a physical system, which statements ϕ can the demon infer, in the sense that $S \cup L \vdash \phi$ for some $S \subseteq A$?

Some further definitions are useful in order to clarify which classes A we regard as reasonable here. First, we suppose we have identified a class R of

[25] In the case of a theory with variables, a formula ϕ with a free variable n could play more or less the same role.

[26] Note that for a given $\check{n}$ there may be many possible choices of the $\check{b}_i$ here — this is what makes our condition a 'collection principle' rather than a 'choice principle'.

propositions that are *relevant* to the initial state of the system (or any other 'given' conditions). Typically, this might include all atomic propositions that describe conditions at time 0. The idea is that we do not expect to be able to deduce any such facts that are not given to us already, since we suppose A to be a complete description of the system with respect to such facts. Moreover, if we are able to deduce some disjunction of such facts, it is reasonable to expect that we can directly observe at least one of the facts. This motivates the following definitions:

- A *relevance class* is a class R of formulae closed under arbitrary (existing) disjunctions and surveyable conjunctions. (Intuitively, it is natural to extend our notion of 'observable fact' in this way, since the facts we can infer will always be closed under these operations.)
- A *model* for our theory $\mathcal{T}$ relative to R is a class A of atomic propositions[27] satisfying the following *plenitude condition*:

  If $T \subseteq R$ is disjoinable and $A \cup L \vdash \bigvee T$, then there is some $\phi \in T$ such that $A \cup L \vdash \phi$.

The plenitude condition is not itself something the demon needs to know about or use in its deductions; rather, it is a meta-law we use to ensure that the class A of given facts is compatible with the theory, given that it is supposed to be 'complete' for formulae in R. Note that by specializing the condition to the case $T = \emptyset$, we obtain the *consistency* condition $A \cup L \not\vdash \bot$; this sets that the given facts themselves conform to any constraints imposed by the laws.

We can now see that even non-surveyable conjunctions and disjunctions can be useful. For instance, a theory might contain a law of the form $\bigwedge_{\check{x}}(\bigvee_{\check{y}} \phi[\check{x},\check{y}])$, where the conjunction and all the disjunctions are non-surveyable, and each $\phi[\check{x},\check{y}] \in R$. The presence of such a law imposes a genuine constraint on models: for any particular $\check{x}$ we can infer $\bigvee_{\check{y}} \phi[\check{x},\check{y}]$, and the plenitude condition then ensures that some particular $\phi[\check{x},\check{y}]$ is inferrable. By contrast, it does not seem that a law of the form $\bigwedge_{\check{x}}(\bigvee_{\check{y}}(\bigwedge_{\check{z}} \phi[\check{x},\check{y},\check{z}]))$ succeeds in saying anything if the $\check{z}$-conjunctions are not surveyable. (The situation is reminiscent of the rather special status of Π^0_2 sentences in logic.)

7 Representing a single continuous parameter

As a first example of the use of the above framework, let us consider some theories which seek merely to express the value of a single continuous parameter — for the sake of definiteness, the position of a stationary single point particle in a one-dimensional space. Even for this utterly trivial scenario, some significant issues arise, and we shall here discuss two kinds of representation in particular. Of course, this is not yet a test case for notions of determinism, since there is

[27] For the avoidance of doubt, this is a rather more syntactic notion of model than is usually considered in model theory!

nothing to 'predict'; however, we shall see in the sequel paper that in less trivial scenarios the choice between these representations can indeed make a difference as regards determinism.

There are really two aspects to the problem: that of characterizing the 'background space' (if we decide to assume there is such a thing), and that of characterizing the position of the particle within it. Let us first try to capture some aspects of the background space conceived as a set of points.[28]

7.1 A point-based view of the line

The theory described here will be called the *background theory* $\mathcal{B}$. We will use a language containing a type *Positions*, plus constants $\check{x}$: *Positions* intended to represent points (that is, *potential* positions of the particle). Let us suppose that these constants constitute a set, which we also denote by *Positions*. For convenience we fix on two distinguished constants $\hat{0}, \hat{1}$: *Positions* in order to fix an origin, scale and choice of positive direction for our line. We now take a relation $\mathit{Distinct}(x, x')$ for saying that two positions are not the same.[29]

More generally, we will want a model to contain enough facts to determine the geometry of the line. We are not too concerned here about minimality of presentation, so let us be quite generous in our choice of relations. Consider the language of arithmetical expressions t given by

$$t \ ::= \ x \mid \hat{0} \mid \hat{1} \mid t_0 + t_1 \mid -t_0 \mid t_0 * t_1 \mid q.t_0$$

where x ranges over variables of type *Positions* and q ranges over all rationals, considered here as scalar multipliers. (Note that the type of rationals is not itself present in our theory.) For each pair t, u of such expressions with position variables among $x_1, \ldots, x_r$, we give ourselves a relation symbol $\mathit{Less}_{t,u}(x_1, \ldots, x_r)$, which for readability we usually denote by '$t < u$'. We will always consider our relevance class R to include all atomic propositions constructed from such relations — intuitively, we take it as read that all relationships of this kind between points should be apparent to the demon.

As laws for the theory, we may for convenience take all *true* formulae of the form

$$\bigwedge_{\check{x_1}, \ldots, \check{x_r}} \ \bigvee_{\check{y_1}, \ldots, \check{y_s}} \ \phi(\check{x_1}, \ldots, \check{y_s})$$

where ϕ is some propositional combination of formulae $t < u$ over $x_1, \ldots, y_s$ (we allow $\perp$ to appear in such formulae). By 'true' here we mean 'true in the standard theory of the real numbers'; note that the set of formulae in question is decidable. We also take the two *Archimedean laws*:

$$\bigwedge_{\check{x}} \bigvee_{n \in \mathbb{Z}} x < n.\hat{1} \qquad \bigwedge_{\check{x}, \check{y}} (\mathit{Distinct}(\check{x}, \check{y}) \Rightarrow \bigvee_{n \in \mathbb{Z}} \hat{1} < n.(x - y))$$

[28] That is, we here treat all the points in space as 'existing' in their own right, independently of whether they are occupied by the particle or not. This attitude towards the background space is sometimes referred to as *manifold substantivalism*, *e.g.* in [22].

