Algebraic & Coalgebraic Methods in Semantics

Lecture III

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Operational Models as Coalgebras

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Behaviour:

Operational Models as Coalgebras

Behaviour: endofunctor B

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Operational Model:

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Behaviour: endofunctor B

Operational Model: B-coalgebra

$$k : X \to BX$$

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Operational Model: B-coalgebra

$$k : X \to BX$$

Running Example

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Behaviour: endofunctor B

Operational Model: B-coalgebra

$$k : X \to BX$$

Running Example

$B : \mathbf{Set} \to \mathbf{Set}$

Operational Models as Coalgebras

Behaviour: endofunctor B

Operational Model: B-coalgebra

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Running Example

$B : \mathbf{Set} \to \mathbf{Set} \qquad BX = 1 + A \times X$

Operational Models as Coalgebras

Behaviour: endofunctor B

Operational Model: B-coalgebra

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Running Example

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- $k : X \to 1 + A \times X$

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Behaviour: endofunctor B

Operational Model: B-coalgebra

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Running Example

$B : \mathbf{Set} \to \mathbf{Set} \qquad BX = 1 + A \times X$

- $k : X \to 1 + A \times X$ (strongly) *deterministic transition system*

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$$k(x) = * \quad \Longleftrightarrow \quad x \downarrow$$

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Behaviour: endofunctor B

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- $k : X \to 1 + A \times X$ (strongly) *deterministic transition system*

$$k(x) = * \iff x \downarrow$$

$$k(x) = \langle a, x' \rangle \iff x \stackrel{a}{\to} x'$$

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$B : \mathbf{Set} \to \mathbf{Set} \qquad BX = 1 + A \times X \qquad x \xrightarrow{a} x'$ or $x \downarrow$

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- (terminating)

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Examples of computations:

- (terminating) $\qquad x$

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Examples of computations:

- (terminating) $\quad x \stackrel{a}{\to} x_1$

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Examples of computations:

- (terminating) $\qquad x \xrightarrow{a} x_1 \xrightarrow{b} x_2$

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Examples of computations:

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- (infinite)

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Final Coalgebra:

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Final Coalgebra: $\quad A^\infty \;\cong\; 1 + A \times A^\infty$

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$$\varepsilon \mapsto *$$

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$$a \cdot w \mapsto \langle a, w \rangle$$

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Coinductive Extension

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Maps a state to the word corresponding to its computation

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$k^{@} : X \to A^{\infty}$ $\qquad x \overset{a}{\to} x_1 \overset{b}{\to} x_2 \overset{c}{\to} x_3 \downarrow$

$$k^{@}(x) \;=\; a \cdot b \cdot c$$

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$k^{@} : X \to A^{\infty}$ $\qquad x \xrightarrow{a} x_1 \xrightarrow{b} x_2 \xrightarrow{c} x_3 \downarrow$

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$$A^\infty$$

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$$\begin{array}{ccc} X & \xrightarrow{k} & 1 + A \times X \\ {\scriptstyle k^{@}} \downarrow & & \downarrow {\scriptstyle 1 + A \times k^{@}} \\ A^{\infty} & \underset{\cong}{\longleftrightarrow} & 1 + A \times A^{\infty} \end{array}$$

Existence of Final Coalgebras

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If B preserves limits of ω^{op}-chains

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$\Longrightarrow$ Basic Lemma [*Smyth & Plotkin 82*]

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Final B-coalgebra as limit of $1 \leftarrow B1 \leftarrow B^2 1 \leftarrow \cdots$

Existence of Final Coalgebras

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Problem:

Existence of Final Coalgebras

If B preserves limits of ω^{op}-chains

$\Rightarrow$ Basic Lemma [*Smyth & Plotkin 82*]

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Problem: not true for the finite powerset endofunctor $\mathcal{P}_{\mathrm{fi}}$

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See my thesis, pages 168-9

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Final B-coalgebra as limit of $1 \leftarrow B1 \leftarrow B^2 1 \leftarrow \cdots$

Problem: not true for the finite powerset endofunctor $\mathcal{P}_{\mathrm{fi}}$

See my thesis, pages 168-9

Still, final $\mathcal{P}_{\mathrm{fi}}$-coalgebra exists [*Aczel & Mendler 89, Barr 92*]:

- set of rooted finitely branching trees modulo bisimulation.

Coalgebraic Bisimulation

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$$X \xleftarrow{r_1} R \xrightarrow{r_2} X'$$

Coalgebraic Bisimulation

$$
\begin{array}{ccccc}
X & \xleftarrow{\;r_1\;} & R & \xrightarrow{\;r_2\;} & X' \\
{\scriptstyle k}\downarrow & & \vdots & & \downarrow{\scriptstyle k'} \\
BX & \xleftarrow[\;Br_1\;]{} & BR & \xrightarrow[\;Br_2\;]{} & BX'
\end{array}
$$

Coalgebraic Bisimulation

$$\begin{array}{ccccc} X & \xleftarrow{\;r_1\;} & R & \xrightarrow{\;r_2\;} & X' \\ {\scriptstyle k}\downarrow & & \vdots & & \downarrow{\scriptstyle k'} \\ BX & \xleftarrow[\;Br_1\;]{} & BR & \xrightarrow[\;Br_2\;]{} & BX' \end{array}$$

[*Aczel & Mendler 89*]

Final Coalgebras & Bisimulation

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Final Coalgebras & Bisimulation

- Final Coalgebras are *internally fully abstract*
  - $\star$ $p \sim q \iff p = q$
- Coinductive Extensions identify bisimilar states
  - $\star$ $x \sim y \Rightarrow k^{@}(x) = k^{@}(y)$
  - $\star$ $x \sim y \Leftarrow k^{@}(x) = k^{@}(y)$ *if* B *preserves weak pullbacks*

Cofree Coalgebras

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Final B-Coalgebra: $A^\infty \cong 1 + A \times A^\infty$

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Final B-Coalgebra: $A^\infty \cong 1 + A \times A^\infty$

- abstract computations $\bullet \stackrel{a}{\rightarrow} \bullet \stackrel{b}{\rightarrow} \bullet \stackrel{c}{\rightarrow} \bullet \rightarrow$

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What if we want computations with states in X?

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What if we want computations with states in X?

$$x \overset{a}{\to} x_1 \overset{b}{\to} x_2 \overset{c}{\to} x_3 \to \cdots$$

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Final $X \times B$-Coalgebra:

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Final $X \times B$-Coalgebra: $A_X^\infty \cong X \times (1 + A \times A_X^\infty)$

Cofree Coalgebras

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$$\cong X + X \times A{\times}A^\infty_X$$

Cofree Coalgebras

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Final $X \times B$-Coalgebra: $\begin{aligned} A^\infty_X &\cong X \times (1 + A \times A^\infty_X) \\ &\cong X + X \times A \times A^\infty_X \end{aligned}$

$$Y \to X \times BY$$

Cofree Coalgebras

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Final $X \times B$-Coalgebra: $A_X^\infty \cong X \times (1 + A \times A_X^\infty) \cong X + X \times A \times A_X^\infty$

$$\frac{Y \to X \times BY}{Y \to X \qquad Y \to BY}$$

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Y

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$$\frac{Y \to X \times BY}{Y \to X \qquad Y \to BY}$$

$$Y \xrightarrow{k} 1 + A \times Y$$

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$$Y \xrightarrow{k} 1 + A\times Y$$

X

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$$\frac{Y \to X \times BY}{\overline{Y \to X \qquad Y \to BY}}$$

$$Y \stackrel{k}{\longrightarrow} 1 + A \times Y$$

$$Y \stackrel{f}{\longrightarrow} X \qquad A^\infty_X$$

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$$x \stackrel{a}{\to} x_1 \stackrel{b}{\to} x_2 \stackrel{c}{\to} x_3 \to \cdots$$

Final $X \times B$-Coalgebra:
$$\begin{array}{rcl} A^\infty_X & \cong & X \times (1 + A \times A^\infty_X) \\ & \cong & X + X \times A \times A^\infty_X \end{array}$$

$$\begin{array}{c} Y \to X \times BY \\ \hline\hline Y \to X \qquad Y \to BY \end{array}$$

$$\begin{array}{ccccc} & & Y & \xrightarrow{k} & 1 + A \times Y \\ & \swarrow^{f} & \downarrow{\scriptstyle f^\flat} & & \downarrow{\scriptstyle 1+A\times f^\flat} \\ X & \longleftarrow & A^\infty_X & \longrightarrow & 1 + A \times A^\infty_X \end{array}$$

Coinductive Extension along an Arrow

Coinductive Extension along an Arrow

Example

X

Coinductive Extension along an Arrow

Example

$$X \xrightarrow{k} 1 + A \times X$$

Coinductive Extension along an Arrow

Example

$$X \xrightarrow{k} 1 + A \times X$$

X

Coinductive Extension along an Arrow

Example

$$\begin{array}{ccc} & X & \xrightarrow{k} & 1 + A \times X \\ & {}^{\mathrm{id}_X}\swarrow & & \\ X & & A^{\infty}_{X} & \end{array}$$

Coinductive Extension along an Arrow

Example

$$
\begin{array}{ccccc}
 & & X & \xrightarrow{k} & 1 + A \times X \\
 & \swarrow^{\mathrm{id}_X} & \Big\downarrow{\scriptstyle {\mathrm{id}_X}^\flat} & & \Big\downarrow{\scriptstyle 1+A\times{\mathrm{id}_X}^\flat} \\
X & \longleftarrow & A^\infty_X & \longrightarrow & 1 + A \times A^\infty_X
\end{array}
$$

Cofree Coalgebras (ctd)

