Algebraic & Coalgebraic Methods
in
Semantics

Lecture IV

Distributive Laws for Semantics

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Starting Point:

Starting Point: Signature Σ

Starting Point: Signature Σ and Behaviour B

Starting Point: Signature Σ and Behaviour B

- Syntax:

Starting Point: Signature Σ and Behaviour B

- Syntax: Induction

Starting Point: Signature Σ and Behaviour B

- Syntax: Induction $\quad \Sigma \leadsto T$

Starting Point: Signature Σ and Behaviour B

- Syntax: Induction $\qquad \Sigma \rightsquigarrow T = \langle T, \eta, \mu \rangle$

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- Syntax: Induction $\Sigma \rightsquigarrow T = \langle T, \eta, \mu \rangle$
  - ⋆ Initial Algebra:

Starting Point: Signature Σ and Behaviour B

- Syntax: Induction $\Sigma \rightsquigarrow T = \langle T, \eta, \mu \rangle$
  - ⋆ Initial Algebra: $\Sigma T0 \quad \cong \quad T0$

Starting Point: Signature Σ and Behaviour B

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- Computations:

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$$\Sigma T0 \cong T0$$

$$\mu_0 : T^2 0 \longrightarrow T0$$

- Computations: Coinduction $\quad B \rightsquigarrow D$

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$$D1 \quad \cong \quad BD1$$

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- Computations: Coinduction $\quad B \rightsquigarrow D = \langle D, \varepsilon, \delta \rangle$
  - $\star$ Final Coalgebra:
$$\begin{array}{rccc} & D1 & \cong & BD1 \\ \delta_1 : & D1 & \longrightarrow & D^2 1 \end{array}$$

Semantics

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- $[\![-]\!] : \mathrm{Syntax} \longrightarrow \mathcal{M}$

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$$
\begin{array}{ccc}
T^2 0 & \xrightarrow{\;T[\![-]\!]\;} & TD1 \\
\mu_0 \downarrow & & \downarrow \boxed{?} \\
T0 & \xrightarrow{\;[\![-]\!]\;} & D1 \\
\boxed{?} \downarrow & & \downarrow \delta_1 \\
DT0 & \xrightarrow[\;D[\![-]\!]\;]{} & D^2 1
\end{array}
$$

Distributive Laws

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$\boxed{?} : T0 \longrightarrow\!\!\triangleright\, DT0$

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- *Distributive Law!*

Distributive Laws

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- *Distributive Law!* $\boxed{\lambda : TD \Longrightarrow\!\!\triangleright DT}$

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How does it work?

$X \xrightarrow{k} DX$

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How does it work?

$$X \xrightarrow{k} DX \qquad \mapsto \qquad TX \xrightarrow{Tk} TDX$$

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$$TX \xrightarrow{h} X$$

Distributive Laws

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- $T1 \longrightarrow\!\!\triangleright\, 1 \quad \mapsto \quad TD1 \xrightarrow{\lambda_1} DT1 \longrightarrow\!\!\triangleright\, D1$

Distributive Laws (ctd)

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- T_λ

Distributive Laws (ctd)

- $T_\lambda(X \stackrel{k}{\longrightarrow\!\!\triangleright} DX)$

Distributive Laws (ctd)

- $T_\lambda(X \xrightarrow{k} DX) = TX \xrightarrow{Tk} TDX \xrightarrow{\lambda_X} DTX$

Distributive Laws (ctd)

- $T_\lambda(X \overset{k}{\longrightarrow} DX) = TX \overset{Tk}{\longrightarrow} TDX \overset{\lambda_X}{\longrightarrow} DTX$

- D_λ

Distributive Laws (ctd)

- $T_\lambda(X \stackrel{k}{\longrightarrow} DX) = TX \stackrel{Tk}{\longrightarrow} TDX \stackrel{\lambda_X}{\longrightarrow} DTX$
- $D_\lambda(TX \stackrel{h}{\longrightarrow} X)$

Distributive Laws (ctd)

