Algebraic & Coalgebraic Methods
in
Semantics

Lecture V

Natural Operational Semantics

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- *How do we define*:

$$\lambda : TD \Longrightarrow DT \text{ (distr. law) ?}$$

- *How do we define*:

$$\frac{\lambda : TD \Longrightarrow DT \text{ (distr. law) ?}}{D\text{-Coalg} \xrightarrow{\widetilde{T}} D\text{-Coalg}}$$

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$$\begin{array}{ccc}
D\text{-Coalg} & \xrightarrow{\widetilde{T}} & D\text{-Coalg} \\
{\scriptstyle \text{Forget}}\big\downarrow & & \big\downarrow{\scriptstyle \text{Forget}} \\
\mathcal{C} & \xrightarrow[T]{} & \mathcal{C}
\end{array}$$

- *How do we define*:

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$$T\text{-Alg} \xrightarrow{\widetilde{D}} T\text{-Alg}$$

- *How do we define*:

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- $\Sigma \rightsquigarrow T$

- $\Sigma \rightsquigarrow T = \langle T, \eta, \mu \rangle$

- $\Sigma \rightsquigarrow T = \langle T, \eta, \mu \rangle$ $\quad\quad$ Σ-Alg $\cong T$-Alg

- $\Sigma \rightsquigarrow T = \langle T, \eta, \mu \rangle$ Σ-Alg $\cong$ T-Alg
  - ⋆ *structural* induction

- $\Sigma \rightsquigarrow T = \langle T, \eta, \mu \rangle$ Σ-Alg $\cong$ T-Alg
  - *structural* induction
- $B \rightsquigarrow D$

- $\Sigma \rightsquigarrow T = \langle T, \eta, \mu \rangle$ $\qquad$ Σ-Alg $\cong$ T-Alg
  - $\star$ *structural* induction
- $B \rightsquigarrow D = \langle D, \varepsilon, \delta \rangle$

- $\Sigma \rightsquigarrow T = \langle T, \eta, \mu \rangle$ Σ-Alg $\cong$ T-Alg
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- $B \rightsquigarrow D = \langle D, \varepsilon, \delta \rangle$ B-Coalg $\cong$ D-Coalg

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- $B \rightsquigarrow D = \langle D, \varepsilon, \delta \rangle$ B-Coalg $\cong$ D-Coalg
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$$B\text{-Coalg} \xrightarrow{\widetilde{T}} B\text{-Coalg}$$

- $\Sigma \rightsquigarrow T = \langle T, \eta, \mu \rangle \qquad \Sigma\text{-Alg} \cong T\text{-Alg}$
  - $\star$ *structural* induction
- $B \rightsquigarrow D = \langle D, \varepsilon, \delta \rangle \qquad B\text{-Coalg} \cong D\text{-Coalg}$
  - $\star$ *behavioural* coinduction

$$
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- $\Sigma \rightsquigarrow T = \langle T, \eta, \mu \rangle$ $\qquad$ Σ-Alg $\cong$ T-Alg
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- $B \rightsquigarrow D = \langle D, \varepsilon, \delta \rangle$ $\qquad$ B-Coalg $\cong$ D-Coalg
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- $\Sigma \leadsto T = \langle T, \eta, \mu \rangle$ Σ-Alg $\cong$ T-Alg
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- ⋆ $X \xrightarrow{k} BX$

- $\Sigma \rightsquigarrow T = \langle T, \eta, \mu \rangle$ $\qquad$ Σ-Alg $\cong$ T-Alg
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⋆ $X \xrightarrow{k} BX \qquad \mapsto \qquad TX \xrightarrow{\widetilde{T}(k)} BTX$

- $\Sigma \rightsquigarrow T = \langle T, \eta, \mu \rangle \qquad \Sigma\text{-Alg} \cong T\text{-Alg}$
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- $\star \quad X \xrightarrow{k} BX \qquad \mapsto \qquad TX \xrightarrow{\widetilde{T}(k)} BTX$
- $\star \quad \Sigma X \xrightarrow{h} X$

- $\Sigma \rightsquigarrow T = \langle T, \eta, \mu \rangle$ $\quad$ Σ-Alg $\cong$ T-Alg
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- $\star$ $X \xrightarrow{k} BX \quad \mapsto \quad TX \xrightarrow{\widetilde{T}(k)} BTX$
- $\star$ $\Sigma X \xrightarrow{h} X \quad \mapsto \quad \Sigma DX \xrightarrow{\widetilde{D}(k)} DX$

Structural Operational Semantics

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Operational semantics by *structural induction*?

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- *Plotkin*'s SOS (1981)

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Example: merge operator

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Example: merge operator $x_1 \| x_2$

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$$\frac{x_1 \stackrel{a}{\to} x_1'}{x_1 \| x_2 \stackrel{a}{\to} x_1' \| x_2}$$

Structural Operational Semantics

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Example: merge operator $x_1 \| x_2$

$$\frac{x_1 \stackrel{a}{\to} x_1'}{x_1 \| x_2 \stackrel{a}{\to} x_1' \| x_2} \qquad \frac{x_2 \stackrel{a}{\to} x_2'}{x_1 \| x_2 \stackrel{a}{\to} x_1 \| x_2'}$$

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Structure:

Structural Operational Semantics

Operational semantics by *structural induction*?

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Structure: operator $\|$

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Structure: operator $\|$

Behaviour of $x_1 \| x_2$:

Structural Operational Semantics

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Structure: operator $\|$

Behaviour of $x_1 \| x_2$: defined in terms of the behaviour

Structural Operational Semantics

Operational semantics by *structural induction*?