Y

Cofree Coalgebras (ctd)

$$Y \qquad\qquad\qquad Y \xrightarrow{k} 1 + A \times Y$$

Cofree Coalgebras (ctd)

$$Y \qquad\qquad Y \xrightarrow{k} 1 + A \times Y$$

$$X$$

Cofree Coalgebras (ctd)

$$
\begin{array}{ccc}
 & & Y \\
 & \swarrow^{f} & \downarrow f^\flat \\
X & \longleftarrow & A^\infty_X
\end{array}
\qquad\qquad
\begin{array}{ccc}
Y & \xrightarrow{k} & 1 + A \times Y \\
{\scriptstyle f^\flat}\,\vdots & & \downarrow {\scriptstyle 1+A\times f^\flat} \\
A^\infty_X & \longrightarrow & 1 + A \times A^\infty_X
\end{array}
$$

Cofree Coalgebras (ctd)

$$
\begin{array}{ccc}
 & & Y \\
 & \swarrow^{f} & \downarrow f^{\flat} \\
X & \longleftarrow & A^{\infty}_{X}
\end{array}
\qquad\qquad
\begin{array}{ccc}
Y & \xrightarrow{\;k\;} & 1 + A \times Y \\
{\scriptstyle f^{\flat}} \downarrow & & \downarrow {\scriptstyle 1 + A \times f^{\flat}} \\
A^{\infty}_{X} & \longrightarrow & 1 + A \times A^{\infty}_{X}
\end{array}
$$

Notation:

Cofree Coalgebras (ctd)

$$
\begin{array}{ccc}
 & & Y \\
 & \overset{f}{\swarrow} & \downarrow f^\flat \\
X & \longleftarrow & A^\infty_X
\end{array}
\qquad\qquad
\begin{array}{ccc}
Y & \xrightarrow{k} & 1 + A \times Y \\
f^\flat \downarrow & & \downarrow 1+A\times f^\flat \\
A^\infty_X & \longrightarrow & 1 + A \times A^\infty_X
\end{array}
$$

Notation: DX

Cofree Coalgebras (ctd)

$$\begin{array}{ccc} & & Y \\ & \swarrow^{f} & \downarrow f^\flat \\ X & \longleftarrow & A_X^\infty \end{array} \qquad \begin{array}{ccc} Y & \xrightarrow{k} & 1 + A \times Y \\ f^\flat \downarrow & & \downarrow 1+A\times f^\flat \\ A_X^\infty & \longrightarrow & 1 + A \times A_X^\infty \end{array}$$

Notation: DX $\equiv$ final $X \times B$-coalgebra

Cofree Coalgebras (ctd)

$$
\begin{array}{ccc}
 & & Y \\
 & \swarrow^{f} & \downarrow f^\flat \\
X & \longleftarrow & A^\infty_X
\end{array}
\qquad\qquad
\begin{array}{ccc}
Y & \stackrel{k}{\longrightarrow} & 1 + A\times Y \\
f^\flat \downarrow & & \downarrow 1+A\times f^\flat \\
A^\infty_X & \longrightarrow & 1 + A\times A^\infty_X
\end{array}
$$

Notation: DX $\equiv$ final $X \times B$-coalgebra (X-*computations*)

Cofree Coalgebras (ctd)

$$
\begin{array}{ccc}
 & & Y \\
 & \overset{f}{\swarrow} & \downarrow f^\flat \\
X & \longleftarrow & A^\infty_X
\end{array}
\qquad\qquad
\begin{array}{ccc}
Y & \xrightarrow{k} & 1 + A \times Y \\
f^\flat \downarrow & & \downarrow 1+A\times f^\flat \\
A^\infty_X & \longrightarrow & 1 + A \times A^\infty_X
\end{array}
$$

Notation: $DX \equiv$ final $X \times B$-coalgebra (X-*computations*)

$$B : \mathcal{C} \to \mathcal{C}$$

Cofree Coalgebras (ctd)

$$\begin{array}{ccc} & & Y \\ & \swarrow^{f} & \downarrow f^\flat \\ X & \longleftarrow & A^\infty_X \end{array} \qquad \begin{array}{ccc} Y & \xrightarrow{k} & 1 + A \times Y \\ f^\flat \downarrow & & \downarrow 1+A\times f^\flat \\ A^\infty_X & \longrightarrow & 1 + A \times A^\infty_X \end{array}$$

Notation: DX $\equiv$ final $X \times B$-coalgebra (X*-computations*)

$B : \mathcal{C} \to \mathcal{C}$ $\qquad\qquad$ Y

Cofree Coalgebras (ctd)

$$
\begin{array}{ccc}
 & & Y \\
 & \swarrow^{f} & \downarrow f^\flat \\
X & \longleftarrow & A_X^\infty
\end{array}
\qquad\qquad
\begin{array}{ccc}
Y & \xrightarrow{k} & 1 + A \times Y \\
{\scriptstyle f^\flat} \downarrow & & \downarrow {\scriptstyle 1 + A \times f^\flat} \\
A_X^\infty & \longrightarrow & 1 + A \times A_X^\infty
\end{array}
$$

Notation: $DX \equiv$ final $X \times B$-coalgebra (X*-computations*)

$$
B : \mathcal{C} \to \mathcal{C} \qquad\qquad Y \qquad\qquad Y \xrightarrow{k} BY
$$

Cofree Coalgebras (ctd)

$$
\begin{array}{ccc}
 & & Y \\
 & \swarrow^{f} & \downarrow f^\flat \\
X & \longleftarrow & A^\infty_X
\end{array}
\qquad\qquad
\begin{array}{ccc}
Y & \xrightarrow{k} & 1 + A \times Y \\
{\scriptstyle f^\flat}\downarrow & & \downarrow {\scriptstyle 1+A\times f^\flat} \\
A^\infty_X & \longrightarrow & 1 + A \times A^\infty_X
\end{array}
$$

Notation: DX $\equiv$ final $X \times B$-coalgebra (X-*computations*)

$$
B : \mathcal{C} \to \mathcal{C} \qquad\qquad Y \qquad\qquad Y \xrightarrow{k} BY
$$

$$
X
$$

Cofree Coalgebras (ctd)

$$\begin{array}{ccc} & & Y \\ & \swarrow^{f} & \downarrow f^{\flat} \\ X & \longleftarrow & A^{\infty}_{X} \end{array} \qquad \begin{array}{ccc} Y & \xrightarrow{k} & 1 + A \times Y \\ f^{\flat} \downarrow & & \downarrow 1 + A \times f^{\flat} \\ A^{\infty}_{X} & \longrightarrow & 1 + A \times A^{\infty}_{X} \end{array}$$

Notation: $DX \equiv$ final $X \times B$-coalgebra (X-*computations*)

$$B : \mathcal{C} \to \mathcal{C} \qquad \begin{array}{ccc} & & Y \\ & \swarrow^{f} & \downarrow f^{\flat} \\ X & \longleftarrow & DX \end{array} \qquad \begin{array}{ccc} Y & \xrightarrow{k} & BY \\ f^{\flat} \downarrow & & \downarrow B f^{\flat} \\ DX & \longrightarrow & BDX \end{array}$$

Cofree Coalgebras (ctd)

$$\begin{array}{ccc} & & Y \\ & \overset{f}{\swarrow} & \downarrow f^\flat \\ X & \longleftarrow & A^\infty_X \end{array} \qquad \begin{array}{ccc} Y & \xrightarrow{k} & 1 + A \times Y \\ f^\flat \downarrow & & \downarrow 1+A\times f^\flat \\ A^\infty_X & \longrightarrow & 1 + A \times A^\infty_X \end{array}$$

Notation: $DX \equiv$ final $X \times B$-coalgebra (X-*computations*)

$$B : \mathcal{C} \to \mathcal{C} \qquad \begin{array}{ccc} & & Y \\ & \overset{f}{\swarrow} & \downarrow f^\flat \\ X & \longleftarrow & DX \end{array} \qquad \begin{array}{ccc} Y & \xrightarrow{k} & BY \\ f^\flat \downarrow & & \downarrow Bf^\flat \\ DX & \longrightarrow & BDX \end{array}$$

$DX \to BDX$ is the cofree B-coalgebra over X

Cofree Coalgebras (ctd)

$$
\begin{array}{ccc}
 & & Y \\
 & \swarrow^{f} & \downarrow f^\flat \\
X & \longleftarrow & A^\infty_X
\end{array}
\qquad\qquad
\begin{array}{ccc}
Y & \xrightarrow{k} & 1 + A \times Y \\
f^\flat \downarrow & & \downarrow 1+A\times f^\flat \\
A^\infty_X & \longrightarrow & 1 + A \times A^\infty_X
\end{array}
$$

Notation: $DX \equiv$ final $X \times B$-coalgebra (X*-computations*)

$$
B : \mathcal{C} \to \mathcal{C}
\qquad
\begin{array}{ccc}
 & & Y \\
 & \swarrow^{f} & \downarrow f^\flat \\
X & \longleftarrow & DX
\end{array}
\qquad\qquad
\begin{array}{ccc}
Y & \xrightarrow{k} & BY \\
f^\flat \downarrow & & \downarrow Bf^\flat \\
DX & \longrightarrow & BDX
\end{array}
$$

$DX \to BDX$ is the cofree B-coalgebra over X

i.e. $U_B : B$-Coalg $\to \mathcal{C}$

Cofree Coalgebras (ctd)

$$\begin{array}{ccc} & & Y \\ & \swarrow^{f} & \downarrow f^\flat \\ X & \longleftarrow & A^\infty_X \end{array} \qquad \begin{array}{ccc} Y & \xrightarrow{k} & 1 + A \times Y \\ f^\flat \downarrow & & \downarrow 1+A\times f^\flat \\ A^\infty_X & \longrightarrow & 1 + A \times A^\infty_X \end{array}$$

Notation: DX $\equiv$ final $X \times B$-coalgebra (X*-computations*)

$$B : \mathcal{C} \to \mathcal{C} \qquad \begin{array}{ccc} & & Y \\ & \swarrow^{f} & \downarrow f^\flat \\ X & \longleftarrow & DX \end{array} \qquad \begin{array}{ccc} Y & \xrightarrow{k} & BY \\ f^\flat \downarrow & & \downarrow Bf^\flat \\ DX & \longrightarrow & BDX \end{array}$$