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Distributive Laws (ctd)

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A family of maps $\lambda_X : TDX \longrightarrow DTX$

Distributive Laws (ctd)

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A family of maps $\lambda_X : TDX \longrightarrow DTX$

- respecting η & μ

Distributive Laws (ctd)

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A family of maps $\lambda_X : TDX \longrightarrow\!\!\!\triangleright DTX$

- respecting η & μ $\qquad \rightsquigarrow \qquad$ $D_\lambda(h)$ is a T-algebra

Distributive Laws (ctd)

- $T_\lambda(X \xrightarrow{k} DX) = TX \xrightarrow{Tk} TDX \xrightarrow{\lambda_X} DTX$
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A family of maps $\lambda_X : TDX \longrightarrow DTX$

- respecting η & μ $\quad\leadsto\quad$ $D_\lambda(h)$ is a T-algebra
- respecting ε & δ

Distributive Laws (ctd)

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A family of maps $\lambda_X : TDX \longrightarrow DTX$

- respecting η & μ $\rightsquigarrow$ $D_\lambda(h)$ is a T-algebra
- respecting ε & δ $\rightsquigarrow$ $T_\lambda(k)$ is a D-coalgebra

Distributive Laws (ctd)

- $T_\lambda(X \stackrel{k}{\longrightarrow} DX) = TX \stackrel{Tk}{\longrightarrow} TDX \stackrel{\lambda_X}{\longrightarrow} DTX$
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A family of maps $\lambda_X : TDX \longrightarrow DTX$

- respecting η & μ $\rightsquigarrow$ $D_\lambda(h)$ is a T-algebra
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- natural in X

Distributive Laws (ctd)

- $T_\lambda(X \xrightarrow{k} DX) = TX \xrightarrow{Tk} TDX \xrightarrow{\lambda_X} DTX$
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A family of maps $\lambda_X : TDX \longrightarrow DTX$

- respecting η & μ $\rightsquigarrow$ $D_\lambda(h)$ is a T-algebra
- respecting ε & δ $\rightsquigarrow$ $T_\lambda(k)$ is a D-coalgebra
- natural in X $\rightsquigarrow$ T_λ & D_λ are functors

Distributive Laws (ctd)

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A family of maps $\lambda_X : TDX \longrightarrow DTX$

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$$T_\lambda = \langle T_\lambda, \eta, \mu \rangle$$

Distributive Laws (ctd)

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A family of maps $\lambda_X : TDX \longrightarrow DTX$

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$\mathbb{T}_\lambda = \langle T_\lambda, \eta, \mu \rangle$ is a monad!

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$D_\lambda = \langle D_\lambda, \varepsilon, \delta \rangle$

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A family of maps $\lambda_X : TDX \longrightarrow DTX$

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$\mathbb{T}_\lambda = \langle T_\lambda, \eta, \mu \rangle$ is a monad!

$\mathbb{D}_\lambda = \langle D_\lambda, \varepsilon, \delta \rangle$ is a comonad.

Liftings

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- $T_\lambda(X \xrightarrow{k} DX) = TX \xrightarrow{Tk} TDX \xrightarrow{\lambda_X} DTX$

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D-Coalg

Liftings

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$$D\text{-Coalg} \xrightarrow{T_\lambda} D\text{-Coalg}$$