[29] We here follow the intuitionistic treatment of the real numbers in treating *distinctness* rather than equality of points as a positive property.

(Note that these provide examples of disjunctions that are not parameterized by a type of the theory.) Finally, we have a law

$$\bigwedge_{\check{x},\check{y}}(\mathit{Distinct}(\check{x},\check{y}) \;\Leftrightarrow\; x<y \vee y<x)$$

which in principle allows us to dispense with *Distinct*.

All this is now enough to ensure that, in any model, each $\check{x} \in \mathit{Positions}$ can be uniquely correlated (at the meta-level) with an ordinary real number $\rho\check{x}$. For this purpose, the disjunctions in the Archimedean laws need not be surveyable; however, for most other purposes it is useful to take them to be so.

Note that ρ may map different constants to the same real number. This has nothing to do with the idea of *infinitesimals*, which are ruled out by the Archimedean laws, but simply arises from the fact that it is possible to have many *names* for the same point. This is harmless since, in this case, the names will satisfy precisely the same logical properties. A much deeper issue is that the mapping ρ need not be surjective — this in effect impinges on the question of what we mean by 'the' real numbers. We will examine this issue in detail in Section 8.

A variant $\mathcal{B}_=$ of the above theory may be obtained by also adding relations $Eq_{t,u}(x_1,\ldots,x_r)$, written for readability as '$t=u$', and allowing such formulae to appear in the set of true formulae we take as laws. In this case, we say our theory describes a space with *decidable equality*. For a weak ω-demon, this is in effect true anyway, since we may regard '$t=u$' as merely an abbreviation for the surveyable conjunction $\bigwedge_{n\in\mathbb{Z}} n.(t-u) < \hat{1}$. However, if we do not wish to assume ω-surveyability, it is of interest to see whether decidable equality on points is required for whatever we wish to prove.

7.2 The position of a particle

We now turn to the question of representing the position of the particle itself. We consider two approaches, which differ in whether or not a point position can be captured by a single fact. Perhaps surprisingly, this is somewhat orthogonal to the issue of decidable equality for the points themselves.

In the first approach (which we take to correspond to the common 'naive' conception), we simply introduce a relation $\mathit{Is\_At}(x)$ which allows us to say that the particle is precisely at a given point. We include all the resulting atomic propositions in the class of relevant facts. The 'physical laws' for the system may be taken to be:

- $\bigvee_{\check{x},\check{x}'}(\mathit{Is\_At}(\check{x}) \wedge \mathit{Is\_At}(\check{x}') \wedge \mathit{Distinct}(\check{x},\check{x}')) \Rightarrow \bot$
- $\bigvee_{\check{x}} \mathit{Is\_At}(\check{x})$

(For most purposes, the disjunction need not be surveyable.) A model of this theory will then essentially be a model for $\mathcal{B}$ together with a chosen position. The crucial point here is that the fact $\mathit{Is\_At}(\check{x})$ is considered to be ontologically real — thus, the position of the particle is in effect visible to our demon to infinite precision at a single glance.

The second approach is what we shall call an *interval-based* approach to positions. This may be motivated by more epistemic considerations. A human agent, if confronted with a point particle, could not ascertain its position exactly, but could at best make a succession of ever more precise measurements to within certain diminishing error bounds. We may therefore, by analogy, imagine a demon who is likewise able to observe the position of the particle only by means of such measurements, albeit to any required finite precision. Of course, we might still suppose that the particle 'really' had an exact, determinate point position which the demon's observations were approximating; or we might go a step further and take the view that the supposed unknowable 'real position' was a chimera, and that this 'nest' of individually observable facts was all there was.

We may work with the same background theory $\mathcal{B}$, and take a relation $\mathit{Is\_In}(\check{x}, \check{y})$, informally meaning that the particle is somewhere in the open interval (x, y) (which we take to be empty if $x \not< y$). We work with open intervals since these intuitively correspond to affirmable properties: if the particle is in (x, y) then a sufficiently precise observation will be able to discover this fact (see [89]). We may now give ourselves the following laws:

- $\mathit{Is\_In}(\check{x}, \check{y}) \Rightarrow \check{x} < \check{y}$
- $\mathit{Is\_In}(\check{x}, \check{y}) \wedge \check{x}' < \check{x} \wedge \check{y} < \check{y'} \Rightarrow \mathit{Is\_In}(\check{x}', \check{y}')$
- $\mathit{Is\_In}(\check{x}, \check{y}) \wedge \mathit{Is\_In}(\check{x}', \check{y}') \Rightarrow \mathit{Is\_In}(\check{x}, \check{y}')$

Thus far, our theory allows for the possibility that the particle's position is only determined to within some interval. Indeed, this might be useful for some kinds of physical theories in which the value of some parameter is simply not ontologically determined to arbitrary precision — that is, the universe simply does not bother to make up its mind exactly what the value is.[30] Suppose, however, that we want to enforce the property that the position is determined to within arbitrarily small intervals. This can be achieved *e.g.* by the following laws:

- $\bigvee_{n \in \mathbb{Z}} \mathit{Is\_In}(-n.\hat{1}, n.\hat{1})$
- $\mathit{Is\_In}(\check{x}, \check{y}) \wedge \check{x} < \check{x}' < \check{y}' < \check{y} \Rightarrow \mathit{Is\_In}(\check{x}, \check{y}') \vee \mathit{Is\_In}(\check{x}', \check{y})$

In any model for the theory, it will now be the case that we can (classically) identify a unique real number specifying the position of the particle. However, within the theory itself we will not in general be able to infer any *single* fact

[30] This may sound very reminiscent of the status of parameters such as position and momentum in quantum mechanics, but that is not quite the kind of underdetermination we have in mind here. Even discussions of quantum mechanics tend to be framed in terms of some underlying ontological reality (*e.g.* the wave function) involving determinate real or complex parameters. Here we have in mind a theory in which the bottom level of ontology is one of intervals.

that specifies the position precisely. This will only be possible if enough sets of converging intervals are surveyable. Even for a weak ω-demon this may not be the case, since we cannot specify the facts we wish to conjoin uniformly by means of a single formula. However, to a strong ω-demon, the two approaches to specifying the particle position are essentially equivalent.