$DX \to BDX$ is the cofree B-coalgebra over X

i.e. U_B : B-Coalg $\to \mathcal{C}$ has a right
adjoint

Cofree Coalgebras (ctd)

$$\begin{array}{ccc} & & Y \\ & \swarrow^{f} & \downarrow f^\flat \\ X & \longleftarrow & A^\infty_X \end{array} \qquad \begin{array}{ccc} Y & \xrightarrow{k} & 1 + A \times Y \\ f^\flat \downarrow & & \downarrow 1+A\times f^\flat \\ A^\infty_X & \longrightarrow & 1 + A \times A^\infty_X \end{array}$$

Notation: $DX \equiv$ final $X \times B$-coalgebra (X*-computations*)

$$B : \mathcal{C} \to \mathcal{C} \qquad \begin{array}{ccc} & & Y \\ & \swarrow^{f} & \downarrow f^\flat \\ X & \longleftarrow & DX \end{array} \qquad \begin{array}{ccc} Y & \xrightarrow{k} & BY \\ f^\flat \downarrow & & \downarrow Bf^\flat \\ DX & \longrightarrow & BDX \end{array}$$

$DX \to BDX$ is the cofree B-coalgebra over X

i.e. $U_B : B\text{-Coalg} \to \mathcal{C}$ has a right adjoint $G : \mathcal{C} \to B\text{-Coalg}$

Cofree Coalgebras (ctd)

$$\begin{array}{ccc} & & Y \\ & \overset{f}{\swarrow} & \downarrow f^\flat \\ X & \longleftarrow & A^\infty_X \end{array} \qquad \begin{array}{ccc} Y & \xrightarrow{k} & 1 + A \times Y \\ f^\flat \downarrow & & \downarrow 1{+}A{\times}f^\flat \\ A^\infty_X & \longrightarrow & 1 + A \times A^\infty_X \end{array}$$

Notation: DX $\equiv$ final $X \times B$-coalgebra (X*-computations*)

$$B : \mathcal{C} \to \mathcal{C} \qquad \begin{array}{ccc} & & Y \\ & \overset{f}{\swarrow} & \downarrow f^\flat \\ X & \longleftarrow & DX \end{array} \qquad \begin{array}{ccc} Y & \xrightarrow{k} & BY \\ f^\flat \downarrow & & \downarrow Bf^\flat \\ DX & \longrightarrow & BDX \end{array}$$

$DX \to BDX$ is the cofree B-coalgebra over X

i.e. U_B : B-Coalg $\to \mathcal{C}$ has a right
adjoint G : $\mathcal{C} \quad \to \quad B$-Coalg
X

Cofree Coalgebras (ctd)

$$
\begin{array}{ccc}
 & & Y \\
 & \overset{f}{\swarrow} & \downarrow f^\flat \\
X & \longleftarrow & A_X^\infty
\end{array}
\qquad\qquad
\begin{array}{ccc}
Y & \xrightarrow{k} & 1 + A \times Y \\
f^\flat \downarrow & & \downarrow 1+A\times f^\flat \\
A_X^\infty & \longrightarrow & 1 + A \times A_X^\infty
\end{array}
$$

Notation: $DX \equiv$ final $X \times B$-coalgebra (X*-computations*)

$$
B : \mathcal{C} \to \mathcal{C}
\qquad
\begin{array}{ccc}
 & & Y \\
 & \overset{f}{\swarrow} & \downarrow f^\flat \\
X & \longleftarrow & DX
\end{array}
\qquad\qquad
\begin{array}{ccc}
Y & \xrightarrow{k} & BY \\
f^\flat \downarrow & & \downarrow Bf^\flat \\
DX & \longrightarrow & BDX
\end{array}
$$

$DX \to BDX$ is the cofree B-coalgebra over X

i.e. $U_B : B\text{-Coalg} \to \mathcal{C}$ has a right adjoint $G : \begin{array}{ccc} \mathcal{C} & \to & B\text{-Coalg} \\ X & \mapsto & \end{array}$

Cofree Coalgebras (ctd)

$$\begin{array}{ccc} & & Y \\ & \swarrow^{f} & \downarrow f^\flat \\ X & \longleftarrow & A^\infty_X \end{array} \qquad \begin{array}{ccc} Y & \xrightarrow{k} & 1 + A \times Y \\ f^\flat \downarrow & & \downarrow 1+A\times f^\flat \\ A^\infty_X & \longrightarrow & 1 + A \times A^\infty_X \end{array}$$

Notation: $DX \equiv$ final $X \times B$-coalgebra (X*-computations*)

$$B : \mathcal{C} \to \mathcal{C} \qquad \begin{array}{ccc} & & Y \\ & \swarrow^{f} & \downarrow f^\flat \\ X & \longleftarrow & DX \end{array} \qquad \begin{array}{ccc} Y & \xrightarrow{k} & BY \\ f^\flat \downarrow & & \downarrow Bf^\flat \\ DX & \longrightarrow & BDX \end{array}$$

$DX \to BDX$ is the cofree B-coalgebra over X

i.e. $U_B : B$-Coalg $\to \mathcal{C}$ has a right adjoint G :

$$\begin{array}{ccc} \mathcal{C} & \to & B\text{-Coalg} \\ X & \mapsto & DX \to BDX \end{array}$$

Cofree Coalgebras (ctd)

Cofree Coalgebras (ctd)

DX

Cofree Coalgebras (ctd)

$DX \equiv$ final $X \times B$-coalgebra (X*-computations*)

$$B : \mathcal{C} \to \mathcal{C} \qquad \begin{array}{ccc} & & Y \\ & \overset{f}{\swarrow} & \downarrow f^\flat \\ X & \longleftarrow & DX \end{array} \qquad \begin{array}{ccc} Y & \xrightarrow{k} & BY \\ f^\flat \downarrow & & \downarrow Bf^\flat \\ DX & \longrightarrow & BDX \end{array}$$

$DX \to BDX$ is the cofree B-coalgebra over X

i.e. $U_B : B$-Coalg $\to \mathcal{C}$ has a right adjoint

$$\begin{array}{ccc} \mathcal{C} & \to & B\text{-Coalg} \\ X & \mapsto & DX \to BDX \end{array}$$

Cofree Coalgebras (ctd)

DX $\equiv$ final $X \times B$-coalgebra (X-*computations*)

$$B : \mathcal{C} \to \mathcal{C} \qquad \begin{array}{ccc} & & Y \\ & \swarrow^{f} & \downarrow f^\flat \\ X & \longleftarrow & DX \end{array} \qquad \begin{array}{ccc} Y & \xrightarrow{k} & BY \\ f^\flat \downarrow & & \downarrow Bf^\flat \\ DX & \longrightarrow & BDX \end{array}$$

$DX \to BDX$ is the cofree B-coalgebra over X

i.e. $U_B : B$-Coalg $\to \mathcal{C}$ has a right adjoint

$$\begin{array}{ccc} \mathcal{C} & \to & B\text{-Coalg} \\ X & \mapsto & DX \to BDX \end{array}$$

- What is $D1$?

Cofree Coalgebras (ctd)

$DX \equiv$ final $X \times B$-coalgebra (X*-computations*)

$$B : \mathcal{C} \to \mathcal{C} \qquad \begin{array}{ccc} & & Y \\ & \swarrow^{f} & \downarrow f^\flat \\ X & \longleftarrow & DX \end{array} \qquad \begin{array}{ccc} Y & \xrightarrow{k} & BY \\ f^\flat \downarrow & & \downarrow Bf^\flat \\ DX & \longrightarrow & BDX \end{array}$$

$DX \to BDX$ is the cofree B-coalgebra over X

i.e. $U_B : B$-Coalg $\to \mathcal{C}$ has a right adjoint
$$\begin{array}{ccc} \mathcal{C} & \to & B\text{-Coalg} \\ X & \mapsto & DX \to BDX \end{array}$$

- What is $D1$?
  - ⋆ the final B-coalgebra!

Cofree Coalgebras (ctd)

$DX \equiv$ final $X \times B$-coalgebra (X-*computations*)

$$B : \mathcal{C} \to \mathcal{C} \qquad \begin{array}{ccc} & & Y \\ & \swarrow^{f} & \downarrow f^\flat \\ X & \longleftarrow & DX \end{array} \qquad \begin{array}{ccc} Y & \xrightarrow{k} & BY \\ f^\flat \downarrow & & \downarrow Bf^\flat \\ DX & \longrightarrow & BDX \end{array}$$

$DX \to BDX$ is the cofree B-coalgebra over X

i.e. $U_B : B$-Coalg $\to \mathcal{C}$ has a right adjoint

$$\begin{array}{ccl} \mathcal{C} & \to & B\text{-Coalg} \\ X & \mapsto & DX \to BDX \end{array}$$

- What is $D1$?
  - ⋆ the final B-coalgebra!
  - ⋆ 1-computations

Cofree Coalgebras (ctd)

DX $\equiv$ final $X \times B$-coalgebra (X*-computations*)

$$B : \mathcal{C} \to \mathcal{C} \qquad \begin{array}{ccc} & & Y \\ & \overset{f}{\swarrow} & \downarrow f^\flat \\ X & \longleftarrow & DX \end{array} \qquad \begin{array}{ccc} Y & \xrightarrow{k} & BY \\ f^\flat \downarrow & & \downarrow Bf^\flat \\ DX & \longrightarrow & BDX \end{array}$$

$DX \to BDX$ is the cofree B-coalgebra over X

i.e. $U_B : B$-Coalg $\to \mathcal{C}$ has a right adjoint

$$\begin{array}{ccc} \mathcal{C} & \to & B\text{-Coalg} \\ X & \mapsto & DX \to BDX \end{array}$$