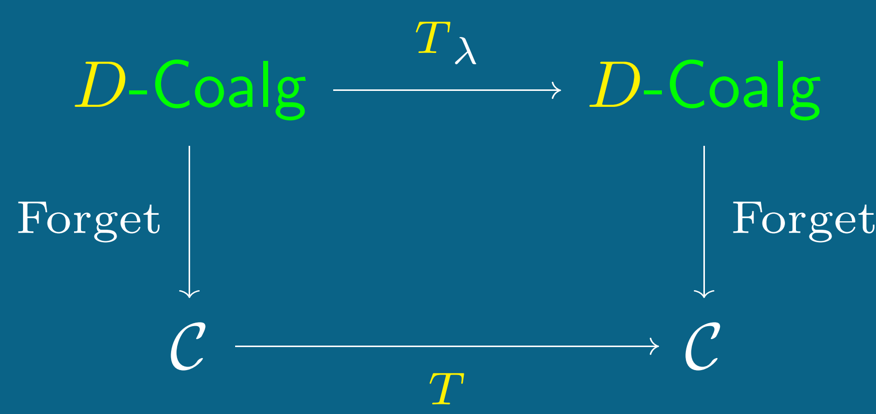
Liftings

- $T_\lambda(X \xrightarrow{k} DX) = TX \xrightarrow{Tk} TDX \xrightarrow{\lambda_X} DTX$

$$\begin{array}{ccc} D\text{-Coalg} & \xrightarrow{\quad T_\lambda \quad} & D\text{-Coalg} \\ {\scriptstyle \text{Forget}}\big\downarrow & & \big\downarrow{\scriptstyle \text{Forget}} \\ \mathcal{C} & \xrightarrow[\quad T \quad]{} & \mathcal{C} \end{array}$$

Liftings

- $T_\lambda(X \xrightarrow{k} DX) = TX \xrightarrow{Tk} TDX \xrightarrow{\lambda_X} DTX$



Liftings

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$\lambda : TD \Longrightarrow DT$ (distr. law)

$$D\text{-}\mathsf{Coalg} \xrightarrow{\quad \widetilde{T} \quad} D\text{-}\mathsf{Coalg}$$

$$\begin{array}{ccc} D\text{-Coalg} & \xrightarrow{\widetilde{T}} & D\text{-Coalg} \\ {\scriptstyle \text{Forget}}\downarrow & & \downarrow{\scriptstyle \text{Forget}} \\ \mathcal{C} & \xrightarrow[T]{} & \mathcal{C} \end{array}$$

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$$\lambda : TD \Longrightarrow DT \text{ (distr. law)}$$

$$T\text{-Alg} \xrightarrow{\widetilde{D}} T\text{-Alg}$$

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Bialgebras

Bialgebras

$$D_\lambda = \langle D_\lambda, \varepsilon, \delta \rangle$$

Bialgebras

$D_\lambda = \langle D_\lambda, \varepsilon, \delta \rangle$ is a comonad on T-Alg:

Bialgebras

$D_\lambda = \langle D_\lambda, \varepsilon, \delta \rangle$ is a comonad on T-Alg:

- what are the D_λ-coalgebras?

Bialgebras

$D_\lambda = \langle D_\lambda, \varepsilon, \delta \rangle$ is a comonad on T-Alg:

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Recall: D_λ

Bialgebras

$D_\lambda = \langle D_\lambda, \varepsilon, \delta \rangle$ is a comonad on T-Alg:

- what are the D_λ-coalgebras?

Recall: $D_\lambda(TX \stackrel{h}{\longrightarrow\!\!\triangleright} X)$

Bialgebras

$D_\lambda = \langle D_\lambda, \varepsilon, \delta \rangle$ is a comonad on T-Alg:

- what are the D_λ-coalgebras?

Recall: $D_\lambda(TX \xrightarrow{h} X) = TDX \xrightarrow{\lambda_X} DTX \xrightarrow{Dh} DX$

Bialgebras

$D_\lambda = \langle D_\lambda, \varepsilon, \delta \rangle$ is a comonad on T-Alg:

- what are the D_λ-coalgebras?

