Algebraic & Coalgebraic Methods
in
Semantics

Lecture IV

Distributive Laws for Semantics

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Starting Point: Signature Σ and Behaviour B

- Syntax: Induction $\Sigma \rightsquigarrow T \;=\langle T, \eta, \mu\rangle$
  - ⋆ Initial Algebra:

$$\Sigma T0 \cong T0$$
$$\mu_0 : T^2 0 \longrightarrow T0$$

- Computations: Coinduction $B \rightsquigarrow D \;=\langle D, \varepsilon, \delta\rangle$
  - ⋆ Final Coalgebra:

$$D1 \cong BD1$$
$$\delta_1 : D1 \longrightarrow D^2 1$$

Semantics

- $[\![-]\!] : \mathrm{Syntax} \longrightarrow \mathcal{M}$
  - ⋆ *compositional* $\qquad [\![f(t_1,\ldots,t_n)]\!] = f^{\mathcal{M}}([\![t_1]\!],\ldots,[\![t_n]\!])$
  - ⋆ *respecting bisimulation* $\qquad t_1 \sim t_2 \iff [\![t_1]\!] = [\![t_2]\!]$
- Syntax $= T0 \qquad \mathcal{M} = D1$

$$\begin{array}{ccc}
T^2 0 & \xrightarrow{T[\![-]\!]} & TD1 \\
\mu_0 \downarrow & & \downarrow \; ? \\
T0 & \xrightarrow{[\![-]\!]} & D1 \\
? \; \downarrow & & \downarrow \delta_1 \\
DT0 & \xrightarrow[D[\![-]\!]]{} & D^2 1
\end{array}$$

Distributive Laws

$\boxed{?} : T0 \longrightarrow\!\triangleright DT0 \qquad \boxed{?} : TD1 \longrightarrow\!\triangleright D1$

- *Distributive Law!* $\qquad \boxed{\lambda : TD \Longrightarrow\!\triangleright DT}$
  - ⋆ A family of maps $\lambda_X : TDX \longrightarrow\!\triangleright DTX$ such that. . .

How does it work?

$X \xrightarrow{k} DX \quad \mapsto \quad TX \xrightarrow{Tk} TDX \xrightarrow{\lambda_X} DTX$

- $0 \longrightarrow\!\triangleright D0 \quad \mapsto \quad T0 \longrightarrow\!\triangleright TD0 \xrightarrow{\lambda_0} DT0$

$TX \xrightarrow{h} X \quad \mapsto \quad TDX \xrightarrow{\lambda_X} DTX \xrightarrow{Dh} DX$

- $T1 \longrightarrow\!\triangleright 1 \quad \mapsto \quad TD1 \xrightarrow{\lambda_1} DT1 \longrightarrow\!\triangleright D1$

Distributive Laws (ctd)

- $T_\lambda\,(X \stackrel{k}{\longrightarrow} DX) \;= TX \stackrel{Tk}{\longrightarrow} TDX \stackrel{\lambda_X}{\longrightarrow} DTX$

- $D_\lambda\,(TX \stackrel{h}{\longrightarrow} X) \;= TDX \stackrel{\lambda_X}{\longrightarrow} DTX \stackrel{Dh}{\longrightarrow} TX$

A family of maps $\lambda_X : TDX \longrightarrow DTX$

- respecting η & μ $\rightsquigarrow$ $D_\lambda(h)$ is a T-algebra
- respecting ε & δ $\rightsquigarrow$ $T_\lambda(k)$ is a D-coalgebra
- natural in X $\rightsquigarrow$ T_λ & D_λ are functors

$T_\lambda = \langle T_\lambda, \eta, \mu\rangle$ is a monad!

$D_\lambda = \langle D_\lambda, \varepsilon, \delta\rangle$ is a comonad.

Liftings

- $T_\lambda(X \stackrel{k}{\longrightarrow} DX) = TX \stackrel{Tk}{\longrightarrow} TDX \stackrel{\lambda_X}{\longrightarrow} DTX$

$$\begin{array}{ccc} D\text{-Coalg} & \xrightarrow{\;T_\lambda\;} & D\text{-Coalg} \\ {\scriptstyle \text{Forget}}\downarrow & & \downarrow{\scriptstyle \text{Forget}} \\ \mathcal{C} & \xrightarrow[\;T\;]{} & \mathcal{C} \end{array}$$

$$\lambda : TD \Longrightarrow DT \text{ (distr. law)}$$

$$\begin{array}{ccc} D\text{-Coalg} & \xrightarrow{\widetilde{T}} & D\text{-Coalg} \\ {\scriptstyle \text{Forget}}\downarrow & & \downarrow{\scriptstyle \text{Forget}} \\ \mathcal{C} & \xrightarrow[T]{} & \mathcal{C} \end{array}$$

$$\lambda : TD \Longrightarrow DT \text{ (distr. law)}$$

$$\begin{array}{ccc} T\text{-Alg} & \xrightarrow{\widetilde{D}} & T\text{-Alg} \\ {\scriptstyle \text{Forget}}\downarrow & & \downarrow{\scriptstyle \text{Forget}} \\ \mathcal{C} & \xrightarrow[D]{} & \mathcal{C} \end{array}$$