Once we have formulated this approach, it is very tempting to abandon the background space altogether and adopt a more 'relationalist' approach in which only the properties of the particle 'exist'. Thus, in place of the binary predicate Is_In we might take a family of nullary predicates (*i.e.* proposition symbols) $Is\_In_{x,y}$ parameterized at the meta-level, satisfying laws precisely analogous to those above.[31] We did not do this to start with, as we wanted to clarify the independence between decidable equality and determinate position. However, the more consistently 'pointless' perspective will be developed further in Part II.[32]

The point- and interval-based views of particle position represent two different responses to the question: 'What information are we given when we are given a particle with position?'. We should briefly mention here that there is yet a third approach, and it is the one commonly taken in many schools of constructive mathematics and computable analysis (*e.g.* [9, 90]). Here, a point on a line may be given by a *Cauchy sequence* of certain 'known' points (*e.g.* the rational ones) with some known rate of convergence. This approach is, in a sense, intermediate between the two discussed above, since given such a Cauchy sequence, it is within the competence of a weak ω-demon to infer a single fact expressing the precise position. However, we tend to favour the interval view to the Cauchy sequence one for the purpose of our investigations, for the reason that a single point may be approached by many different Cauchy sequences, so that a Cauchy sequence embodies not just the position of the point but also a 'choice of representation'. Thus, if we model the position by a Cauchy sequence, we are providing more information than would appear to be present in the physical situation (unless some exotic physical ontology is adopted). The interval view seems closer to an intuitive model of what the demon might 'see' as he looks at the particle.[33]

Even a fourth and fifth possibility might be mentioned as curiosities. For instance, a position might be specified by an oracle to which one feeds some affirmable property one wishes to test (specified by *e.g.* an open rational interval). If the oracle manages to affirm the observation, it will return 'yes'; otherwise, it will keep on trying forever, and one will not have the opportunity to ask further questions. (This is the sort of thing that could happen if the demon is following instructions in a purely *sequential* programming language.) This severely limits the kinds of computations that one can perform. The fifth possibility is that

[31] Of course, we will lose almost all of the geometry from the theory if we do this, and will have to incorporate it in some other way.

[32] In the meantime, the reader may wish to consult [6, 5, 10] to gain an idea of where we are heading.

[33] Actually, the interval and sequence approaches turn out to be very closely related, and it can be shown that in many contexts the difference does not matter (see [5, 80]). There are, however, some situations in which it *does* matter: see, for example, the discussion of the dichotomy principle in Section 8.5 below.

the position is specified by a finite 'program' that can compute approximations to the desired point as closely as desired. This, again, gives us much more information than we would normally consider to be physically present, and is also limited in that it can only represent the 'computable' points. We mention these possibilities not as serious candidates for consideration here, but simply to emphasize the fact that the way a continuous parameter is given to us makes a big difference to what we are able to do with it.

8 Derivatives and the time continuum

8.1 A simple Newtonian problem

We are now at last in a position to investigate the question of determinism for some very simple physical theories within our framework, with a view to scrutinizing the implicit physical and metamathematical assumptions. We shall start with a childishly simple problem: that of the motion of a single point particle of mass 1 in one dimension under a constant force F, according to Newton's second law of motion.

We will try to formalize the 'classical' conception of this scenario. The state of the system at any instant in time will be given by two parameters: the position and momentum of the particle. There are thus three 'continua' involved: position, momentum and time. We will model each of these in the classical point-based way:[34] that is, we postulate the background theory $\mathcal{B}$ for the type *Positions* exactly as in Section 7.1, and similarly for *Momenta* and *Times*. We use x, p, t as variables ranging over *Positions*, *Momenta* and *Times* respectively, and $\check{x}, \check{p}, \check{t}$ as metavariables ranging over the constants of these types. We also take relations $PosAtTime(t, x)$ and $MomAtTime(t, p)$ to express the (exact) state of the particle at time t, and give us laws analogous to those for Is_At to say that the particle has a unique position and momentum at each time t.

Next, we may take $\hat{0}$: *Times* as our starting time, and stipulate that our relevance class P consists of all atomic propositions of the form $PosAtTime(\hat{0}, \check{x})$ and $MomAtTime(\hat{0}, \check{p})$, together with all atomic propositions relating to the geometry of our three continua. A model A for our theory w.r.t. P will thus consist of two facts $PosAtTime(\hat{0}, x_0)$ and $MomAtTime(\hat{0}, p_0)$ giving the initial state of the system, together with enough facts to provide models of the background theory for each of the continua.

Now everything is in place except the laws that determine the actual dynamics of the system. Of course, the solution we are hoping to arrive at (in

[34] We will start by considering a point-based approach since the concepts involved are already familiar in this setting. An interval-based treatment is also interesting, but requires the development of some further ideas, *e.g.* because the treatment of differentiation is not entirely straightforward. We will comment briefly on the interval-based approach at the end of this section, and will study it in much more detail in the sequel paper.