- What is $D1$?
  - ⋆ the final B-coalgebra!
  - ⋆ 1-computations $\equiv$ most *abstract* computations

Dualities

Dualities

Behaviour B

Dualities

Behaviour B Signature Σ

Dualities

Behaviour B

B-Coalgebras

Signature Σ

Dualities

Behaviour B	Signature Σ
B-Coalgebras	Σ-Algebras

Dualities

Behaviour B

B-Coalgebras

$k : X \to BX$

Signature Σ

Σ-Algebras

Dualities

Behaviour B	Signature Σ
B-Coalgebras	Σ-Algebras
$k : X \to BX$	$h : \Sigma X \to X$

Dualities

Behaviour B	Signature Σ
B-Coalgebras	Σ-Algebras
$k : X \to BX$	$h : \Sigma X \to X$
Operational Models	

Dualities

Behaviour B	Signature Σ
B-Coalgebras	Σ-Algebras
$k : X \to BX$	$h : \Sigma X \to X$
Operational Models	Denotational Models

Dualities

Behaviour B	Signature Σ
B-Coalgebras	Σ-Algebras
$k : X \to BX$	$h : \Sigma X \to X$
Operational Models	Denotational Models
(Coalgebraic) Bisimilarity	

Dualities

Behaviour B	Signature Σ
B-Coalgebras	Σ-Algebras
$k : X \to BX$	$h : \Sigma X \to X$
Operational Models	Denotational Models
(Coalgebraic) Bisimilarity	Compositionality

Dualities

Behaviour B	Signature Σ
B-Coalgebras	Σ-Algebras
$k : X \to BX$	$h : \Sigma X \to X$
Operational Models	Denotational Models
(Coalgebraic) Bisimilarity	Compositionality
Final Coalgebra	

Dualities

Behaviour B	Signature Σ
B-Coalgebras	Σ-Algebras
$k : X \to BX$	$h : \Sigma X \to X$
Operational Models	Denotational Models
(Coalgebraic) Bisimilarity	Compositionality
Final Coalgebra	Initial Algebra

Dualities

Behaviour B	Signature Σ
B-Coalgebras	Σ-Algebras
$k : X \to BX$	$h : \Sigma X \to X$
Operational Models	Denotational Models
(Coalgebraic) Bisimilarity	Compositionality
Final Coalgebra	Initial Algebra
State-less Computations	

Dualities

Behaviour B	Signature Σ
B-Coalgebras	Σ-Algebras
$k : X \to BX$	$h : \Sigma X \to X$
Operational Models	Denotational Models
(Coalgebraic) Bisimilarity	Compositionality
Final Coalgebra	Initial Algebra
State-less Computations	Closed Terms

Dualities

Behaviour B	Signature Σ
B-Coalgebras	Σ-Algebras
$k : X \to BX$	$h : \Sigma X \to X$
Operational Models	Denotational Models
(Coalgebraic) Bisimilarity	Compositionality
Final Coalgebra	Initial Algebra
State-less Computations	Closed Terms
Cofree Coalgebra over X	

Dualities

Behaviour B	Signature Σ
B-Coalgebras	Σ-Algebras
$k : X \to BX$	$h : \Sigma X \to X$
Operational Models	Denotational Models
(Coalgebraic) Bisimilarity	Compositionality
Final Coalgebra	Initial Algebra
State-less Computations	Closed Terms
Cofree Coalgebra over X	?

Dualities

Behaviour B	Signature Σ
B-Coalgebras	Σ-Algebras
$k : X \to BX$	$h : \Sigma X \to X$
Operational Models	Denotational Models
(Coalgebraic) Bisimilarity	Compositionality
Final Coalgebra	Initial Algebra
State-less Computations	Closed Terms
Cofree Coalgebra over X	?
Computations with states in X	

Dualities

Behaviour B	Signature Σ
B-Coalgebras	Σ-Algebras
$k : X \to BX$	$h : \Sigma X \to X$
Operational Models	Denotational Models
(Coalgebraic) Bisimilarity	Compositionality
Final Coalgebra	Initial Algebra
State-less Computations	Closed Terms
Cofree Coalgebra over X	?
Computations with states in X	?

Dualities

Behaviour B	Signature Σ
B-Coalgebras	Σ-Algebras
$k : X \to BX$	$h : \Sigma X \to X$
Operational Models	Denotational Models
(Coalgebraic) Bisimilarity	Compositionality
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State-less Computations	Closed Terms
Cofree Coalgebra over X	?
Computations with states in X	?

Free Algebras

Free Algebras

Closed Terms (example)

Free Algebras

Closed Terms (example)

- $\mathcal{E}$

Free Algebras

Closed Terms (example)

- $\mathcal{E} \ni e ::=$ zero

Free Algebras

Closed Terms (example)

- $\mathcal{E} \ni e ::= \mathsf{zero} \mid \mathsf{plus}(e_1, e_2)$

Free Algebras

Closed Terms (example)

- $\mathcal{E} \ni e ::= \mathsf{zero} \mid \mathsf{plus}(e_1, e_2) \mid \mathsf{times}(e_1, e_2)$

Free Algebras

Closed Terms (example)

- $\mathcal{E} \ni e ::= \mathsf{zero} \mid \mathsf{plus}(e_1, e_2) \mid \mathsf{times}(e_1, e_2)$
  - $\star$ 0

Free Algebras

Closed Terms (example)

- $\mathcal{E} \ni e ::= \mathsf{zero} \mid \mathsf{plus}(e_1, e_2) \mid \mathsf{times}(e_1, e_2)$
  - ⋆ $0 \subseteq \Sigma 0$

Free Algebras

Closed Terms (example)

- $\mathcal{E} \ni e ::= \mathsf{zero} \mid \mathsf{plus}(e_1, e_2) \mid \mathsf{times}(e_1, e_2)$
  - ⋆ $0 \subseteq \Sigma 0 \subseteq \Sigma^2 0$

Free Algebras

Closed Terms (example)

- $\mathcal{E} \ni e ::= \mathsf{zero} \mid \mathsf{plus}(e_1, e_2) \mid \mathsf{times}(e_1, e_2)$
  - ⋆ $0 \subseteq \Sigma 0 \subseteq \Sigma^2 0 \subseteq \Sigma^3 0$

Free Algebras

Closed Terms (example)

- $\mathcal{E} \ni e ::= \mathsf{zero} \mid \mathsf{plus}(e_1, e_2) \mid \mathsf{times}(e_1, e_2)$
  - ⋆ $0 \subseteq \Sigma 0 \subseteq \Sigma^2 0 \subseteq \Sigma^3 0 \subseteq \cdots$

Free Algebras

Closed Terms (example)

- $\mathcal{E} \ni e ::= \textsf{zero} \mid \textsf{plus}(e_1, e_2) \mid \textsf{times}(e_1, e_2)$

  ⋆ $0 \subseteq \Sigma 0 \subseteq \Sigma^2 0 \subseteq \Sigma^3 0 \subseteq \cdots$ $\mathcal{E}$

Free Algebras

Closed Terms (example)

- $\mathcal{E} \ni e ::=$ zero | plus(e_1, e_2) | times(e_1, e_2)
  - $0 \subseteq \Sigma 0 \subseteq \Sigma^2 0 \subseteq \Sigma^3 0 \subseteq \cdots$ $\mathcal{E}$

Terms with Variables in X

Free Algebras

Closed Terms (example)

- $\mathcal{E} \ni e ::= \mathsf{zero} \mid \mathsf{plus}(e_1, e_2) \mid \mathsf{times}(e_1, e_2)$
  - ⋆ $0 \subseteq \Sigma 0 \subseteq \Sigma^2 0 \subseteq \Sigma^3 0 \subseteq \cdots$ $\mathcal{E}$

Terms with Variables in X

- $\mathcal{E}_X$

Free Algebras

Closed Terms (example)

- $\mathcal{E} \ni e ::= \text{zero} \mid \text{plus}(e_1, e_2) \mid \text{times}(e_1, e_2)$
  - ⋆ $0 \subseteq \Sigma 0 \subseteq \Sigma^2 0 \subseteq \Sigma^3 0 \subseteq \cdots$ $\qquad \mathcal{E}$

Terms with Variables in X

- $\mathcal{E}_X \ni e ::=$

Free Algebras

Closed Terms (example)

- $\mathcal{E} \ni e ::= \mathsf{zero} \mid \mathsf{plus}(e_1, e_2) \mid \mathsf{times}(e_1, e_2)$
  - ⋆ $0 \subseteq \Sigma 0 \subseteq \Sigma^2 0 \subseteq \Sigma^3 0 \subseteq \cdots$ $\mathcal{E}$

Terms with Variables in X

- $\mathcal{E}_X \ni e ::= x$

Free Algebras

Closed Terms (example)

- $\mathcal{E} \ni e ::= \mathsf{zero} \mid \mathsf{plus}(e_1, e_2) \mid \mathsf{times}(e_1, e_2)$
  - $\star$ $0 \subseteq \Sigma 0 \subseteq \Sigma^2 0 \subseteq \Sigma^3 0 \subseteq \cdots$ $\mathcal{E}$

Terms with Variables in X

- $\mathcal{E}_X \ni e ::= x \mid \mathsf{zero}$

Free Algebras

Closed Terms (example)

- $\mathcal{E} \ni e ::= \mathsf{zero} \mid \mathsf{plus}(e_1, e_2) \mid \mathsf{times}(e_1, e_2)$

  - ⋆ $0 \subseteq \Sigma 0 \subseteq \Sigma^2 0 \subseteq \Sigma^3 0 \subseteq \cdots$ $\mathcal{E}$

Terms with Variables in X

- $\mathcal{E}_X \ni e ::= x \mid \mathsf{zero} \mid \mathsf{plus}(e_1, e_2)$

Free Algebras

Closed Terms (example)