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Example: merge operator $x_1 \| x_2$

$$\frac{x_1 \xrightarrow{a} x_1'}{x_1 \| x_2 \xrightarrow{a} x_1' \| x_2} \qquad \frac{x_2 \xrightarrow{a} x_2'}{x_1 \| x_2 \xrightarrow{a} x_1 \| x_2'}$$

Structure: operator $\|$

Behaviour of $x_1 \| x_2$: defined in terms of the behaviour of the components x_1 and x_2

A simple Process Algebra

A simple Process Algebra

- $t ::= x \mid$

A simple Process Algebra

- $t ::= x \mid \mathsf{nil}$

A simple Process Algebra

- $t ::= x \mid \mathsf{nil} \mid a.t$

A simple Process Algebra

- $t ::= x \mid \mathsf{nil} \mid a.t \mid t \| t$

A simple Process Algebra

- $t ::= x \mid \mathsf{nil} \mid a.t \mid t \| t$
  - ⋆ $\Sigma X = 1 + A \cdot X + X^2$

A simple Process Algebra

- $t ::= x \mid \mathsf{nil} \mid a.t \mid t \| t$
  - $\star$ $\Sigma X = 1 + A \cdot X + X^2$
- *Rules* $\mathcal{R}$:

A simple Process Algebra

- $t ::= x \mid \mathsf{nil} \mid a.t \mid t \| t$
  - $\Sigma X = 1 + A{\cdot}X + X^2$
- *Rules* $\mathcal{R}$: $a.x \stackrel{a}{\rightarrow} x$

A simple Process Algebra

- $t ::= x \mid \mathsf{nil} \mid a.t \mid t \| t$
  - $\star$ $\Sigma X = 1 + A{\cdot}X + X^2$
- *Rules* $\mathcal{R}$: $a.x \stackrel{a}{\rightarrow} x$ $\qquad \dfrac{x_1 \stackrel{a}{\rightarrow} x_1'}{x_1 \| x_2 \stackrel{a}{\rightarrow} x_1' \| x_2}$

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- *Behaviour*:

A simple Process Algebra

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- *Behaviour*: $\boxed{BX = (\mathcal{P}_{\mathrm{fi}} X)^A}$

A simple Process Algebra

- $t ::= x \mid \mathsf{nil} \mid a.t \mid t \| t$
  - $\star$ $\Sigma X = 1 + A{\cdot}X + X^2$

- *Rules* $\mathcal{R}$: $a.x \xrightarrow{a} x$ $\quad \dfrac{x_1 \xrightarrow{a} x_1'}{x_1 \| x_2 \xrightarrow{a} x_1' \| x_2}$ $\quad \dfrac{x_2 \xrightarrow{a} x_2'}{x_1 \| x_2 \xrightarrow{a} x_1 \| x_2'}$

- *Behaviour*: $BX = (\mathcal{P}_{\mathrm{fi}} X)^A$ (for each $a \in A$ only a finite choice)

A simple Process Algebra

- $t ::= x \mid \textsf{nil} \mid a.t \mid t \| t$
  - $\star$ $\Sigma X = 1 + A \cdot X + X^2$
- *Rules* $\mathcal{R}$: $a.x \xrightarrow{a} x \qquad \dfrac{x_1 \xrightarrow{a} x_1'}{x_1 \| x_2 \xrightarrow{a} x_1' \| x_2} \qquad \dfrac{x_2 \xrightarrow{a} x_2'}{x_1 \| x_2 \xrightarrow{a} x_1 \| x_2'}$
- *Behaviour*: $\boxed{BX = (\mathcal{P}_{\mathrm{fi}} X)^A}$ (for each $a \in A$ only a finite choice)
  - $\star$ B-*coalgebras*:

A simple Process Algebra

- $t ::= x \mid \mathsf{nil} \mid a.t \mid t \| t$
  - $\Sigma X = 1 + A \cdot X + X^2$
- *Rules* $\mathcal{R}$: $\quad a.x \stackrel{a}{\to} x \qquad \dfrac{x_1 \stackrel{a}{\to} x_1'}{x_1 \| x_2 \stackrel{a}{\to} x_1' \| x_2} \qquad \dfrac{x_2 \stackrel{a}{\to} x_2'}{x_1 \| x_2 \stackrel{a}{\to} x_1 \| x_2'}$
- *Behaviour*: $\boxed{BX = (\mathcal{P}_{\mathrm{fi}} X)^A}$ (for each $a \in A$ only a finite choice)
  - B-*coalgebras*: image finite transition systems

A simple Process Algebra

- $t ::= x \mid \mathsf{nil} \mid a.t \mid t \| t$
  - $\star$ $\Sigma X = 1 + A \cdot X + X^2$
- *Rules* $\mathcal{R}$: $a.x \overset{a}{\to} x$ $\quad \dfrac{x_1 \overset{a}{\to} x_1'}{x_1 \| x_2 \overset{a}{\to} x_1' \| x_2}$ $\quad \dfrac{x_2 \overset{a}{\to} x_2'}{x_1 \| x_2 \overset{a}{\to} x_1 \| x_2'}$
- *Behaviour*: $\boxed{BX = (\mathcal{P}_{\mathrm{fi}} X)^A}$ (for each $a \in A$ only a finite choice)
  - $\star$ B-*coalgebras*: image finite transition systems
    - $*$ $X \overset{k}{\longrightarrow} (\mathcal{P}_{\mathrm{fi}} X)^A$