traditional notation) is

$$x(t) \;=\; x_0 + p_0 t + F t^2/2, \qquad p(t) \;=\; p_0 + Ft$$

More precisely, for any triple $(\check{t}, \check{x}, \check{p})$ satisfying these relations, we might hope to derive

$$A \cup L \vdash \mathit{PosAtTime}(\check{t}, \check{x}) \wedge \mathit{MomAtTime}(\check{t}, \check{p})$$

where L consists of all the laws of our theory. It would be perfectly possible to achieve this by simply adopting suitable translations of the above equations as 'laws' of our theory — this would be tantamount to postulating some kind of unexplained or 'unmediated' connection between the state at time $\hat{0}$ and that at time $\check{t}$. However, it is more in the spirit of traditional physics (and certainly more Laplacian in spirit) to try show how this global behaviour arises from *local* laws and properties, and this is what Newton's second law seeks to do. In traditional notation, the dynamical laws may be written

$$\dot{x}(t) \;=\; p(t), \qquad \dot{p}(t) \;=\; F$$

(where a dot denotes differentiation with respect to time). With the affine geometry of our various continua in place, we may express the above laws in our framework in Weierstrass style, *e.g.* as follows (helping ourselves to a little syntactic sugar):

$$\forall t.\, \forall \epsilon > 0.\, \exists \delta > 0.\, \forall t'.\; |t' - t| < \delta \Rightarrow |(p(t') - p(t)) - F(t' - t)| < \epsilon |t' - t|$$

Here we straightaway encounter a dilemma: how do we treat the quantifiers in such a formula? For each of the quantifiers independently, we in principle have the following options (listed here roughly in decreasing order of metaphysical commitment):

1. Translate the quantifier by a corresponding surveyable conjunction or disjunction in our system. This would succeed in capturing the naive 'classical' meaning of the quantifier, at the price of some surveyability assumptions.

2. Translate the quantifier by a non-surveyable conjunction or disjunction. This weakens the required metaphysical commitment, but may result in a formula from which less can be deduced. For example, if this choice is adopted for *all* the quantifiers, it seems that the formula does not succeed in saying anything at all (see the discussion at the end of Section 6.3).

3. Treat the quantifier as an object-level syntactic construct of the theory, equipping it with suitable laws, such as the usual intuitionistic rules for $\forall$ and $\exists$, and try to find some way of assigning a meaning to the quantifier other than by appealing to a survey principle. In particular, one might propose some kind of constructive reading of $\exists\delta$, saying that we have some way of finding a δ. For example, we might postulate that some kind of 'modulus information' was actually present in the physical state of the

system — a rather bizarre proposal from a physical point of view.[35] More generally, any 'way' of finding a δ would seem to depend on importing some further kind of ontological or metaphysical assumption.

4. Treat the quantifier as in 3, but do not attempt to assign any meaning to it. This would correspond to a formalist or instrumentalist attitude, which treats the theory as an uninterpreted piece of syntax that happens to yield the right answers under certain formal manipulations.

There is not much to say about option 4 here — it can be made to work, but the author's strong preference is to hope that a more intelligible and conceptually satisfying *explanation* of physics might be forthcoming. We regard 3 as a somewhat exotic proposal, taking us well beyond the classical Newtonian conception — we will not pursue it here, although something of a similar flavour will resurface when we consider an interval-based approach. This leaves us with 1 and 2. In the case of the outer quantifiers $\forall t, \forall \epsilon$, it makes no difference to provability which we choose, so for economy let us take these to be non-surveyable. As we have remarked, we then need to treat at least one of $\exists \delta$ and $\forall t'$ as surveyable. Here, it is natural to ask what is needed in order to be able to apply the theory. To anticipate what we shall discover below, it would seem to be necessary and sufficient to treat the $\exists \delta$ quantification as surveyable, and this is the choice we provisionally adopt.

In fact, we do not require the full strength of surveyability over all possible time lengths δ, but can get away with just a surveyable quantification over ω, of a kind that is acceptable to a weak ω-demon:

$$\bigwedge_{t} \bigwedge_{\epsilon>0} \bigvee_{n} \bigwedge_{t'} |t'-t| < 2^{-n} \Rightarrow |p(t') - p(t) - F(t'-t)| < \epsilon |t'-t|$$

(Technically, we do not even need to postulate a type ω here, only the disjoinability and surveyability of a certain family of formulae.) The 'δ-version' of the above statement, if desired, can then be deduced using a version of the Archimedean axiom (see Section 7.1) in which the disjunction over n is taken to be surveyable.

8.2 Some problems for logical determinism

We may now ask: in what sense is this theory deterministic? Here it is not the computability of the intended solution that is in doubt, nor the fact that this solution does indeed satisfy the above laws, but rather the question of uniqueness. Are our laws sufficient to ensure that the intended solution is the *only* solution (metaphysical determinism)? More particularly, can the values of x and p at a given time $\check{t}$ be inferred from x_0, p_0 in the theory as we have presented

[35] This particular proposal also seems to involve a kind of prescience: the state can only 'know' that a particular δ is suitable for a given t and ϵ if it is sure that the force is not going to increase wildly at time $t + (\delta/2)$. But perhaps there are other versions of the idea that do not suffer from this problem.

it (logical determinism)? For the sake of concreteness, we will concentrate here on the latter question, although many of the same issues apply to the former question as well.

As things stand, the answer is 'not always', since there is nothing to say that *Times* constitutes a *complete* continuum. The problem arises even for the 'trivial' case where $x_0 = \hat{0}$, $p_0 = \hat{0}$ and $\check{t} = \hat{1}$. For instance, a perfectly reasonable model can be given in which all points in time correspond to *rational* numbers; one may then consider the set of all facts arising from the trajectory given by

$$x(t) \;=\; \begin{cases} 0 & \text{if } t < 1/\sqrt{2} \\ 1 & \text{if } t > 1/\sqrt{2} \end{cases} \qquad\qquad p(t) \;=\; 0$$

This satisfies all the laws of the theory — in particular the dynamical laws hold at all points in time, since the 'time' at which the discontinuity occurs does not exist! Of course, this particular trajectory can easily be excluded by postulating further properties of *Times* (*e.g.* that it is algebraically closed); but in fact, essentially the same problem will arise for *any* set of laws for *Times* that is framed in terms of first-order logic. (We have in mind attempts to add in some countable set of 'completeness laws' for *Times*, none of which involve the relations *PosAtTime* and *MomAtTime*.) Indeed, such a set of laws will always have a model in which the set of instants in time is countable, and by replacing $1/\sqrt{2}$ by any classical real not represented by an instant in time, one may construct a complete model for the whole theory as before. (This shows, incidentally, that we cannot infer $PosAtTime(\hat{1}, \hat{0})$ no matter how many surveyable sets are admitted.)

