Algebraic & Coalgebraic Methods in Semantics

Lecture III

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Operational Models as Coalgebras

Behaviour: endofunctor B

Operational Model: B-coalgebra

$$k : X \to BX$$

Running Example

$B : \mathbf{Set} \to \mathbf{Set}$ $\qquad BX = 1 + A \times X$

- $k : X \to 1 + A \times X$ (strongly) *deterministic transition system*

$$k(x) = * \iff x \downarrow$$

$$k(x) = \langle a, x' \rangle \iff x \xrightarrow{a} x'$$

Determinism

$B : \mathbf{Set} \to \mathbf{Set} \qquad BX = 1 + A \times X \qquad x \stackrel{a}{\to} x'$ or $x \downarrow$

Examples of computations:

- (terminating) $\qquad x \stackrel{a}{\to} x_1 \stackrel{b}{\to} x_2 \stackrel{c}{\to} x_3 \downarrow$
- (infinite) $\qquad y \stackrel{a}{\to} y' \stackrel{b}{\to} y \stackrel{a}{\to} y' \stackrel{b}{\to} y \cdots$

Final Coalgebra:

$$A^\infty \cong 1 + A \times A^\infty$$

$$\varepsilon \mapsto *$$

$$a \cdot w \mapsto \langle a, w \rangle$$

$$a \cdot b \cdot c \quad \rightsquigarrow \quad \bullet \stackrel{a}{\to} \bullet \stackrel{b}{\to} \bullet \stackrel{c}{\to} \bullet \downarrow$$

Coinductive Extension

Maps a state to the word corresponding to its computation

$k^{@} : X \to A^{\infty}$

$$x \overset{a}{\to} x_1 \overset{b}{\to} x_2 \overset{c}{\to} x_3 \downarrow$$

$$\begin{aligned} k^{@}(x) &= a \cdot b \cdot c \\ &= \bullet \overset{a}{\to} \bullet \overset{b}{\to} \bullet \overset{c}{\to} \bullet \downarrow \end{aligned}$$

$$k^{@}(x) = \begin{cases} \varepsilon & \text{if } x \downarrow \\ a \cdot k^{@}(x') & \text{if } x \overset{a}{\to} x' \end{cases}$$

$$\begin{array}{ccc} X & \xrightarrow{k} & 1 + A \times X \\ {\scriptstyle k^{@}} \downarrow & & \downarrow {\scriptstyle 1 + A \times k^{@}} \\ A^{\infty} & \underset{\cong}{\longleftrightarrow} & 1 + A \times A^{\infty} \end{array}$$

Existence of Final Coalgebras

If B preserves limits of ω^{op}-chains

$\Rightarrow$ Basic Lemma [*Smyth & Plotkin 82*]

Final B-coalgebra as limit of $1 \leftarrow B1 \leftarrow B^2 1 \leftarrow \cdots$

Problem: not true for the finite powerset endofunctor $\mathcal{P}_{\mathrm{fi}}$

See my thesis, pages 168-9

Still, final $\mathcal{P}_{\mathrm{fi}}$-coalgebra exists [*Aczel & Mendler 89, Barr 92*]:

- set of rooted finitely branching trees modulo bisimulation.

Coalgebraic Bisimulation

$$
\begin{array}{ccccc}
X & \xleftarrow{r_1} & R & \xrightarrow{r_2} & X' \\
{\scriptstyle k}\downarrow & & \vdots & & \downarrow{\scriptstyle k'} \\
BX & \xleftarrow[Br_1]{} & BR & \xrightarrow[Br_2]{} & BX'
\end{array}
$$

[*Aczel & Mendler 89*]

Final Coalgebras & Bisimulation

- Final Coalgebras are *internally fully abstract*
  - ⋆ $p \sim q \iff p = q$
- Coinductive Extensions identify bisimilar states
  - ⋆ $x \sim y \Rightarrow k^{@}(x) = k^{@}(y)$
  - ⋆ $x \sim y \Leftarrow k^{@}(x) = k^{@}(y)$ *if* B *preserves weak pullbacks*

Cofree Coalgebras

Final B-Coalgebra: $A^\infty \cong 1 + A \times A^\infty$

- abstract computations $\bullet \stackrel{a}{\to} \bullet \stackrel{b}{\to} \bullet \stackrel{c}{\to} \bullet \to$

What if we want computations with states in X?

$$x \stackrel{a}{\to} x_1 \stackrel{b}{\to} x_2 \stackrel{c}{\to} x_3 \to \cdots$$