A simple Process Algebra

- $t ::= x \mid \mathsf{nil} \mid a.t \mid t \| t$
  - $\star$ $\Sigma X = 1 + A \cdot X + X^2$
- *Rules* $\mathcal{R}$: $a.x \stackrel{a}{\rightarrow} x \qquad \dfrac{x_1 \stackrel{a}{\rightarrow} x_1'}{x_1 \| x_2 \stackrel{a}{\rightarrow} x_1' \| x_2} \qquad \dfrac{x_2 \stackrel{a}{\rightarrow} x_2'}{x_1 \| x_2 \stackrel{a}{\rightarrow} x_1 \| x_2'}$
- *Behaviour*: $BX = (\mathcal{P}_{\mathrm{fi}} X)^A$ (for each $a \in A$ only a finite choice)
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A simple Process Algebra

- $t ::= x \mid \mathsf{nil} \mid a.t \mid t \| t$
  - ⋆ $\Sigma X = 1 + A{\cdot}X + X^2$
- *Rules* $\mathcal{R}$: $a.x \stackrel{a}{\to} x$ $\quad \dfrac{x_1 \stackrel{a}{\to} x_1'}{x_1 \| x_2 \stackrel{a}{\to} x_1' \| x_2} \quad \dfrac{x_2 \stackrel{a}{\to} x_2'}{x_1 \| x_2 \stackrel{a}{\to} x_1 \| x_2'}$
- *Behaviour*: $\boxed{BX = (\mathcal{P}_{\mathrm{fi}} X)^A}$ (for each $a \in A$ only a finite choice)
  - ⋆ B-*coalgebras*: image finite transition systems
    - ∗ $X \stackrel{k}{\longrightarrow} (\mathcal{P}_{\mathrm{fi}} X)^A \qquad x' \in k$

A simple Process Algebra

- $t ::= x \mid \mathsf{nil} \mid a.t \mid t \| t$
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- *Rules* $\mathcal{R}$: $a.x \stackrel{a}{\to} x$ $\quad \dfrac{x_1 \stackrel{a}{\to} x_1'}{x_1 \| x_2 \stackrel{a}{\to} x_1' \| x_2}$ $\quad \dfrac{x_2 \stackrel{a}{\to} x_2'}{x_1 \| x_2 \stackrel{a}{\to} x_1 \| x_2'}$
- *Behaviour*: $\boxed{BX = (\mathcal{P}_{\mathrm{fi}} X)^A}$ (for each $a \in A$ only a finite choice)
  - ⋆ B-*coalgebras*: image finite transition systems
    - ∗ $X \stackrel{k}{\longrightarrow} (\mathcal{P}_{\mathrm{fi}} X)^A$ $\qquad x' \in k(x)$

A simple Process Algebra

- $t ::= x \mid \mathsf{nil} \mid a.t \mid t \| t$
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- *Rules* $\mathcal{R}$: $a.x \overset{a}{\to} x \qquad \dfrac{x_1 \overset{a}{\to} x_1'}{x_1 \| x_2 \overset{a}{\to} x_1' \| x_2} \qquad \dfrac{x_2 \overset{a}{\to} x_2'}{x_1 \| x_2 \overset{a}{\to} x_1 \| x_2'}$
- *Behaviour*: $BX = (\mathcal{P}_{\text{fi}} X)^A$ (for each $a \in A$ only a finite choice)
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    - $*$ $X \overset{k}{\longrightarrow} (\mathcal{P}_{\text{fi}} X)^A \qquad x' \in k(x)(a)$

A simple Process Algebra

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- *Behaviour*: $\boxed{BX = (\mathcal{P}_{\mathrm{fi}} X)^A}$ (for each $a \in A$ only a finite choice)
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    - ∗ $X \overset{k}{\longrightarrow} (\mathcal{P}_{\mathrm{fi}} X)^A \qquad x' \in k(x)(a) \iff$

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- $t ::= x \mid \mathsf{nil} \mid a.t \mid t \| t$
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- *Behaviour*: $BX = (\mathcal{P}_{\mathrm{fi}} X)^A$ (for each $a \in A$ only a finite choice)
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    - $*$ $X \overset{k}{\longrightarrow} (\mathcal{P}_{\mathrm{fi}} X)^A$ $\qquad x' \in k(x)(a) \iff x \overset{a}{\to} x'$

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  - ⋆ B-*coalgebras*: image finite transition systems
    - ∗ $X \stackrel{k}{\longrightarrow} (\mathcal{P}_{\mathrm{fi}} X)^A \qquad x' \in k(x)(a) \iff x \stackrel{a}{\to} x'$

$$\boxed{[\![\mathcal{R}]\!]_X : \Sigma(X \times BX) \longrightarrow BTX}$$

Modelling the Rules thru' Σ and B

Modelling the Rules thru' Σ and B

$t ::= x \mid \mathsf{nil} \mid a.t \mid t \| t \qquad \Sigma X = 1 + A \cdot X + X^2 \qquad BX = (\mathcal{P}_{\mathrm{fi}} X)^A$

Modelling the Rules thru' Σ and B

$t ::= x \mid \mathsf{nil} \mid a.t \mid t \| t \qquad \Sigma X = 1 + A{\cdot}X + X^2 \qquad BX = (\mathcal{P}_{\mathrm{fi}} X)^A$

$$[\![\mathcal{R}]\!]_X : \Sigma(X \times BX) \longrightarrow BTX$$

Modelling the Rules thru' Σ and B

$t ::= x \mid \mathsf{nil} \mid a.t \mid t \| t \qquad \Sigma X = 1 + A{\cdot}X + X^2 \qquad BX = (\mathcal{P}_{\mathrm{fi}} X)^A$

$$[\![\mathcal{R}]\!]_X : \Sigma(X \times BX) \longrightarrow BTX$$

$$[\![\mathcal{R}]\!]_X : 1 + A{\cdot}(X \times (\mathcal{P}_{\mathrm{fi}} X)^A) + (X \times (\mathcal{P}_{\mathrm{fi}} X)^A X)^2 \longrightarrow (\mathcal{P}_{\mathrm{fi}} TX)^A$$