Recall: $D_\lambda(TX \xrightarrow{h} X) = TDX \xrightarrow{\lambda_X} DTX \xrightarrow{Dh} DX$

$$TX \xrightarrow{\quad h \quad} X$$

Bialgebras

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Bialgebras

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$T_\lambda(X \xrightarrow{k} DX) = TX \xrightarrow{Tk} TDX \xrightarrow{\lambda_X} DTX$

λ-bialgebras

λ-bialgebras $\stackrel{\text{def}}{=}$ T_λ-algebras

λ-bialgebras $\stackrel{\text{def}}{=}$ T_λ-algebras $=$ D_λ-coalgebras

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$$TX \xrightarrow{h} X$$

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$$TX \xrightarrow{h} X \xrightarrow{k} DX$$

λ-bialgebras $\stackrel{\text{def}}{=}$ T_λ-algebras $=$ D_λ-coalgebras

$$\begin{array}{ccccc} TX & \xrightarrow{h} & X & \xrightarrow{k} & DX \\ {\scriptstyle Tk}\downarrow & & & & \uparrow{\scriptstyle Dh} \\ TDX & & \xrightarrow[\lambda_X]{\qquad\qquad} & & DTX \end{array}$$

λ-bialgebras $\stackrel{\text{def}}{=}$ T_λ-algebras $=$ D_λ-coalgebras

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λ-Bialg

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λ-Bialg $\cong$ T_λ-Alg

λ-bialgebras $\stackrel{\text{def}}{=}$ T_λ-algebras $=$ D_λ-coalgebras

$$\begin{array}{ccccc} TX & \xrightarrow{h} & X & \xrightarrow{k} & DX \\ {\scriptstyle Tk}\downarrow & & & & \uparrow{\scriptstyle Dh} \\ TDX & & \xrightarrow[\lambda_X]{\hspace{3em}} & & DTX \end{array}$$

λ-Bialg $\cong$ T_λ-Alg $\cong$ D_λ-Coalg

λ-bialgebras $\stackrel{\text{def}}{=}$ T_λ-algebras $=$ D_λ-coalgebras

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λ-Bialg $\cong$ T_λ-Alg $\cong$ D_λ-Coalg

$$\begin{array}{ccc} TX & \xrightarrow{Tf} & TX' \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle h'} \\ X & \xrightarrow{f} & X' \\ {\scriptstyle k}\downarrow & & \downarrow{\scriptstyle k'} \\ DX & \xrightarrow[Df]{} & DX' \end{array}$$

Free and Cofree Bialgebras

Free and Cofree Bialgebras

λ-Bialg

Free and Cofree Bialgebras

λ-Bialg $\cong T_\lambda$-Alg

Free and Cofree Bialgebras

$$\lambda\text{-Bialg} \cong T_\lambda\text{-Alg} \xrightarrow{\text{Forget}} D\text{-Coalg}$$

Free and Cofree Bialgebras

λ-Bialg $\cong T_\lambda$-Alg $\xrightarrow{\text{Forget}}$ D-Coalg

- *<u>Left</u> Adjoint*

Free and Cofree Bialgebras

$$\lambda\text{-Bialg} \cong T_\lambda\text{-Alg} \xrightarrow{\text{Forget}} D\text{-Coalg}$$

- *Left Adjoint*: D-Coalg $\longrightarrow$ λ-Bialg

Free and Cofree Bialgebras

λ-Bialg $\cong T_\lambda$-Alg $\overset{\text{Forget}}{\longrightarrow\!\triangleright}$ D-Coalg

- *Left Adjoint*: D-Coalg $\longrightarrow\!\triangleright$ λ-Bialg

  $\star$ $X \overset{k}{\longrightarrow\!\triangleright} DX$

Free and Cofree Bialgebras

λ-Bialg $\cong T_\lambda$-Alg $\xrightarrow{\text{Forget}}$ D-Coalg

- *Left Adjoint*: D-Coalg $\longrightarrow$ λ-Bialg

  ⋆ $X \xrightarrow{k} DX \quad \mapsto \quad T^2X \xrightarrow{\mu_X} TX$

Free and Cofree Bialgebras

λ-Bialg $\cong T_\lambda$-Alg $\xrightarrow{\text{Forget}}$ D-Coalg

- *Left Adjoint*: D-Coalg $\longrightarrow$ λ-Bialg

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Free and Cofree Bialgebras

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λ-Bialg

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λ-Bialg $\cong D_\lambda$-Coalg