Final $X \times B$-Coalgebra:
$$\begin{aligned} A^\infty_X &\cong X \times (1 + A \times A^\infty_X) \\ &\cong X + X \times A \times A^\infty_X \end{aligned}$$

$$\frac{Y \to X \times BY}{Y \to X \qquad Y \to BY}$$

$$\begin{array}{ccccc} & & Y & \stackrel{k}{\longrightarrow} & 1 + A \times Y \\ & \swarrow^{f} & \downarrow f^\flat & & \downarrow 1 + A \times f^\flat \\ X & \longleftarrow & A^\infty_X & \longrightarrow & 1 + A \times A^\infty_X \end{array}$$

Coinductive Extension along an Arrow

Example

$$
\begin{array}{ccccc}
 & & X & \xrightarrow{k} & 1 + A \times X \\
 & \swarrow^{\mathrm{id}_X} & \downarrow{\scriptstyle {\mathrm{id}_X}^{\flat}} & & \downarrow{\scriptstyle 1 + A \times {\mathrm{id}_X}^{\flat}} \\
X & \longleftarrow & A^{\infty}_{X} & \longrightarrow & 1 + A \times A^{\infty}_{X}
\end{array}
$$

Cofree Coalgebras (ctd)

$$
\begin{array}{ccc}
 & & Y \\
 & \swarrow^{f} & \downarrow f^\flat \\
X & \longleftarrow & A_X^\infty
\end{array}
\qquad\qquad
\begin{array}{ccc}
Y & \xrightarrow{k} & 1 + A \times Y \\
f^\flat \downarrow & & \downarrow 1+A\times f^\flat \\
A_X^\infty & \longrightarrow & 1 + A \times A_X^\infty
\end{array}
$$

Notation: $DX \equiv$ final $X \times B$-coalgebra (X*-computations*)

$$
B : \mathcal{C} \to \mathcal{C}
\qquad
\begin{array}{ccc}
 & & Y \\
 & \swarrow^{f} & \downarrow f^\flat \\
X & \longleftarrow & DX
\end{array}
\qquad\qquad
\begin{array}{ccc}
Y & \xrightarrow{k} & BY \\
f^\flat \downarrow & & \downarrow Bf^\flat \\
DX & \longrightarrow & BDX
\end{array}
$$

$DX \to BDX$ is the cofree B-coalgebra over X

i.e. $U_B : B\text{-Coalg} \to \mathcal{C}$ has a right adjoint

$$
\begin{array}{rccc}
G : & \mathcal{C} & \to & B\text{-Coalg} \\
 & X & \mapsto & DX \to BDX
\end{array}
$$

Cofree Coalgebras (ctd)

$DX \equiv$ final $X \times B$-coalgebra (X*-computations*)

$$B : \mathcal{C} \to \mathcal{C} \qquad \begin{array}{ccc} & & Y \\ & \overset{f}{\swarrow} & \downarrow f^\flat \\ X & \longleftarrow & DX \end{array} \qquad \begin{array}{ccc} Y & \overset{k}{\longrightarrow} & BY \\ f^\flat \downarrow & & \downarrow Bf^\flat \\ DX & \longrightarrow & BDX \end{array}$$

$DX \to BDX$ is the cofree B-coalgebra over X

i.e. $U_B : B\text{-Coalg} \to \mathcal{C}$ has a right adjoint

$$\begin{array}{ccc} \mathcal{C} & \to & B\text{-Coalg} \\ X & \mapsto & DX \to BDX \end{array}$$

- What is $D1$?
  - ⋆ the final B-coalgebra!
  - ⋆ 1-computations $\equiv$ most *abstract* computations

Dualities

Behaviour B	Signature Σ
B-Coalgebras	Σ-Algebras
$k : X \to BX$	$h : \Sigma X \to X$
Operational Models	Denotational Models
(Coalgebraic) Bisimilarity	Compositionality
Final Coalgebra	Initial Algebra
State-less Computations	Closed Terms
Cofree Coalgebra over X	?
Computations with states in X	?

Free Algebras

Closed Terms (example)

- $\mathcal{E} \ni e ::=$ zero $\mid$ plus(e_1, e_2) $\mid$ times(e_1, e_2)
  - $\star$ $0 \subseteq \Sigma 0 \subseteq \Sigma^2 0 \subseteq \Sigma^3 0 \subseteq \cdots$ $\mathcal{E}$

Terms with Variables in X

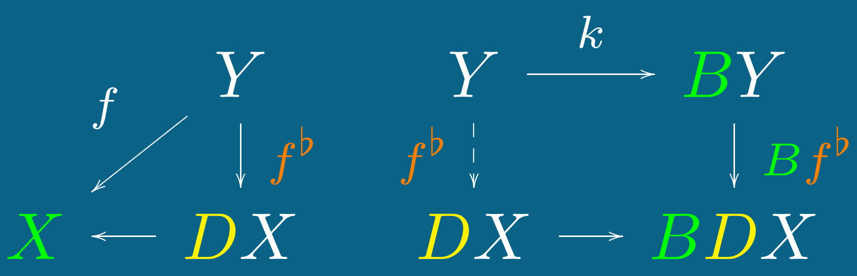
- $\mathcal{E}_X \ni e ::= x \mid$ zero $\mid$ plus(e_1, e_2) $\mid$ times(e_1, e_2)
  - $\star$ $X \subseteq X + \Sigma X \subseteq X + \Sigma^2 X \subseteq X + \Sigma^3 X \subseteq \cdots$ $\mathcal{E}_X$