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- $[\![a.]\!]_X$

Modelling the Rules thru' Σ and B

$t ::= x \mid \mathsf{nil} \mid a.t \mid t \| t \qquad \Sigma X = 1 + A{\cdot}X + X^2 \qquad BX = (\mathcal{P}_{\mathrm{fi}} X)^A$

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$t ::= x \mid \mathsf{nil} \mid a.t \mid t \| t$ $\qquad \Sigma X = 1 + A{\cdot}X + X^2$ $\qquad BX = (\mathcal{P}_{\mathrm{fi}} X)^A$

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$$[\![\mathsf{nil}]\!]_X : 1 \longrightarrow\!\!\!\rhd (\mathcal{P}_{\mathrm{fi}} T X)^A$$

$$[\![\mathsf{nil}]\!]_X : 1 \longrightarrow (\mathcal{P}_{\mathrm{fi}} T X)^A$$

- *no rule* $[\![\mathsf{nil}]\!]_X$

$$[\![\mathsf{nil}]\!]_X : 1 \longrightarrow\!\!\triangleright (\mathcal{P}_{\mathrm{fi}} T X)^A$$

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- $a.x \stackrel{a}{\rightarrow} x$

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- $a.x \stackrel{a}{\to} x \quad [\![c.]\!]_X(x, \beta)$

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- $a.x \stackrel{a}{\to} x \quad [\![c.]\!]_X(x, \beta) = a \mapsto \begin{cases} \{x\} & \text{if } a = c \\ \emptyset & \text{otherwise} \end{cases}$

$[\![\mathsf{nil}]\!]_X : 1 \longrightarrow\!\!\triangleright (\mathcal{P}_{\mathrm{fi}} T X)^A$

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$[\![\ \|\ ]\!]_X : (X \times (\mathcal{P}_{\mathrm{fi}} X)^A)^2 \longrightarrow\!\!\triangleright (\mathcal{P}_{\mathrm{fi}} T X)^A$

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$[\![\ \|\ ]\!]_X : (X \times (\mathcal{P}_{\mathrm{fi}} X)^A)^2 \longrightarrow\!\!\!\triangleright (\mathcal{P}_{\mathrm{fi}} T X)^A$

- $\dfrac{x_1 \stackrel{a}{\rightarrow} x_1'}{x_1 \| x_2 \stackrel{a}{\rightarrow} x_1' \| x_2} \qquad \dfrac{x_2 \stackrel{a}{\rightarrow} x_2'}{x_1 \| x_2 \stackrel{a}{\rightarrow} x_1 \| x_2'}$

$[\![\mathsf{nil}]\!]_X : 1 \longrightarrow\!\!\triangleright (\mathcal{P}_{\mathrm{fi}} T X)^A$

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$$(x_1, \beta_1)[\![\ \|\ ]\!]_X(x_2, \beta_2)$$

$[\![\mathsf{nil}]\!]_X : 1 \longrightarrow\!\!\triangleright (\mathcal{P}_{\mathrm{fi}} T X)^A$

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$$(x_1, \beta_1) [\![\ \|\ ]\!]_X (x_2, \beta_2) = a \mapsto \{x_1' \| x_2 \mid x_1' \in \beta_1(a)\}$$

$[\![\mathsf{nil}]\!]_X : 1 \longrightarrow\!\!\!\rhd (\mathcal{P}_{\mathrm{fi}} T X)^A$

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- $a.x \overset{a}{\to} x \quad [\![c.]\!]_X(x,\beta) = a \mapsto \begin{cases} \{x\} & \text{if } a = c \\ \emptyset & \text{otherwise} \end{cases}$

$[\![\ \|\ ]\!]_X : (X \times (\mathcal{P}_{\mathrm{fi}} X)^A)^2 \longrightarrow\!\!\!\rhd (\mathcal{P}_{\mathrm{fi}} T X)^A$

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$$(x_1, \beta_1)[\![\ \|\ ]\!]_X(x_2, \beta_2) = a \mapsto \begin{array}{c} \{x_1' \| x_2 \mid x_1' \in \beta_1(a)\} \\ \cup \end{array}$$

$[\![\mathsf{nil}]\!]_X : 1 \longrightarrow\!\!\!\triangleright (\mathcal{P}_{\mathrm{fi}} T X)^A$

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Naturality

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$$[\![\mathcal{R}]\!]_X : 1 + A \cdot (X \times (\mathcal{P}_{\mathrm{fi}} X)^A) + (X \times (\mathcal{P}_{\mathrm{fi}} X)^A)^2 \longrightarrow\!\!\!\triangleright (\mathcal{P}_{\mathrm{fi}} T X)^A$$

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$$[\![\mathcal{R}]\!]_X : 1 + A\cdot(X \times (\mathcal{P}_{\mathrm{fi}}X)^A) + (X \times (\mathcal{P}_{\mathrm{fi}}X)^A)^2 \longrightarrow (\mathcal{P}_{\mathrm{fi}}TX)^A$$

- *natural* in X

Naturality

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- *natural* in X
  - $\star$ $f : X \longrightarrow\!\triangleright Y$

Naturality

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  - $\star$ $f : X \longrightarrow\!\!\triangleright Y \equiv$ *renaming*