- $\mathcal{E} \ni e ::= \mathsf{zero} \mid \mathsf{plus}(e_1, e_2) \mid \mathsf{times}(e_1, e_2)$
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- $\mathcal{E}_X \ni e ::= x \mid \mathsf{zero} \mid \mathsf{plus}(e_1, e_2) \mid \mathsf{times}(e_1, e_2)$

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Terms with Variables in X

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  - ⋆ X

Free Algebras

Closed Terms (example)

- $\mathcal{E} \ni e ::=$ zero | plus(e_1, e_2) | times(e_1, e_2)
  - ⋆ $0 \subseteq \Sigma 0 \subseteq \Sigma^2 0 \subseteq \Sigma^3 0 \subseteq \cdots$ $\mathcal{E}$

Terms with Variables in X

- $\mathcal{E}_X \ni e ::= x$ | zero | plus(e_1, e_2) | times(e_1, e_2)
  - ⋆ $X \subseteq X + \Sigma X$

Free Algebras

Closed Terms (example)

- $\mathcal{E} \ni e ::= \textsf{zero} \mid \textsf{plus}(e_1, e_2) \mid \textsf{times}(e_1, e_2)$
  - ⋆ $0 \subseteq \Sigma 0 \subseteq \Sigma^2 0 \subseteq \Sigma^3 0 \subseteq \cdots$ $\mathcal{E}$

Terms with Variables in X

- $\mathcal{E}_X \ni e ::= x \mid \textsf{zero} \mid \textsf{plus}(e_1, e_2) \mid \textsf{times}(e_1, e_2)$
  - ⋆ $X \subseteq X + \Sigma X \subseteq X + \Sigma^2 X$

Free Algebras

Closed Terms (example)

- $\mathcal{E} \ni e ::= \textsf{zero} \mid \textsf{plus}(e_1, e_2) \mid \textsf{times}(e_1, e_2)$
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Terms with Variables in X

- $\mathcal{E}_X \ni e ::= x \mid \textsf{zero} \mid \textsf{plus}(e_1, e_2) \mid \textsf{times}(e_1, e_2)$
  - ⋆ $X \subseteq X + \Sigma X \subseteq X + \Sigma^2 X \subseteq X + \Sigma^3 X$

Free Algebras

Closed Terms (example)

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Terms with Variables in X

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Free Algebras

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Terms with Variables in X

- $\mathcal{E}_X \ni e ::= x \mid \mathsf{zero} \mid \mathsf{plus}(e_1, e_2) \mid \mathsf{times}(e_1, e_2)$
  - ⋆ $X \subseteq X + \Sigma X \subseteq X + \Sigma^2 X \subseteq X + \Sigma^3 X \subseteq \cdots$ $\mathcal{E}_X$

Free Algebras

Closed Terms (example)

- $\mathcal{E} \ni e ::= \mathsf{zero} \mid \mathsf{plus}(e_1, e_2) \mid \mathsf{times}(e_1, e_2)$
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Terms with Variables in X

- $\mathcal{E}_X \ni e ::= x \mid \mathsf{zero} \mid \mathsf{plus}(e_1, e_2) \mid \mathsf{times}(e_1, e_2)$
  - $\star \quad X \subseteq X + \Sigma X \subseteq X + \Sigma^2 X \subseteq X + \Sigma^3 X \subseteq \cdots \qquad \mathcal{E}_X$

Notation:

Free Algebras

Closed Terms (example)

- $\mathcal{E} \ni e ::=$ zero | plus(e_1, e_2) | times(e_1, e_2)
  - $\star \quad 0 \subseteq \Sigma 0 \subseteq \Sigma^2 0 \subseteq \Sigma^3 0 \subseteq \cdots$ $\qquad \mathcal{E}$

Terms with Variables in X

- $\mathcal{E}_X \ni e ::= x$ | zero | plus(e_1, e_2) | times(e_1, e_2)
  - $\star \quad X \subseteq X + \Sigma X \subseteq X + \Sigma^2 X \subseteq X + \Sigma^3 X \subseteq \cdots$ $\qquad \mathcal{E}_X$

Notation: TX

Free Algebras

Closed Terms (example)

- $\mathcal{E} \ni e ::= \text{zero} \mid \text{plus}(e_1, e_2) \mid \text{times}(e_1, e_2)$
  - ⋆ $0 \subseteq \Sigma 0 \subseteq \Sigma^2 0 \subseteq \Sigma^3 0 \subseteq \cdots$ $\mathcal{E}$

Terms with Variables in X

- $\mathcal{E}_X \ni e ::= x \mid \text{zero} \mid \text{plus}(e_1, e_2) \mid \text{times}(e_1, e_2)$
  - ⋆ $X \subseteq X + \Sigma X \subseteq X + \Sigma^2 X \subseteq X + \Sigma^3 X \subseteq \cdots$ $\mathcal{E}_X$

Notation: $TX \equiv$ initial $X + \Sigma$-algebra

Final $X \times B$-coalgebra

Final $X \times B$-coalgebra

Initial $X + \Sigma$-Algebra

Final $X \times B$-coalgebra

$$DX \cong X \times BDX$$

Initial $X + \Sigma$-Algebra

Final $X \times B$-coalgebra

$$DX \cong X \times BDX$$

Initial $X + \Sigma$-Algebra

$$TX \cong X + \Sigma TX$$

Final $X \times B$-coalgebra

$$DX \cong X \times BDX$$

$$B : \mathcal{C} \to \mathcal{C}$$

Initial $X + \Sigma$-Algebra

$$TX \cong X + \Sigma TX$$

Final $X \times B$-coalgebra

$$DX \cong X{\times}BDX$$

$$B : \mathcal{C} \to \mathcal{C}$$

Initial $X + \Sigma$-Algebra

$$TX \cong X{+}\Sigma TX$$

$$\Sigma : \mathcal{C} \to \mathcal{C}$$

Final $X \times B$-coalgebra

$$DX \cong X \times B D X$$

$$B : \mathcal{C} \to \mathcal{C}$$

$$\begin{array}{ccc} & & Y \\ & \swarrow^{f} & \downarrow f^{\flat} \\ X & \longleftarrow & DX \end{array} \qquad \begin{array}{ccc} Y & \xrightarrow{k} & BY \\ f^{\flat} \downarrow & & \downarrow B f^{\flat} \\ DX & \longrightarrow & BDX \end{array}$$

Initial $X + \Sigma$-Algebra

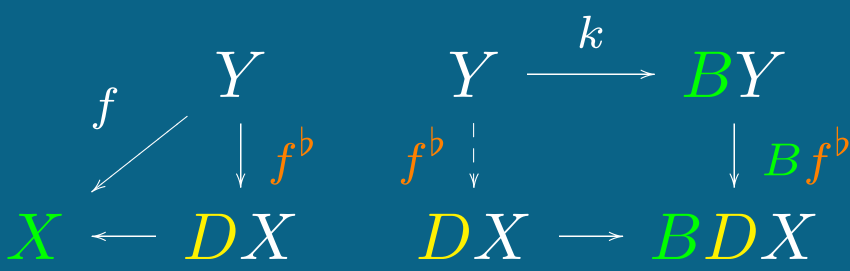
$$TX \cong X + \Sigma T X$$

$$\Sigma : \mathcal{C} \to \mathcal{C}$$

Final $X \times B$-coalgebra

$$DX \cong X \times BDX$$

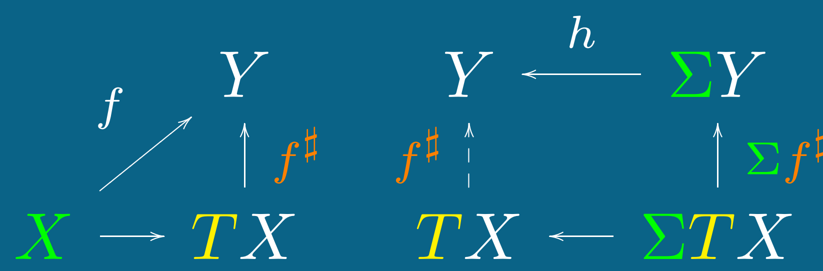
$$B : \mathcal{C} \to \mathcal{C}$$



Initial $X + \Sigma$-Algebra

$$TX \cong X + \Sigma TX$$

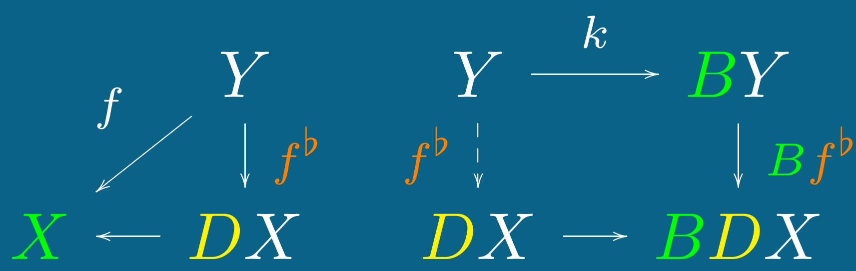
$$\Sigma : \mathcal{C} \to \mathcal{C}$$