As we shall see below, the problem can be overcome by postulating certain higher-order entities of an intuitively 'non-physical' nature, and (more essentially) by adding completeness laws which *do* involve the state of the particle. Let us suppose, however, that we are reluctant to do either of these things before we have exhausted all other options. Let us see what else we can try.

Firstly, the following objection to our argument might be raised. Recall that our relevance class includes all atomic propositions that pertain to the geometry of *Times*; thus, any model A will have to contain, among other things, a complete description of this geometry via atomic facts. (Informally, the demon is assumed to know exactly what the future part of the time-line is going to consist of before it arrives.) Granted that there are *some* models that describe time-lines with gaps; but we are only really interested in what can be inferred from models A that describe the time continuum as she actually is. Suppose that A includes a point-by-point description of a time-line that is *in fact* complete, whether or not the demon has any way of seeing this or even of expressing the concept of completeness. Can we then infer $PosAtTime(\hat{1}, \hat{0})$ from $A \cup L$?

Surprisingly, the answer appears still to be no. It is fairly easy to see this if all surveyable sets are at most countable: a 'proof' of $PosAtTime(\hat{1}, \hat{0})$ will then have the form of a well-founded countably branching tree, and can therefore mention only countably many of the geometrical facts. We may then construct a countable submodel validating all these facts; re-running the above argument then leads to a contradiction. For the case where arbitrary surveyable sets are admitted, it seems likely that a similar conclusion may be proved by means of a set-theoretic reflection argument, though we have not checked the details. The reader will get a feel for the problem by trying to construct, relative to the above scenario, a derivation of $A \cup L \vdash \mathit{PosAtTime}(\check{t}, \hat{0})$ for *any* time $\check{t}$ later than $\check{0}$; there appears to be a 'Zeno-like' problem even in getting off the starting block.

8.3 Locality and physical laws

One might also wonder whether the problem could be fixed simply by varying or strengthening our statements of the dynamical laws in some way. However, it is easy to see that exactly the same problems will beset any proposed dynamical laws which admit the intended solutions and are 'local' in character — that is, laws of the form $\forall t.\, L(t)$, where $L(t)$ is a local predicate. By a local predicate we informally mean a formula $L(t)$ such that for any time $\check{t}$ and any two possible histories (*i.e.* trajectories) h, h' that agree on some ϵ-neighbourhood of $\check{t}$, $L(\check{t})$ holds for h iff it holds for h'. We may capture a rather mild version of this notion without reference to possible histories as follows: $L(t)$ is a local predicate, if whenever A, A' are both complete models for the theory *without* the dynamical laws, containing exactly the same geometrical facts, and such that for all $\check{t'}$ in some ϵ-neighbourhood of $\check{t}$ we have

$$\begin{aligned} \mathit{PosAtTime}(\check{t'}, \check{x}) \in A \quad &\text{iff} \quad \mathit{PosAtTime}(\check{t'}, \check{x}) \in A', \\ \mathit{MomAtTime}(\check{t'}, \check{p}) \in A \quad &\text{iff} \quad \mathit{MomAtTime}(\check{t'}, \check{p}) \in A' \end{aligned}$$

we have $A \cup \{L(\check{t})\} \vdash \bot$ iff $A' \cup \{L(\check{t})\} \vdash \bot$. The point is that the pathological trajectories involving jumps that were used in the above arguments are locally just fine — that is, each point in time has some neighbourhood in which these trajectories agree with some globally constant trajectory which we do wish to allow. Thus, no set of local dynamical laws can rule out the jumping trajectories without ruling out these constant trajectories too.

An example of a possible set of non-local laws for this situation was mentioned earlier: the laws that simply say the particle follows the expected trajectory. However, we contend that any non-local theory can hardly be called a formalization of the original Newtonian theory — if we accept non-locality, then we are really considering a physical theory of a very different kind. Indeed, the idea that the laws of physics ought to be local in character (with respect to both time and space) seems to us to be an important ingredient of the informal Laplacian idea of a *mechanistic* universe. Certainly, the discovery of (experimentally confirmed) non-local phenomena in quantum theory was widely considered to be deeply shocking, and represented a profound shift in our understanding of the kind of universe we are in.

It is tempting to think that the jumping trajectories could be excluded by postulating some upper bound on the speed at which our particle can move (such as the 'speed of light'). Taken literally, this is a fallacy: there is *no* time when the speed of the particle is anything exceptional. However, one might still ask whether something like the following (mildly non-local) *Lipschitz condition* might suffice (here $\lambda > 0$ is some constant specified by the theory):

$$PosAtTime(t, x) \wedge PosAtTime(t', x') \Rightarrow |x' - x| \leq \lambda |t' - t|$$

This condition expresses a particularly strong kind of uniform continuity. However, it turns out that not even such a condition as this will suffice. In fact, given any dense countable subset D of the classical reals and any $\lambda > 0$, one may construct a function $f : D \to \mathbb{R}$ which is locally constant and satisfies the above Lipschitz condition, but is not globally constant. We have in mind a 'Devil's staircase' construction, similar in flavour to the construction of the Cantor space.[36] This example would seem to suggest that quite a strong kind of non-locality is required to rule out the jumping trajectories.