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  - ⋆ $f : X \longrightarrow\!\!\triangleright Y \equiv$ *renaming* (possibly identifying variables)

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    - $*$ first *renaming* and then *applying* $\mathcal{R}$

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      $\equiv$

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      $\equiv$

      first *applying* $\mathcal{R}$ and then *renaming*

GSOS

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Which rules can we model as a transformation

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$$\Sigma(X \times (\mathcal{P}_{\mathrm{fi}}X)^A) \longrightarrow\!\!\triangleright (\mathcal{P}_{\mathrm{fi}}TX)^A \quad ?$$

GSOS

Which rules can we model as a transformation

$$\Sigma(X \times (\mathcal{P}_{\mathrm{fi}} X)^A) \longrightarrow\!\!\triangleright (\mathcal{P}_{\mathrm{fi}} T X)^A \quad ?$$

$$\frac{\{x_i \stackrel{a}{\to} y_{ij}^a\}_{1 \le j \le m_i^a}^{1 \le i \le n,\, a \in A_i} \quad \{x_i \stackrel{b}{\not\to}\}_{b \in B_i}^{1 \le i \le n}}{\sigma(x_1, \ldots, x_n) \stackrel{c}{\to} t}$$

GSOS

Which rules can we model as a transformation

$$\Sigma(X \times (\mathcal{P}_{\mathrm{fi}}X)^A) \longrightarrow (\mathcal{P}_{\mathrm{fi}}TX)^A \quad ?$$

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What about *naturality* in X?

GSOS

Which rules can we model as a transformation

$$\Sigma(X \times (\mathcal{P}_{\mathrm{fi}}X)^A) \longrightarrow (\mathcal{P}_{\mathrm{fi}}TX)^A \quad ?$$

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What about *naturality* in X?

- ⋆ x_i and y_{ij}^a all distinct

GSOS

Which rules can we model as a transformation

$$\Sigma(X \times (\mathcal{P}_{\mathrm{fi}} X)^A) \longrightarrow (\mathcal{P}_{\mathrm{fi}} T X)^A \quad ?$$

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What about *naturality* in X?

- x_i and y^a_{ij} all distinct
- x_i and y^a_{ij} the only variables occurring in t

GSOS

Which rules can we model as a transformation

$$\Sigma(X \times (\mathcal{P}_{\text{fi}} X)^A) \longrightarrow (\mathcal{P}_{\text{fi}} T X)^A \quad ?$$

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What about *naturality* in X?

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GSOS rules [*Bloom, Istrail, Meyer 88*]!

Theorem

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(A finite)

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There is a correspondence between transformations of type

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$$\Sigma(X \times (\mathcal{P}_{\mathrm{fi}} X)^A) \longrightarrow\!\!\!\!\triangleright\ (\mathcal{P}_{\mathrm{fi}} T X)^A$$

Theorem

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$$\Sigma(X \times (\mathcal{P}_{\mathrm{fi}} X)^A) \longrightarrow (\mathcal{P}_{\mathrm{fi}} T X)^A$$

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image finite sets of GSOS rules for Σ

Theorem

(A finite)

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and

image finite sets of GSOS rules for Σ

(over a fixed denumerably infinite set of variables V).

Theorem

(A finite)

There is a correspondence between transformations of type

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natural in X

and

image finite sets of GSOS rules for Σ

(over a fixed denumerably infinite set of variables V).

- This correspondence is 1-1 up to equivalence of sets of rules.

From Rules to Distributive Laws

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$$\rho_X : \Sigma(X \times BX) \longrightarrow BTX$$

From Rules to Distributive Laws

$\rho_X : \Sigma(X \times BX) \longrightarrow BTX$ $\qquad$ B-Coalg

From Rules to Distributive Laws

$$\rho_X : \Sigma(X \times BX) \longrightarrow BTX \qquad\qquad B\text{-Coalg} \xrightarrow{\;T_\rho\;} B\text{-Coalg}$$

From Rules to Distributive Laws

$$\rho_X : \Sigma(X \times BX) \longrightarrow BTX$$

$$\begin{array}{ccc} B\text{-Coalg} & \xrightarrow{T_\rho} & B\text{-Coalg} \\ {\scriptstyle \text{Forget}}\downarrow & & \downarrow{\scriptstyle \text{Forget}} \\ \mathcal{C} & \xrightarrow[T]{} & \mathcal{C} \end{array}$$

From Rules to Distributive Laws

$$\rho_X : \Sigma(X \times BX) \longrightarrow\!\!\triangleright\, BTX$$

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- $X \xrightarrow{\;k\;}\!\!\triangleright\, BX$

From Rules to Distributive Laws

$$\rho_X : \Sigma(X \times BX) \longrightarrow BTX$$

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- $X \xrightarrow{k} BX \quad \mapsto \quad TX \xrightarrow{T_\rho(k)} BTX$

From Rules to Distributive Laws

$\rho_X : \Sigma(X \times BX) \longrightarrow BTX$

$$\begin{array}{ccc} B\text{-Coalg} & \xrightarrow{T_\rho} & B\text{-Coalg} \\ {\scriptstyle \text{Forget}}\downarrow & & \downarrow{\scriptstyle \text{Forget}} \\ \mathcal{C} & \xrightarrow[T]{} & \mathcal{C} \end{array}$$

- $X \xrightarrow{k} BX \quad \mapsto \quad TX \xrightarrow{T_\rho(k)} BTX$

If $\rho_X : \Sigma BX \longrightarrow BTX$

From Rules to Distributive Laws

$\rho_X : \Sigma(X \times BX) \longrightarrow\!\!\triangleright\, BTX$

$$\begin{array}{ccc} B\text{-Coalg} & \xrightarrow{\;T_\rho\;} & B\text{-Coalg} \\ {\scriptstyle \text{Forget}}\downarrow & & \downarrow{\scriptstyle \text{Forget}} \\ \mathcal{C} & \xrightarrow[\;T\;]{} & \mathcal{C} \end{array}$$

- $X \xrightarrow{k}\!\!\triangleright\, BX \qquad \mapsto \qquad TX \xrightarrow{T_\rho(k)}\!\!\triangleright\, BTX$

If $\rho_X : \Sigma BX \longrightarrow\!\!\triangleright\, BTX$

- Derive a Σ-algebra structure for BTX!