Final $X \times B$-coalgebra

$$DX \cong X \times BDX$$

$$B : \mathcal{C} \to \mathcal{C}$$

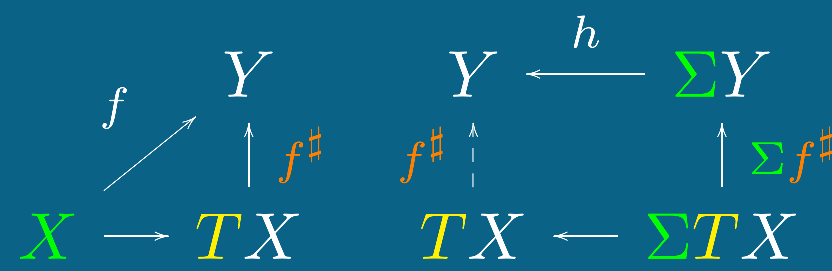


Cofree B-Coalgebra over X

Initial $X + \Sigma$-Algebra

$$TX \cong X + \Sigma TX$$

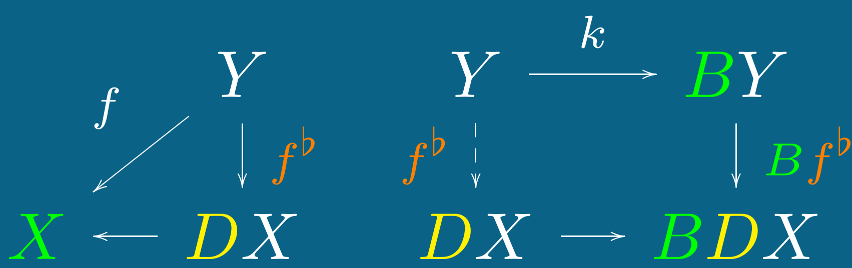
$$\Sigma : \mathcal{C} \to \mathcal{C}$$



Final $X \times B$-coalgebra

$$DX \cong X \times BDX$$

$$B : \mathcal{C} \to \mathcal{C}$$

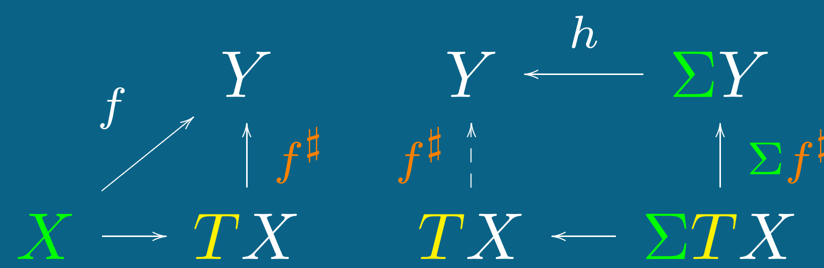


Cofree B-Coalgebra over X

Initial $X + \Sigma$-Algebra

$$TX \cong X + \Sigma TX$$

$$\Sigma : \mathcal{C} \to \mathcal{C}$$

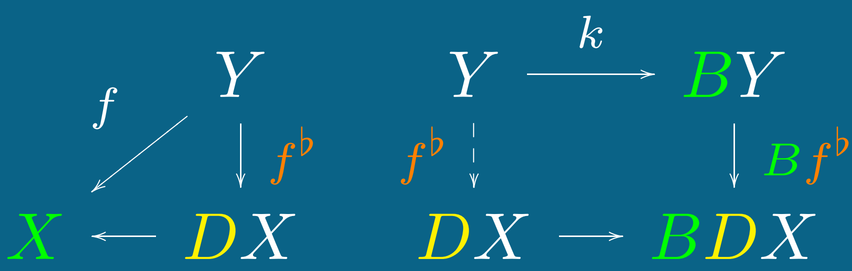


Free Σ-Algebra over X

Final $X \times B$-coalgebra

$$DX \cong X \times BDX$$

$$B : \mathcal{C} \to \mathcal{C}$$



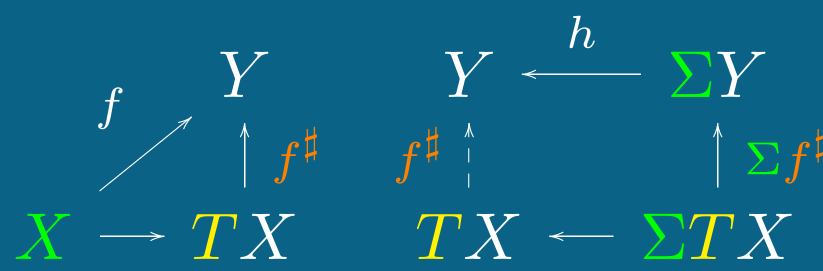
Cofree B-Coalgebra over X

$$DX \to BDX$$

Initial $X + \Sigma$-Algebra

$$TX \cong X + \Sigma TX$$

$$\Sigma : \mathcal{C} \to \mathcal{C}$$



Free Σ-Algebra over X

Final $X \times B$-coalgebra

$$DX \cong X \times BDX$$

$$B : \mathcal{C} \to \mathcal{C}$$

$$\begin{array}{ccc} & & Y \\ & \swarrow^{f} & \downarrow f^{\flat} \\ X & \longleftarrow & DX \end{array} \qquad \begin{array}{ccc} Y & \xrightarrow{k} & BY \\ f^{\flat} \downarrow & & \downarrow Bf^{\flat} \\ DX & \longrightarrow & BDX \end{array}$$

Cofree B-Coalgebra over X

$$DX \to BDX$$

Initial $X + \Sigma$-Algebra

$$TX \cong X + \Sigma TX$$

$$\Sigma : \mathcal{C} \to \mathcal{C}$$

$$\begin{array}{ccc} & & Y \\ & \nearrow^{f} & \uparrow f^{\sharp} \\ X & \longrightarrow & TX \end{array} \qquad \begin{array}{ccc} Y & \xleftarrow{h} & \Sigma Y \\ f^{\sharp} \uparrow & & \uparrow \Sigma f^{\sharp} \\ TX & \longleftarrow & \Sigma TX \end{array}$$

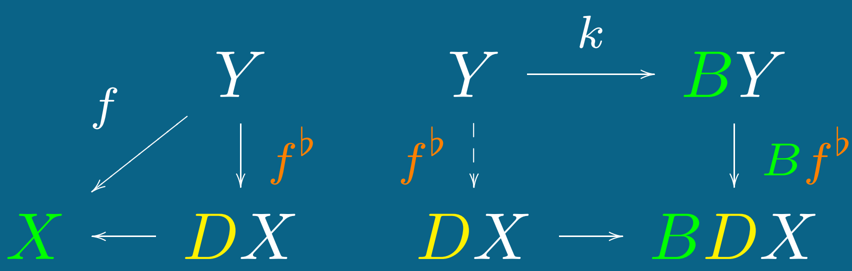
Free Σ-Algebra over X

$$\Sigma TX \to TX$$

Final $X \times B$-coalgebra

$$DX \cong X \times BDX$$

$$B : \mathcal{C} \to \mathcal{C}$$



Cofree B-Coalgebra over X

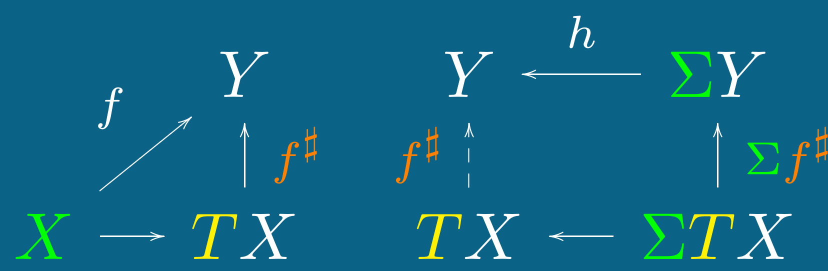
$$DX \to BDX$$

$$U_B$$

Initial $X + \Sigma$-Algebra

$$TX \cong X + \Sigma TX$$

$$\Sigma : \mathcal{C} \to \mathcal{C}$$



Free Σ-Algebra over X

$$\Sigma TX \to TX$$

Final $X \times B$-coalgebra

$$DX \cong X \times BDX$$

$$B : \mathcal{C} \to \mathcal{C}$$

$$\begin{array}{ccc} & & Y \\ & \overset{f}{\swarrow} & \downarrow f^{\flat} \\ X & \longleftarrow & DX \end{array} \qquad \begin{array}{ccc} Y & \xrightarrow{k} & BY \\ f^{\flat} \downarrow & & \downarrow Bf^{\flat} \\ DX & \longrightarrow & BDX \end{array}$$

Cofree B-Coalgebra over X

$$DX \to BDX$$

$$U_B \dashv G$$

Initial $X + \Sigma$-Algebra

$$TX \cong X + \Sigma TX$$

$$\Sigma : \mathcal{C} \to \mathcal{C}$$

$$\begin{array}{ccc} & & Y \\ & \overset{f}{\nearrow} & \uparrow f^{\sharp} \\ X & \longrightarrow & TX \end{array} \qquad \begin{array}{ccc} Y & \xleftarrow{h} & \Sigma Y \\ f^{\sharp} \uparrow & & \uparrow \Sigma f^{\sharp} \\ TX & \longleftarrow & \Sigma TX \end{array}$$

Free Σ-Algebra over X

$$\Sigma TX \to TX$$

Final $X \times B$-coalgebra

$$DX \cong X \times BDX$$

$$B : \mathcal{C} \to \mathcal{C}$$

$$\begin{array}{ccc} & & Y \\ & \swarrow^{f} & \downarrow f^{\flat} \\ X & \longleftarrow & DX \end{array} \qquad \begin{array}{ccc} Y & \xrightarrow{k} & BY \\ f^{\flat} \downarrow & & \downarrow Bf^{\flat} \\ DX & \longrightarrow & BDX \end{array}$$

Cofree B-Coalgebra over X

$$DX \to BDX$$

$$U_B \dashv G : B\text{-Coalg} \to \mathcal{C}$$

Initial $X + \Sigma$-Algebra

$$TX \cong X + \Sigma TX$$

$$\Sigma : \mathcal{C} \to \mathcal{C}$$

$$\begin{array}{ccc} & & Y \\ & \nearrow^{f} & \uparrow f^{\sharp} \\ X & \longrightarrow & TX \end{array} \qquad \begin{array}{ccc} Y & \xleftarrow{h} & \Sigma Y \\ f^{\sharp} \uparrow & & \uparrow \Sigma f^{\sharp} \\ TX & \longleftarrow & \Sigma TX \end{array}$$