8.4 How much completeness do we need?

Suppose that at this point we concede that there is a case for admitting some non-physical, *e.g.* 'set-theoretic' entities into our theory. For instance, in classical real analysis one assumes a least upper bound axiom: every bounded subset of $\mathbb{R}$ has a least upper bound. For this, we need to admit the notion of an arbitrary 'subset of $\mathbb{R}$' into our discourse. In fact this is overkill: it is sufficient, and more ontologically economical, to admit the notion of a countable *sequence* $\mathbb{N} \to \mathbb{R}$, and postulate that every Cauchy sequence has a limit (see [28]; cf. the discussion of Section 5.1). The concept of 'sequence' here is still a second-order one, but of a milder kind. Let us now see how these ideas may be deployed in our setting.

First, since in Newtonian physics one typically conceives time and space continua as an 'inert backdrop' against which events take place, it is natural to try to give a self-contained description of the nature of the time continuum without reference to the state of the particle. For instance, we might postulate a type $\mathcal{P}(Times)$ of subsets of *Times*, and perhaps other types as well, together with certain relations and laws (including a least upper bound axiom for *Times*), but without any use of the relations *PosAtTime* and *MomAtTime*. Perhaps surprisingly, such an approach is still bound to fail: despite the seemingly higher-order character of our system, and despite the fact that it may even be possible to prove the existence of 'uncountable sets' within the system, the whole theory *still* has a countable model,[37] and our earlier 'jumping' arguments can still be applied.

The reader may be getting worried by this point, since after all, in classical mathematics it *is* possible to prove that the differential equations in question have unique solutions subject to the initial conditions. Indeed, by directly translating this classical theory to the physical setting of *Times*, we do in fact obtain

[36] Superficially, at least, our construction uses the axiom of choice. However, we are here only using this construction in a heuristic role, to argue that there is no hope of showing logical determinism for the theory in question. For this, it suffices to assume that the required metatheory is *consistent* — and it would be clutching at straws to hope that it is not!

[37] This is often known as the *Skolem paradox*.

a formulation for which logical determinism holds. So what is the catch? It is that this classical approach, when analysed, turns out to depend crucially on the fact that points in time may be specified *in terms of* the trajectories that are supposed to take place within time! Thus, a 'standalone' approach to characterizing the points of the time continuum will not do — we are driven to accept that our theory of the content of this continuum must be (directly or indirectly) intertwined with our theory of the events that might occur in it. The idea of this kind of interdependence might not seem so unfamiliar to a modern relativity theorist, but it is perhaps surprising that we seem to be driven to this idea even to express the naive Newtonian conception precisely.

To emphasize this idea yet further, one may consider the effect of imposing (possibly global) *a priori* constraints on the class of permissible trajectories. The stronger the constraints, the less completeness we need to assume for *Times* in order to make logical determinism work. For example, if the trajectory is assumed to be given by a polynomial in t, the density assumptions on *Times* already suffice. If the trajectory is merely assumed to be computable, it suffices to suppose that *Times* contains points for all computable reals. If one just assumes the trajectory is continuous (a local condition), a stronger completeness principle for *Times* becomes necessary.

In the classical conception, the implicit interdependence between times and trajectories is not generally noticed, precisely because it is naturally mediated by the postulated 'set-theoretic' entities to which both sides are independently related. For example, a sequence may be constructed from a trajectory in various ways, and a point in time constructed from a sequence by taking its limit.

We will now outline what we consider to be a natural formalization of this common classical picture, using the minimum of ontology needed to do this without distorting the informal picture too much. Even within these constraints, some interesting choices arise, as we shall see.

8.5 A classical approach

Our approach will be inspired by the existence of a simple *constructive* proof of the following result: if $f : [0, 1] \to \mathbb{R}$ is continuous and differentiable with zero derivative everywhere, then f is constant. It might come as a mild surprise that this is possible, since theorems of constructive analysis (as developed *e.g.* in [9]) usually require the stronger hypothesis of *uniform* continuity on every compact interval (a non-local condition).[38] Let us sketch how the proof goes (a more detailed presentation is given in [1, Section 8.2]). We will content ourselves with deriving a contradiction from the hypothesis that $f(0) = 0$ and $f(1) = 1$. We will construct a sequence of closed intervals $[x_0, y_0], [x_1, y_1], \ldots$ such that, for each i we have $y_i - x_i = 2^{-i}$; $f(y_i) - f(x_i) > 2^{i+1}$; and $[x_{i+1}, y_{i+1}]$ is either the

[38] In the conference version of this paper [55], we claimed, erroneously, that the result in question was *not* constructively provable, and that this could be seen by means of a recursive counterexample based on the Kleene tree (or singular coverings). Regrettably, the construction we then had in mind was fatally flawed.

left or right half of $[x_i, y_i]$. We start by setting $[x_0, y_0] = [0, 1]$. Given $[x_i, y_i]$ as above, let $\epsilon_i = 2^i(f(y_i) - f(x_i)) - 1/2$, let $z_i = (x_i + y_i)/2$, and compute $f(z_i)$ to within $2^{-i-2}\epsilon_i$. By this stage, either we know that $f(z_i) - f(x_i) > 2^{i+2}$, or we know that $f(y_i) - f(z_i) > 2^{i+2}$. We may therefore set $[x_{i+1}, y_{i+1}]$ to be $[x_i, z_i]$ in the first case, and $[z_i, y_i]$ otherwise. Now let r be the real number determined by this nest of intervals. It is easy to see that f cannot have zero derivative at r, since there are r' arbitrarily close to r with $f(r') - f(r) > (r' - r)/2$.

Our theory is, in effect, designed to be the minimal natural theory that suffices to support this proof. In effect, our theory simply requires that the time continuum is just as complete as it needs to be to deal with the possible histories that can arise. We will not give full details of our theory, but merely point out the principles of interest, and will freely use sugared notation where it makes the ideas clearer.

We will consider that we have achieved our goal if, in the case $F = 0$, given a model in which $p(0) = p_0$, we can deduce $\mathit{Distinct}(p(1), p_0)) \Rightarrow \bot$. If we wish to go on to infer that $p(1) = p_0$ (*i.e.* $\mathit{MomAtTime}(\check{1}, p_0)$), we require dedicable equality for the space *Momenta* (see Section **??**).