Free and Cofree Bialgebras

λ-Bialg $\cong T_\lambda$-Alg $\xrightarrow{\text{Forget}} D$-Coalg

- *Left Adjoint*: D-Coalg $\longrightarrow \lambda$-Bialg

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λ-Bialg $\cong D_\lambda$-Coalg $\xrightarrow{\text{Forget}} T$-Alg

Free and Cofree Bialgebras

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λ-Bialg $\cong D_\lambda$-Coalg $\xrightarrow{\text{Forget}}$ T-Alg

- *Right Adjoint*

Free and Cofree Bialgebras

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- *Left Adjoint*: D-Coalg $\longrightarrow$ λ-Bialg

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λ-Bialg $\cong D_\lambda$-Coalg $\xrightarrow{\text{Forget}}$ T-Alg

- *Right Adjoint*: T-Alg $\longrightarrow$ λ-Bialg

Free and Cofree Bialgebras

λ-Bialg $\cong T_\lambda$-Alg $\overset{\text{Forget}}{\longrightarrow} D$-Coalg

- *Left Adjoint*: D-Coalg $\longrightarrow \lambda$-Bialg
  - $\star$ $X \xrightarrow{k} DX \quad \mapsto \quad T^2X \xrightarrow{\mu_X} TX \xrightarrow{T_\lambda k} DTX$

λ-Bialg $\cong D_\lambda$-Coalg $\overset{\text{Forget}}{\longrightarrow} T$-Alg

- *Right Adjoint*: T-Alg $\longrightarrow \lambda$-Bialg
  - $\star$ $TX \xrightarrow{h} X$

Free and Cofree Bialgebras

λ-Bialg $\cong T_\lambda$-Alg $\stackrel{\text{Forget}}{\longrightarrow} D$-Coalg

- *Left Adjoint*: D-Coalg $\longrightarrow$ λ-Bialg

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Free and Cofree Bialgebras

λ-Bialg $\cong$ T_λ-Alg $\overset{\text{Forget}}{\longrightarrow}$ D-Coalg

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λ-Bialg $\cong$ D_λ-Coalg $\overset{\text{Forget}}{\longrightarrow}$ T-Alg

- *Right Adjoint*: T-Alg $\longrightarrow$ λ-Bialg

  $\star$ $TX \overset{h}{\longrightarrow} X \quad \mapsto \quad TDX \overset{D_\lambda h}{\longrightarrow} DX \overset{\delta_X}{\longrightarrow} D^2X$

Free and Cofree Bialgebras

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- *Left Adjoint*: D-Coalg $\longrightarrow$ λ-Bialg
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Use blackboard

Bialgebraic Semantics

Bialgebraic Semantics

$$
\begin{array}{ccc}
T^2 0 & \xrightarrow{T[\![-]\!]} & TD1 \\
\mu_0 \big\downarrow & & \big\downarrow T_\lambda 1 \\
T0 & \xrightarrow{[\![-]\!]} & D1 \\
D_\lambda 0 \big\downarrow & & \big\downarrow \delta_1 \\
DT0 & \xrightarrow[D[\![-]\!]]{} & D^2 1
\end{array}
$$

Bialgebraic Semantics

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- $[\![-]\!] : T0 \longrightarrow\!\!\triangleright\, D1$

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- $[\![-]\!] : T0 \longrightarrow\!\!\triangleright\, D1$
  - ⋆ “*superunique*” !

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- $[\![-]\!] : T0 \longrightarrow\!\!\triangleright\, D1$
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How do we define $\lambda : TD \Longrightarrow DT$?

Bialgebraic Semantics

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Next Lecture!

Bialgebraic Semantics

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How do we define $\lambda : TD \Longrightarrow DT$?

Next Lecture!

Back to START

λ respects μ

$$\begin{array}{ccccc}
T^2D & \xrightarrow{T\lambda} & TDT & \xrightarrow{\lambda_T} & DT^2 \\
\mu_D \downarrow & & & & \downarrow D\mu \\
TD & & \xrightarrow[\lambda]{\qquad\qquad} & & DT
\end{array}$$