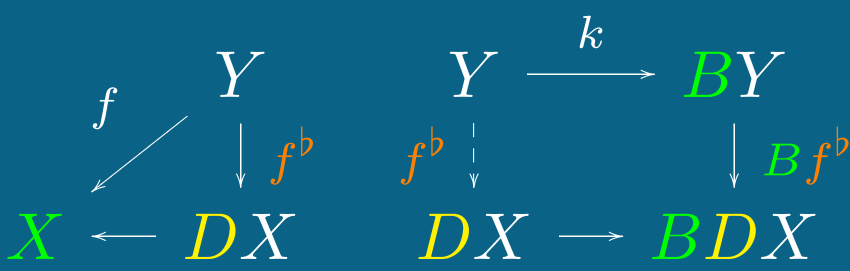
Free Σ-Algebra over X

$$\Sigma TX \to TX$$

Final $X \times B$-coalgebra

$$DX \cong X \times BDX$$

$$B : \mathcal{C} \to \mathcal{C}$$



Cofree B-Coalgebra over X

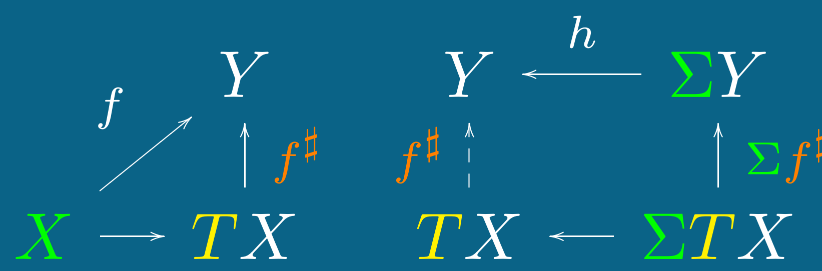
$$DX \to BDX$$

$$U_B \dashv G : B\text{-Coalg} \to \mathcal{C}$$

Initial $X + \Sigma$-Algebra

$$TX \cong X + \Sigma TX$$

$$\Sigma : \mathcal{C} \to \mathcal{C}$$



Free Σ-Algebra over X

$$\Sigma TX \to TX$$

$$F$$

Final $X \times B$-coalgebra

$$DX \cong X \times BDX$$

$$B : \mathcal{C} \to \mathcal{C}$$

$$\begin{array}{ccc} & & Y \\ & \swarrow^{f} & \downarrow f^\flat \\ X & \longleftarrow & DX \end{array} \qquad \begin{array}{ccc} Y & \xrightarrow{k} & BY \\ f^\flat \downarrow & & \downarrow Bf^\flat \\ DX & \longrightarrow & BDX \end{array}$$

Cofree B-Coalgebra over X

$$DX \to BDX$$

$$U_B \dashv G : B\text{-Coalg} \to \mathcal{C}$$

Initial $X + \Sigma$-Algebra

$$TX \cong X + \Sigma TX$$

$$\Sigma : \mathcal{C} \to \mathcal{C}$$

$$\begin{array}{ccc} & & Y \\ & \nearrow^{f} & \uparrow f^\sharp \\ X & \longrightarrow & TX \end{array} \qquad \begin{array}{ccc} Y & \xleftarrow{h} & \Sigma Y \\ f^\sharp \uparrow & & \uparrow \Sigma f^\sharp \\ TX & \longleftarrow & \Sigma TX \end{array}$$

Free Σ-Algebra over X

$$\Sigma TX \to TX$$

$$F \dashv U_\Sigma$$

Final $X \times B$-coalgebra

$$DX \cong X \times BDX$$

$$B : \mathcal{C} \to \mathcal{C}$$

$$\begin{array}{ccc} & & Y \\ & \swarrow^{f} & \downarrow f^{\flat} \\ X & \leftarrow & DX \end{array} \qquad \begin{array}{ccc} Y & \xrightarrow{k} & BY \\ f^{\flat} \downarrow & & \downarrow Bf^{\flat} \\ DX & \longrightarrow & BDX \end{array}$$

Cofree B-Coalgebra over X

$$DX \to BDX$$

$$U_B \dashv G : B\text{-Coalg} \to \mathcal{C}$$

Initial $X + \Sigma$-Algebra

$$TX \cong X + \Sigma TX$$

$$\Sigma : \mathcal{C} \to \mathcal{C}$$

$$\begin{array}{ccc} & & Y \\ & \nearrow^{f} & \uparrow f^{\sharp} \\ X & \longrightarrow & TX \end{array} \qquad \begin{array}{ccc} Y & \xleftarrow{h} & \Sigma Y \\ f^{\sharp} \uparrow & & \uparrow \Sigma f^{\sharp} \\ TX & \longleftarrow & \Sigma TX \end{array}$$

Free Σ-Algebra over X

$$\Sigma TX \to TX$$

$$F \dashv U_{\Sigma} : \mathcal{C} \to \Sigma\text{-Alg}$$

Final $X \times B$-coalgebra

$$DX \cong X \times BDX$$

$$B : \mathcal{C} \to \mathcal{C}$$

$$\begin{array}{ccccccc} & & Y & & Y & \xrightarrow{k} & BY \\ & \swarrow^{f} & \downarrow f^{\flat} & & \downarrow f^{\flat} & & \downarrow Bf^{\flat} \\ X & \longleftarrow & DX & & DX & \longrightarrow & BDX \end{array}$$

Cofree B-Coalgebra over X

$$DX \to BDX$$

$$U_B \dashv G : B\text{-Coalg} \to \mathcal{C}$$

$$GX = DX \to BDX$$

Initial $X + \Sigma$-Algebra

$$TX \cong X + \Sigma TX$$

$$\Sigma : \mathcal{C} \to \mathcal{C}$$

$$\begin{array}{ccccccc} & & Y & & Y & \xleftarrow{h} & \Sigma Y \\ & \nearrow^{f} & \uparrow f^{\sharp} & & \uparrow f^{\sharp} & & \uparrow \Sigma f^{\sharp} \\ X & \longrightarrow & TX & & TX & \longleftarrow & \Sigma TX \end{array}$$

Free Σ-Algebra over X

$$\Sigma TX \to TX$$

$$F \dashv U_\Sigma : \mathcal{C} \to \Sigma\text{-Alg}$$

Final $X \times B$-coalgebra

$$DX \cong X \times BDX$$

$$B : \mathcal{C} \to \mathcal{C}$$

$$\begin{array}{ccc} & & Y \\ & \swarrow^{f} & \downarrow f^\flat \\ X & \longleftarrow & DX \end{array} \qquad \begin{array}{ccc} Y & \xrightarrow{k} & BY \\ f^\flat \downarrow & & \downarrow Bf^\flat \\ DX & \longrightarrow & BDX \end{array}$$

Cofree B-Coalgebra over X

$$DX \to BDX$$

$$U_B \dashv G : B\text{-Coalg} \to \mathcal{C}$$

$$GX = DX \to BDX$$

Initial $X + \Sigma$-Algebra

$$TX \cong X + \Sigma TX$$

$$\Sigma : \mathcal{C} \to \mathcal{C}$$

$$\begin{array}{ccc} & & Y \\ & \nearrow^{f} & \uparrow f^\sharp \\ X & \longrightarrow & TX \end{array} \qquad \begin{array}{ccc} Y & \xleftarrow{h} & \Sigma Y \\ f^\sharp \uparrow & & \uparrow \Sigma f^\sharp \\ TX & \longleftarrow & \Sigma TX \end{array}$$

Free Σ-Algebra over X

$$\Sigma TX \to TX$$

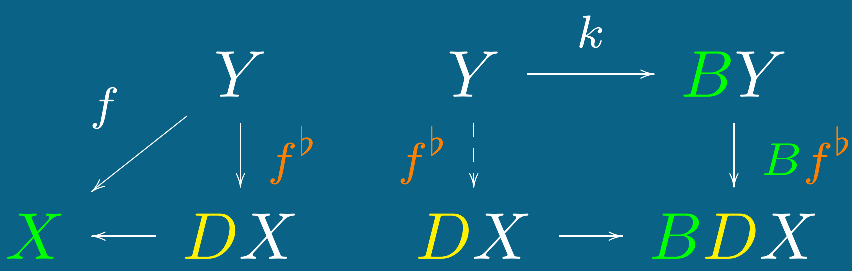
$$F \dashv U_\Sigma : \mathcal{C} \to \Sigma\text{-Alg}$$

$$FX = \Sigma TX \to TX$$

Final $X \times B$-coalgebra

$$DX \cong X \times BDX$$

$$B : \mathcal{C} \to \mathcal{C}$$



Cofree B-Coalgebra over X

$$DX \to BDX$$

$$U_B \dashv G : B\text{-Coalg} \to \mathcal{C}$$

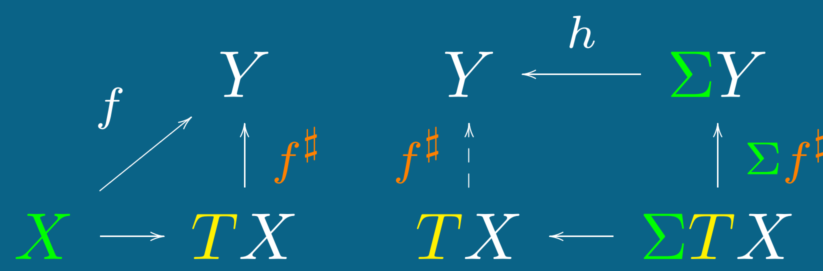
$$GX = DX \to BDX$$

$$U_B G X = DX$$

Initial $X + \Sigma$-Algebra

$$TX \cong X + \Sigma TX$$

$$\Sigma : \mathcal{C} \to \mathcal{C}$$



Free Σ-Algebra over X

$$\Sigma TX \to TX$$

$$F \dashv U_\Sigma : \mathcal{C} \to \Sigma\text{-Alg}$$

$$FX = \Sigma TX \to TX$$

Final $X \times B$-coalgebra

$$DX \cong X \times BDX$$

$$B : \mathcal{C} \to \mathcal{C}$$

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Cofree B-Coalgebra over X

$$DX \to BDX$$

$$U_B \dashv G : B\text{-Coalg} \to \mathcal{C}$$

$$GX = DX \to BDX$$

$$U_B G X = DX$$

Initial $X + \Sigma$-Algebra

$$TX \cong X + \Sigma TX$$

$$\Sigma : \mathcal{C} \to \mathcal{C}$$

$$\begin{array}{ccc} & & Y \\ & \nearrow^{f} & \uparrow f^\sharp \\ X & \longrightarrow & TX \end{array} \qquad \begin{array}{ccc} Y & \xleftarrow{h} & \Sigma Y \\ f^\sharp \uparrow & & \uparrow \Sigma f^\sharp \\ TX & \longleftarrow & \Sigma TX \end{array}$$