The first interesting principle we require is the following dichotomy principle for *Momenta*:

$$p < q \;\Rightarrow\; (x > p \,\vee\, x < q)$$

There is no intrinsic problem understanding this 'classically', in that the formula $x > p \vee x < q$ already has meaning as a surveyable disjunction of affirmable facts. However, the assumptions we will need to adopt below would be weakened if the principle could be invested with constructive content which gave us a way of *picking* an appropriate disjunct for given p, q, x. The principle is typically accepted in constructive approaches to analysis in which reals are presented in terms of more intensional objects such as Cauchy sequences (as in [9]) or recursive indices (as in [1]). However, the principle seems less innocuous in our present setting, in which x is given to us simply as a 'point in momentum space', since any choice operation for the disjunction in question will necessarily be discontinuous.[39] (In effect, we are led to ask whether we should endow our demon with the ability to arbitrate in a 'race' between two observation processes, perhaps taking place in the demon's 'virtual time'.) From a formal point of view, it will be enough if we can define predicates $\psi_p[p, q, x], \psi_q[p, q, x]$ such that we can derive

$$p < q \;\Rightarrow\; ((\psi_p \wedge x > p) \vee (\psi_q \wedge x < q)) \wedge \neg(\psi_p \wedge \psi_q)$$

This is indeed possible if we have decidable equality for momenta, but if this route is taken, our proof is relying on this fact in a very essential way.

Next, we see that in the proof above, a succession of disjunctive choices of the above kind is assembled into an infinite sequence. To capture this, we may postulate a 'mathematical' type *InfSeq* of infinite binary sequences over 0,1,

[39]The dichotomy principle fails in some models for constructive mathematics, such as the realizability model over Scott's $\mathcal{P}\omega$ with the extensional real number object (see [5]). This model will play an important role in our sequel paper.

ranged over by g. We also use a type *FinSeq* of finite binary sequences, ranged over by s, though this is of course mild by comparison. We will write e for the empty finite sequence, $s.0$ and $s.1$ for the sequences obtained by appending a single bit to s, and $g\mid_n$ for the finite sequence obtained as the first n bits of g. The sequence formation principle we require may now be framed as follows (subject to some qualifications which we are about to discuss).

$$\phi(e) \ \wedge \ (\forall s.\phi(s) \Rightarrow \phi(s.0) \vee \phi(s.1)) \ \Rightarrow \ (\exists g.\forall n.\phi(g\mid_n))$$

Of course, to conduct the above proof we only require this for a particular formula $\phi(s)$, which says something about the momentum of the particle at the endpoints of the time interval corresponding to s.

In fact, this is more or less the right formulation if we follow the *classical* treatment of the dichotomy principle — note that the subformula $\phi(s) \Rightarrow \phi(s.0) \vee \phi(s.1)$ meshes with the simpler of the two dichotomy formulae above. In this setting, the present formula expresses the principle of *countable dependent choice*, interpreted classically. We regard this as a definitely non-constructive and not even semi-constructive principle, since it asserts the existence of completed infinite objects which it does not give us any way to define.

Alternatively, we may follow the constructive interpretation of dichotomy, which can be justified on semi-constructive grounds. For this, we need to replace $\phi(s) \Rightarrow \phi(s.0) \vee \phi(s.1)$ by one of the same shape as the more complex dichotomy formula. Our formula now expresses a much more modest principle of *unique choice*, of a kind that can be verified by a *strong* ω-demon.

Another issue concerns the quantifier $\exists g$. To express this via a surveyable disjunction over *InfSeq* would certainly be overkill: since the premises of the formula are supposed to give us the wherewithal to construct such a g, what we really want our formula to say is that some particular g has the required property. We can achieve this effect by *Skolemizing* the existential quantifier — that is, by postulating a function which constructs a suitable g from values of the free variables of $\phi(s)$ (excluding s itself). Of course, the Skolem function will itself have to be described in terms of relations; the end result is that we replace the quantifier $\exists g$ by a unique-existential quantifier, for which no surveyability properties are needed.

Finally, we come to the completeness axiom for *Times* itself. This says that every infinite sequence is a binary representation for some actual point in time. We will assume we have (a relation for) a function $\zeta : \mathit{FinSeq} \to \mathit{Times}$ mapping finite sequences to dyadic points in an obvious way.

$$\forall f : \mathit{InfSeq}.\, \exists t : \mathit{Times}.\, \forall n.\, |t - \zeta(f|_n)| < 2^{-n}$$

With this in place, we can now complete the proof, provided that in our statement of differentiability the disjunction over n is surveyable (see Section 8.1). We thus (at last) have a reasonable candidate for a logically deterministic formulation of our Newtonian problem.

Let us now briefly return to the question of *metaphysical* determinism for our theory. By means of the above ideas, one can in fact prove an assertion

$$L \vdash PosAtTime(t_0, x_0) \wedge MomAtTime(t_0, p_0) \Rightarrow \\ \forall t : Times.\ PosAtTime(t, x_0 + p_0 t + Ft^2/2)$$

and this by itself captures the essence of metaphysical determinism with no reference to histories at all. (This is rather dependent on the fact that we can give a complete logical description of the solution for this particular problem.) We will thus obtain (**MD**) itself in any reasonable extension of the theory that admits the type *Histories*. Moreover, the whole of the above discussion on what is needed to yield (**LD**) carries over to (**MD**), so it would appear that the necessary prerequisites for obtaining (**LD**) and (**MD**) are essentially the same.

8.6 Some remarks on indispensability

Having reached this point, it is natural to ask whether the ontological prerequisites of the theory can be pruned down still further. This question clearly has relevance to the indispensability arguments which we discussed in Section 5.1.

The author's view is that, on the whole, this can only be done at the cost of some artificiality and arbitrariness, and to the detriment of the explanatory power of the theory. (To take an extreme case, we can eliminate all uses of surveyability and all mathematical entities by the device we mentioned at the outset: we adopt the theory that simply says that the particle's trajectory follows the expected formula.) As we noted, this is unsatisfying in that it does not offer any kind of *explanation* of how or why this behaviour arises. Most of the obvious ways of applying Occam's razor to the above theory, we claim, represent a step in this direction.