Notation: $TX \equiv$ initial $X + \Sigma$-algebra

Final $X \times B$-coalgebra

$$DX \cong X \times BDX$$

$$B : \mathcal{C} \to \mathcal{C}$$



Cofree B-Coalgebra over X

$$DX \to BDX$$

$$U_B \dashv G \quad : B\text{-Coalg} \to \mathcal{C}$$

$$GX = DX \to BDX$$

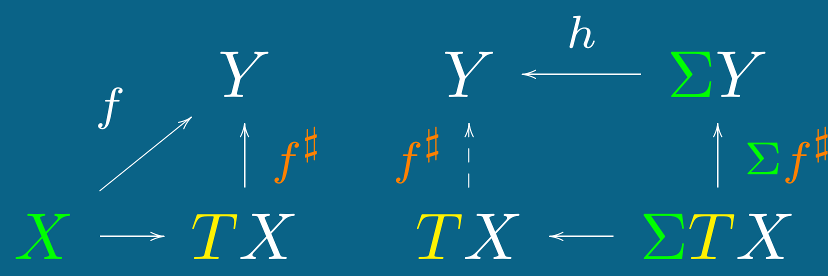
$$U_B G X = DX$$

D is a comonad

Initial $X + \Sigma$-Algebra

$$TX \cong X + \Sigma TX$$

$$\Sigma : \mathcal{C} \to \mathcal{C}$$



Free Σ-Algebra over X

$$\Sigma TX \to TX$$

$$F \dashv U_\Sigma \quad : \mathcal{C} \to \Sigma\text{-Alg}$$

$$FX = \Sigma TX \to TX$$

$$U_\Sigma F X = TX$$

T is a monad

Algebras of Monads

$T = \langle T, \eta, \mu \rangle$ monad on $\mathcal{C}$

- T-algebras $h : TX \to X$ such that

$$\begin{array}{ccc} X & \xrightarrow{\eta_X} & TX \\ & {\scriptstyle \mathrm{id}_X} \searrow & \downarrow h \\ & & X \end{array} \qquad \begin{array}{ccc} T^2X & \xrightarrow{Th} & TX \\ \mu_X \downarrow & & \downarrow h \\ TX & \xrightarrow[h]{} & X \end{array}$$

- $T = U_\Sigma F$

$$\begin{array}{ccccc} & {\scriptstyle \mathrm{id}_{TX}} \nearrow & TX & \longleftarrow & \Sigma TX \\ & & \uparrow {\scriptstyle \mathrm{id}_{TX}{}^\sharp} & & \uparrow {\scriptstyle \Sigma \mathrm{id}_{TX}{}^\sharp} \\ TX & \longrightarrow & T^2X & \longleftarrow & \Sigma T^2 X \end{array}$$

The Category of T-Algebras

The (homo)morphisms $f : \langle X, h \rangle \to \langle X', h' \rangle$ of T-Alg are given by morphisms $f : X \to X'$ such that

$$\begin{array}{ccc} TX & \xrightarrow{Tf} & TX' \\ h \downarrow & & \downarrow h' \\ X & \xrightarrow[f]{} & X' \end{array}$$

Σ-algebras are T-algebras

- $T = U_\Sigma F \quad \Rightarrow \quad \Sigma\text{-Alg} \cong T\text{-Alg}$

$$\begin{array}{ccccc} & & X & \xleftarrow{\;h\;} & \Sigma X \\ & \nearrow^{\mathrm{id}_X} & \uparrow {\scriptstyle \mathrm{id}_X{}^\sharp} & & \uparrow {\scriptstyle \Sigma \mathrm{id}_X{}^\sharp} \\ X & \longrightarrow & TX & \longleftarrow & \Sigma T X \end{array}$$

- $D = U_B G \quad \Rightarrow \quad B\text{-Coalg} \cong D\text{-Coalg}$

$$\begin{array}{ccccc} & & X & \xrightarrow{\;k\;} & BX \\ & \swarrow^{\mathrm{id}_X} & \downarrow {\scriptstyle \mathrm{id}_X{}^\flat} & & \downarrow {\scriptstyle B\mathrm{id}_X{}^\flat} \\ X & \longleftarrow & DX & \longrightarrow & BDX \end{array}$$

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