Free Σ-Algebra over X

$$\Sigma TX \to TX$$

$$F \dashv U_\Sigma : \mathcal{C} \to \Sigma\text{-Alg}$$

$$FX = \Sigma TX \to TX$$

$$U_\Sigma F X = TX$$

Final $X \times B$-coalgebra

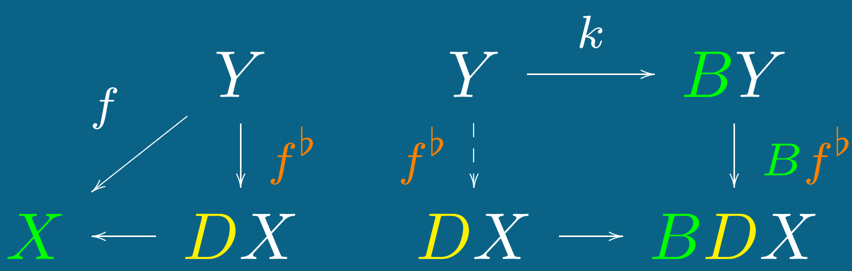
$$DX \cong X \times BDX$$

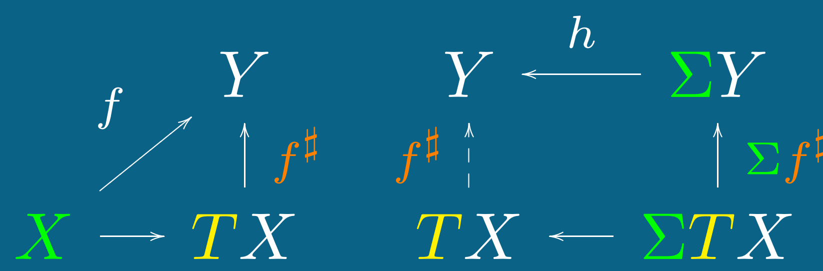
$$B : \mathcal{C} \to \mathcal{C}$$

Initial $X + \Sigma$-Algebra

$$TX \cong X + \Sigma TX$$

$$\Sigma : \mathcal{C} \to \mathcal{C}$$





Cofree B-Coalgebra over X

$$DX \to BDX$$

$$U_B \dashv G : B\text{-Coalg} \to \mathcal{C}$$

$$GX = DX \to BDX$$

$$U_B G X = DX$$

D is a comonad

Free Σ-Algebra over X

$$\Sigma TX \to TX$$

$$F \dashv U_\Sigma : \mathcal{C} \to \Sigma\text{-Alg}$$

$$FX = \Sigma TX \to TX$$

$$U_\Sigma F X = TX$$

T is a monad

Algebras of Monads

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$T = \langle T, \eta, \mu \rangle$ monad on $\mathcal{C}$

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- T-algebras

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- $\mathbb{T}$-algebras $h : TX \to X$ such that

$$\begin{array}{ccc} X & \xrightarrow{\eta_X} & TX \\ & {\scriptstyle \mathrm{id}_X} \searrow & \downarrow h \\ & & X \end{array} \qquad \begin{array}{ccc} T^2X & \xrightarrow{Th} & TX \\ \mu_X \downarrow & & \downarrow h \\ TX & \xrightarrow[h]{} & X \end{array}$$

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$$TX$$

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$$TX \longleftarrow \Sigma TX$$

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$$
\begin{array}{ccc}
& TX & \longleftarrow \; \Sigma TX \\
{\scriptstyle \mathrm{id}_{TX}} \nearrow & & \\
TX \longrightarrow & T^2X &
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$$
\begin{array}{ccccc}
 & \mathrm{id}_{TX} & TX & \longleftarrow & \Sigma TX \\
 & \nearrow & \uparrow \mathrm{id}_{TX}{}^\sharp & & \uparrow \Sigma\mathrm{id}_{TX}{}^\sharp \\
TX & \longrightarrow & T^2X & \longleftarrow & \Sigma T^2X
\end{array}
$$

The Category of T-Algebras

The (homo)morphisms $f : \langle X, h \rangle \to \langle X', h' \rangle$ of T-Alg are given by morphisms $f : X \to X'$ such that

$$\begin{array}{ccc} TX & \xrightarrow{Tf} & TX' \\ h \downarrow & & \downarrow h' \\ X & \xrightarrow[f]{} & X' \end{array}$$

Σ-algebras are T-algebras

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- $T = U_\Sigma F \quad \Rightarrow$

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- $T = U_\Sigma F \quad \Rightarrow \quad \Sigma\text{-Alg} \cong T\text{-Alg}$

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$$X$$

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$$X \qquad\qquad \Sigma X$$

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$$X \xleftarrow{h} \Sigma X$$

$$X$$

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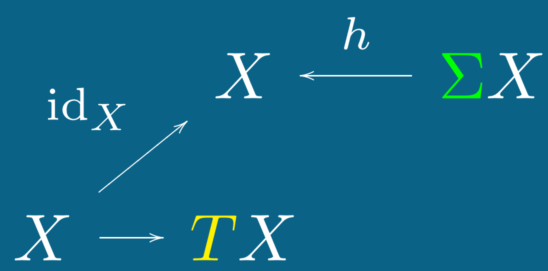

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$$\begin{array}{ccccc} & & X & \xleftarrow{\;h\;} & \Sigma X \\ & {}^{\mathrm{id}_X}\nearrow & \uparrow {\scriptstyle \mathrm{id}_X{}^\sharp} & & \uparrow {\scriptstyle \Sigma \mathrm{id}_X{}^\sharp} \\ X & \longrightarrow & TX & \longleftarrow & \Sigma TX \end{array}$$

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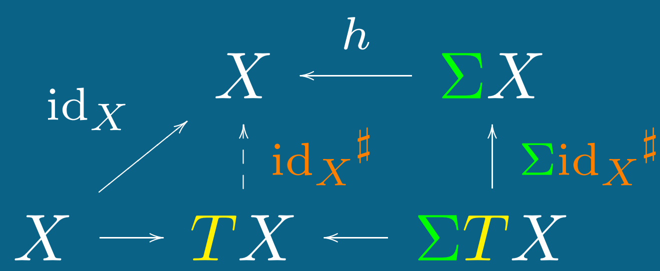

- $D = U_B G$

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$$
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\end{array}
$$

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$$X \xrightarrow{k} BX$$

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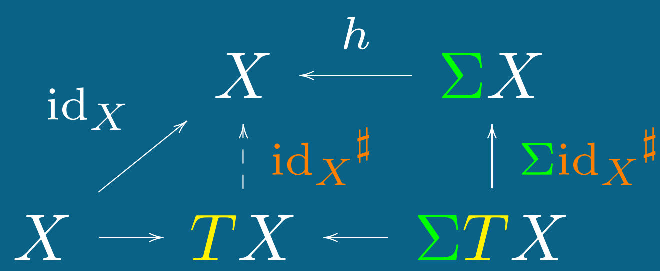

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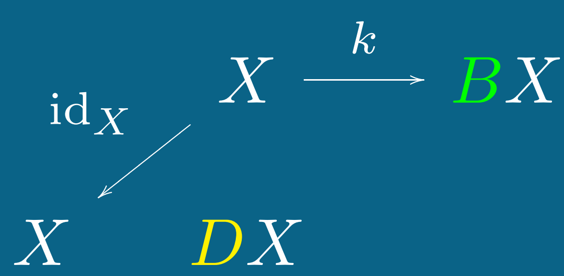

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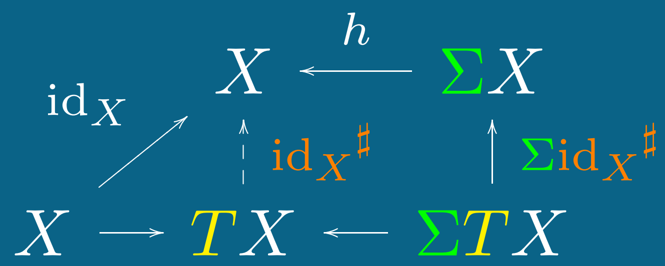

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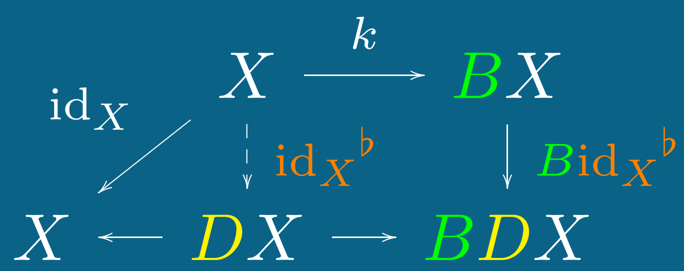

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$$\begin{array}{ccccc} & & X & \xrightarrow{\;k\;} & BX \\ & \overset{\mathrm{id}_X}{\swarrow} & \downarrow {\scriptstyle \mathrm{id}_X{}^\flat} & & \downarrow {\scriptstyle B\mathrm{id}_X{}^\flat} \\ X & \longleftarrow & DX & \longrightarrow & BDX \end{array}$$

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