Bialgebras

$D_\lambda = \langle D_\lambda, \varepsilon, \delta \rangle$ is a comonad on T-Alg:

- what are the D_λ-coalgebras?

Recall: $D_\lambda\,(TX \xrightarrow{h} X) \;= TDX \xrightarrow{\lambda_X} DTX \xrightarrow{Dh} DX$

$$
\begin{array}{ccc}
TX & \xrightarrow{\quad h \quad} & X \\
{\scriptstyle Tk}\downarrow & & \downarrow{\scriptstyle k} \\
TDX \xrightarrow[\lambda_X]{} DTX & \xrightarrow[Dh]{} & DX
\end{array}
\qquad
\begin{array}{ccc}
X & \xrightarrow{\quad k \quad} & DX \\
{\scriptstyle h}\uparrow & & \uparrow{\scriptstyle Dh} \\
TX \xrightarrow[Tk]{} TDX & \xrightarrow[\lambda_X]{} & DTX
\end{array}
$$

$T_\lambda(X \xrightarrow{k} DX) = TX \xrightarrow{Tk} TDX \xrightarrow{\lambda_X} DTX$

λ-bialgebras $\stackrel{\text{def}}{=}$ T_λ-algebras $=$ D_λ-coalgebras

$$\begin{array}{ccccc} TX & \xrightarrow{h} & X & \xrightarrow{k} & DX \\ {\scriptstyle Tk}\downarrow & & & & \uparrow{\scriptstyle Dh} \\ TDX & & \xrightarrow[\lambda_X]{\qquad\qquad} & & DTX \end{array}$$

λ-Bialg $\cong T_\lambda$-Alg $\cong D_\lambda$-Coalg

$$\begin{array}{ccc} TX & \xrightarrow{Tf} & TX' \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle h'} \\ X & \xrightarrow{f} & X' \\ {\scriptstyle k}\downarrow & & \downarrow{\scriptstyle k'} \\ DX & \xrightarrow[Df]{} & DX' \end{array}$$

Free and Cofree Bialgebras

λ-Bialg $\cong T_\lambda$-Alg $\xrightarrow{\text{Forget}}$ D-Coalg

- *Left Adjoint* : D-Coalg $\longrightarrow$ λ-Bialg

  ⋆ $X \xrightarrow{k} DX \quad \mapsto \quad T^2X \xrightarrow{\mu_X} TX \xrightarrow{T_\lambda k} DTX$

λ-Bialg $\cong D_\lambda$-Coalg $\xrightarrow{\text{Forget}}$ T-Alg

- *Right Adjoint* : T-Alg $\longrightarrow$ λ-Bialg

  ⋆ $TX \xrightarrow{h} X \quad \mapsto \quad TDX \xrightarrow{D_\lambda h} DX \xrightarrow{\delta_X} D^2X$

Use blackboard

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Bialgebraic Semantics

$$
\begin{array}{ccc}
T^2 0 & \xrightarrow{T[\![-]\!]} & TD1 \\
{\scriptstyle \mu_0}\downarrow & & \downarrow{\scriptstyle T\lambda 1} \\
T0 & \xrightarrow{[\![-]\!]} & D1 \\
{\scriptstyle D\lambda 0}\downarrow & & \downarrow{\scriptstyle \delta_1} \\
DT0 & \xrightarrow[D[\![-]\!]]{} & D^2 1
\end{array}
$$

- $[\![-]\!] : T0 \longrightarrow\!\!\triangleright D1$
  - ⋆ "*superunique*"! (initiality and finality)

How do we define $\lambda : TD \Longrightarrow\!\!\triangleright DT$?

Next Lecture!

Back to START

λ respects μ

$$\begin{array}{ccccc}
T^2D & \xrightarrow{T\lambda} & TDT & \xrightarrow{\lambda_T} & DT^2 \\
\mu_D \downarrow & & & & \downarrow D\mu \\
TD & & \xrightarrow[\lambda]{\qquad\qquad} & & DT
\end{array}$$