From Rules to Distributive Laws

$$\rho_X : \Sigma(X \times BX) \longrightarrow\!\!\!\!\triangleright\ BTX$$

$$\begin{array}{ccc} B\text{-Coalg} & \xrightarrow{T_\rho} & B\text{-Coalg} \\ {\scriptstyle \text{Forget}}\downarrow & & \downarrow{\scriptstyle \text{Forget}} \\ \mathcal{C} & \xrightarrow[T]{} & \mathcal{C} \end{array}$$

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  - ⋆ $\Sigma BTX \stackrel{\rho_{TX}}{\longrightarrow\!\!\!\!\triangleright} BT^2X \stackrel{B\mu_X}{\longrightarrow\!\!\!\!\triangleright} BTX$

- $\Sigma B T X \xrightarrow{\rho_{T X}} B T^2 X \xrightarrow{B \mu_X} T X$

- $\Sigma BTX \xrightarrow{\rho_{TX}} BT^2X \xrightarrow{B\mu_X} TX$

$$X \xrightarrow{\eta_X} TX \xleftarrow{\text{free algebra}} \Sigma TX$$

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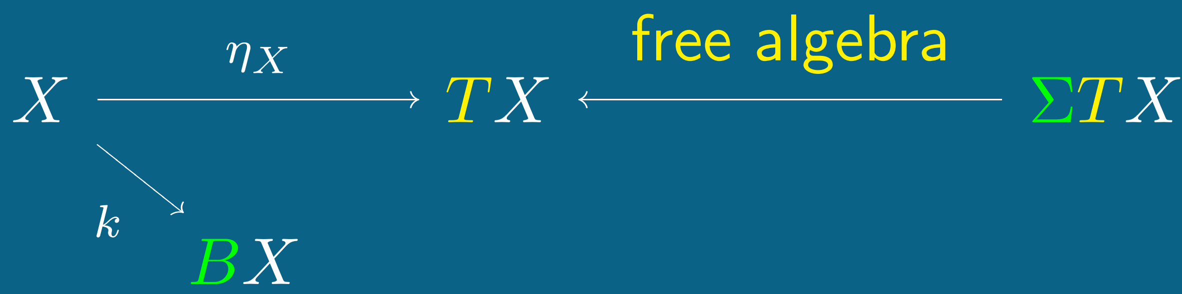

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$$X \xrightarrow{k} BX$$

$$BTX$$

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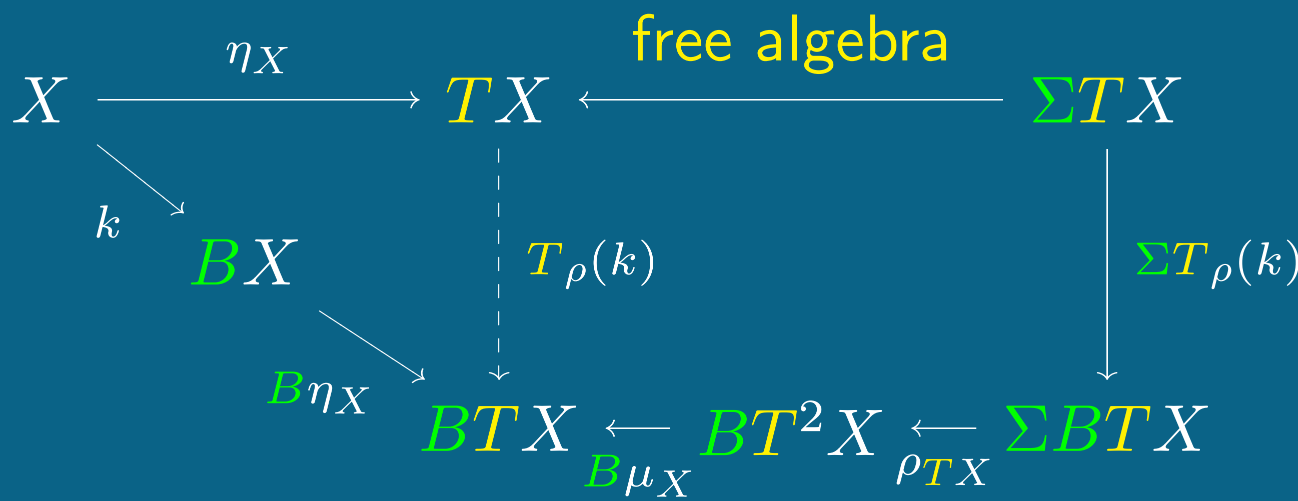