The prime candidate for elimination is the type of infinite sequences. One can, in principle, short-circuit the reference to sequences by rolling together the sequence formation law with the completeness law for *Times*, along the following lines (here $\zeta(s), \xi(s)$ are the endpoints of the evident dyadic time interval corresponding to s):

$$(\phi(e) \wedge (\forall s.\ \phi(s) \Rightarrow \phi(s.0) \vee \phi(s.1)) \implies \\ \exists t. \forall n. \exists s.\ \phi(s) \wedge (|t - \zeta(s)| < 2^{-n} \vee |t - \xi(s)| < 2^{-n})$$

It seems difficult to assess the status of this proposal. To the author, the above formula looks bizarre if it is considered purely as a purely *physical* law (that is, if we are able to banish any idea of sequences from our mind while contemplating it). We also feel that there is some loss in explanatory power resulting from the elision of an intermediate step in the above proof. On the other hand, one might argue that there is not much essential metaphysical difference between the two formulations anyway, since one could take the view that *Times* itself offers a 'model' for some (rather moderate) mathematical theory of the reals as completed objects.

What seems more clear is that attempts to eliminate all uses of *surveyable* infinities from the theory tend to result in a radical depletion of its explanatory power, in that they are forced to build in as assumptions some significant principles for which we would rather have an intelligible explanation. This tends to point towards the conclusion that the philosophical premises of strict constructivism, for instance, do not provide an adequate basis for a satisfactory understanding of physics. Of course, this conclusion could in principle be refuted by finding some 'satisfying' logically deterministic formalization not requiring ω-surveyability.

We remark briefly here on a few other approaches that might be tried. One might ask whether the situation is any different for a theory based on intervals rather than points, in view of the close connection between points and completed infinite sequences. We will discuss interval formulations for various physical theories in the sequel paper, and in particular will investigate the advantages of such formulations for obtaining a good computability theory. In the meantime, suffice it to say that for the Newtonian problem we have been considering, an interval formulation does not by itself seem to dispel the problems we have been discussing, and in particular some ingredient such as ω-surveyability is still required in order to obtain logical determinism.[40] Similar remarks apply to attempts to capture the desired uniqueness of solutions by postulating properties of the abstract *sheaf* of (local) continuous functions over the time continuum.

Another possibility, naturally suggested by the interval setting, is to overcome the obstacles by postulating some Brouwerian principle such as *bar induction* for the time continuum. However, it turns out that these principles, whose original home was a theory of (idealized) mental constructions, give rise to some rather strange effects if applied to the description of an 'external' physical reality. Though we shall not attempt to argue the case in detail here, this approach would seem to lead to a theory in which the properties of physical entities are inextricably linked with the possible mental constructions or 'proofs' that the ideal human mathematician can perform. (We believe this conclusion could be made to stand out more clearly by a careful analysis of possible formulations of the kind we have undertaken for the point-based approach.) Such a view, we suggest, is only plausible in the context of a highly mentalistic or solipsistic attitude to the physical world.

Strictly constructive approaches, then, may not be enough. However, our theory of Section 8.5 would seem to sit very comfortably with *e.g.* a *predicativist* position (at least in spirit — as we have seen, the fact that we are trying to reason directly about physical entities affects the formal details of our theory in various ways). Our findings are thus very consonant with those of Hellman [41], who concludes that

> Classical, non-constructive concepts and reasoning would seem indispensable to some of our best science. However, [...] none of the objections to constructivism reviewed here tells against *predicativism*, or at least not without further argument.

However, we have arrived at our conclusions by a somewhat different route,

[40] There does seem to be one strictly constructive approach which seems very satisfactory from a physical point of view, although it is a rather surprising one. It will be presented in the sequel paper.

namely the close scrutiny of the relationships between mathematical and physical concepts in one very simple physical situation. It seems clear that if even the simplest Newtonian theory is problematic, similar issues will arise for *any* physical theory based on continua and the ideas of differential calculus.

9 Conclusions and prospectus

In this paper we have discussed various possible ways of making precise the idea of physical determinism and the kinds of choices that are involved. We have emphasized in particular the varying degrees of physical and metaphysical commitment that underwrite these notions of determinism. We have proposed a general logical framework for the precise articulation of physical theories, with the intention of allowing questions of determinism to be rigorously investigated. We have found that many issues and choices arise from the attempt to formulate even the simplest physical situations.

We believe that the discussion of Section 8 (in particular) amply demonstrates the subtlety and complexity of the issues involved, and the ways in which questions of determinism are highly sensitive to precise details of the formulation and of the ontological assumptions made. In general terms, one may informally observe a kind of trade-off or complementarity between the respective strengths of the *physical* and *metaphysical* assumptions needed to yield a logically deterministic theory (as one would surely hope to obtain in the case of our simple Newtonian scenario). For instance, the problems discussed in Section 8 can be addressed either by postulating stronger physical assumptions (*e.g.* nonlocal physical laws), or by postulating the existence of mathematical entities and the ω-surveyability of certain sets. In other words, it does not pay to be too parsimonious on both fronts. The *existence* of such a trade-off in principle is not in itself surprising (as can be readily seen by reflecting on some extremal positions), but the specific ways in which this kind of complementarity works out in an 'everyday' physical situation would perhaps not have been expected.

In the present paper we have focused largely on problems relating to logical determinism, since in the very simplest physical situations the issue of computability is unproblematic. As we shall see in the sequel paper, this will change when we move on to consider more complex situations, such as the evolution of an electromagnetic field under Maxwell's equations (a situation already studied in some detail in [69, 70, 92]). In this and other situations, our investigations of computability will involve a close scrutiny of possible alternatives regarding physical ontology, along the lines hinted at in Sections 5 and 7.

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