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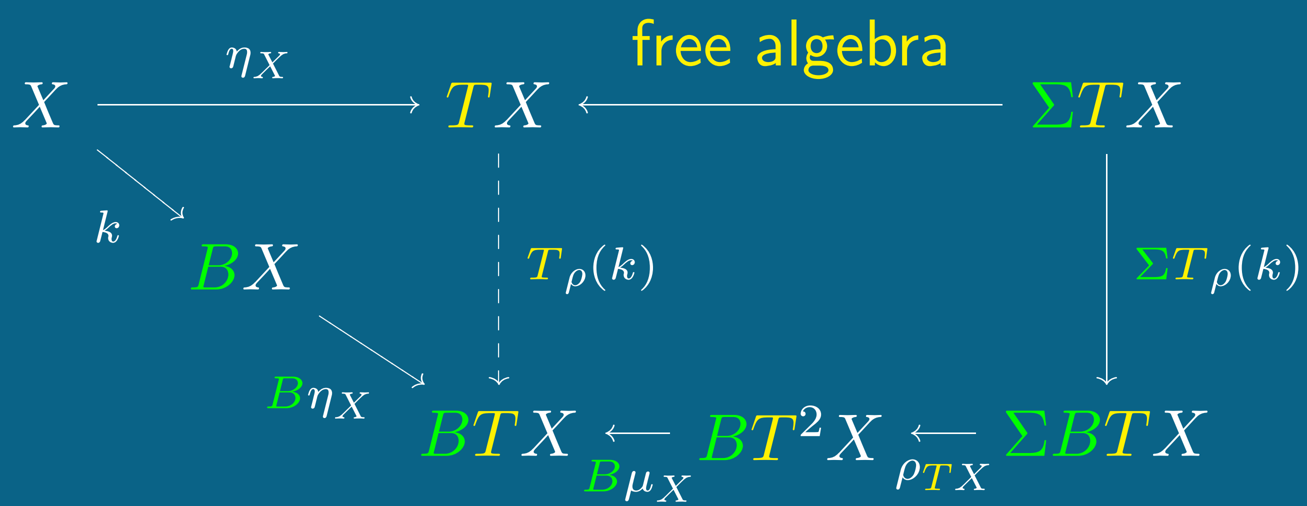
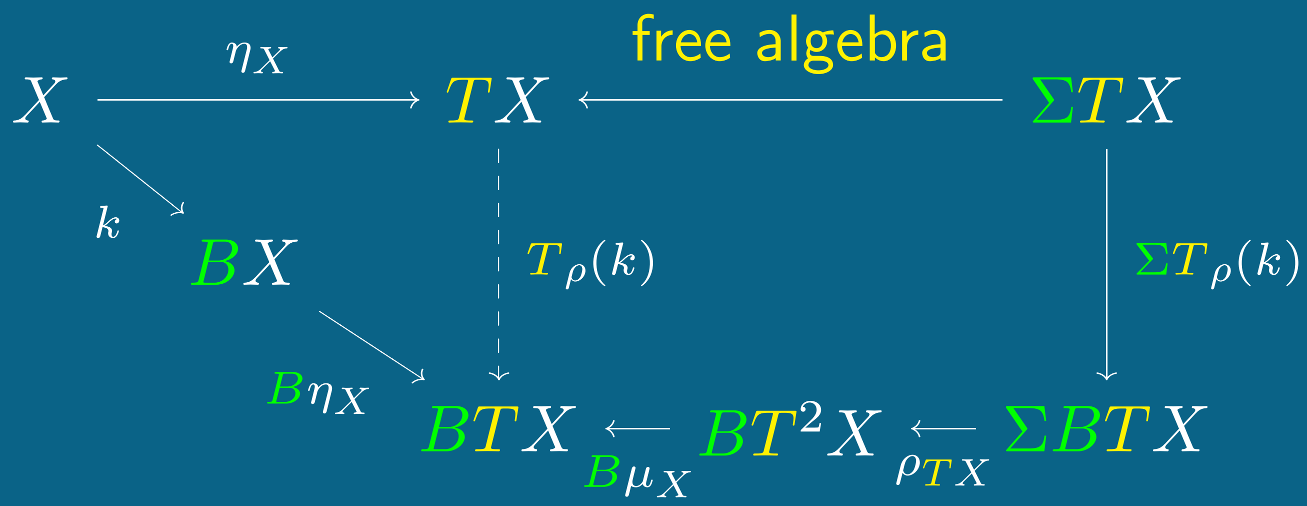


LHS:

- $\Sigma BTX \xrightarrow{\rho_{TX}} BT^2X \xrightarrow{B\mu_X} TX$



LHS: *conservative extension*

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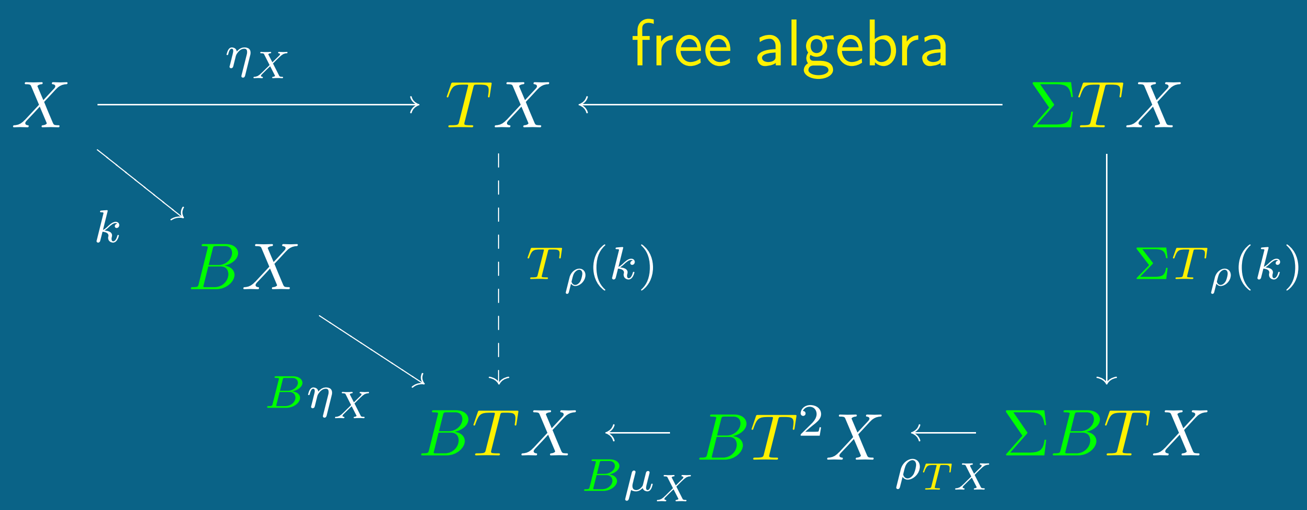

LHS: *conservative extension* (preservation of the behaviour of variables)

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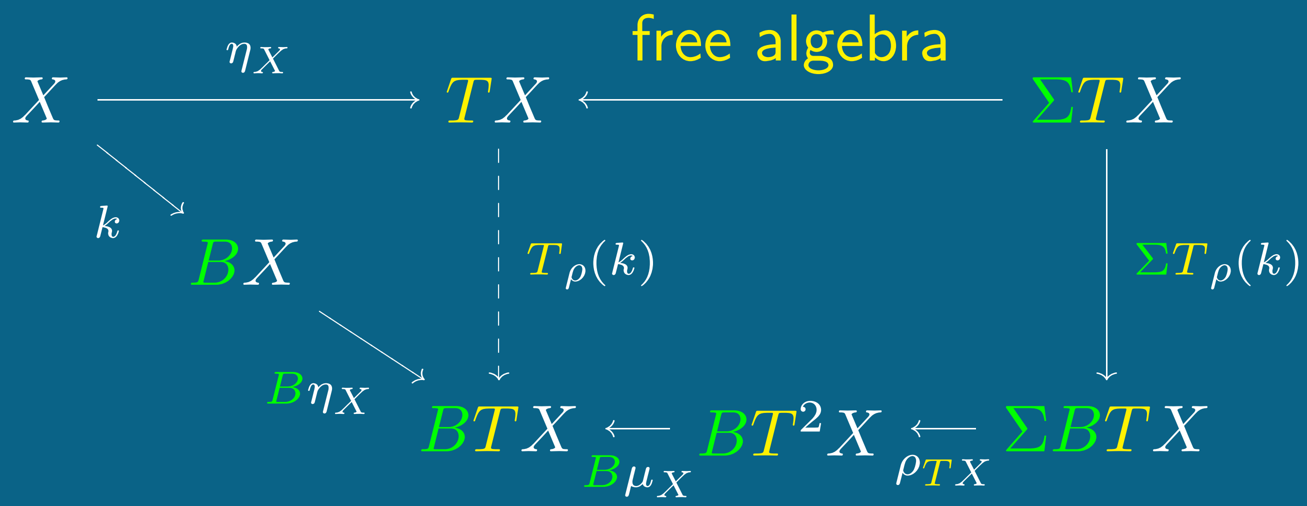

LHS: *conservative extension* (preservation of the behaviour of variables)

$$x' \in k(x)(a)$$

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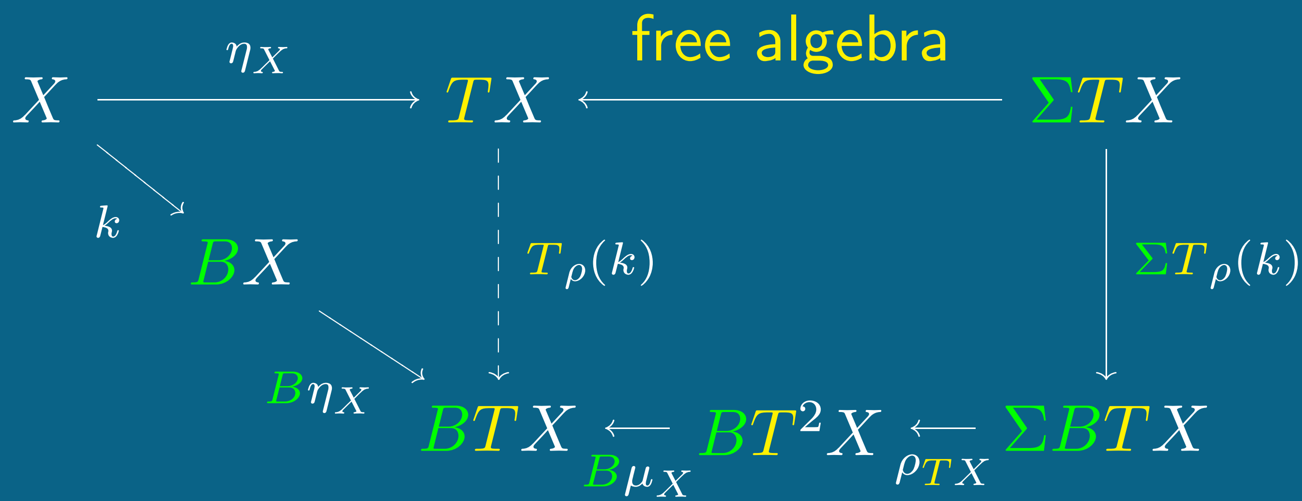

LHS: *conservative extension* (preservation of the behaviour of variables)

$$x' \in k(x)(a) \quad \Longleftrightarrow \quad x' \in T_\rho(k)(x)(a)$$

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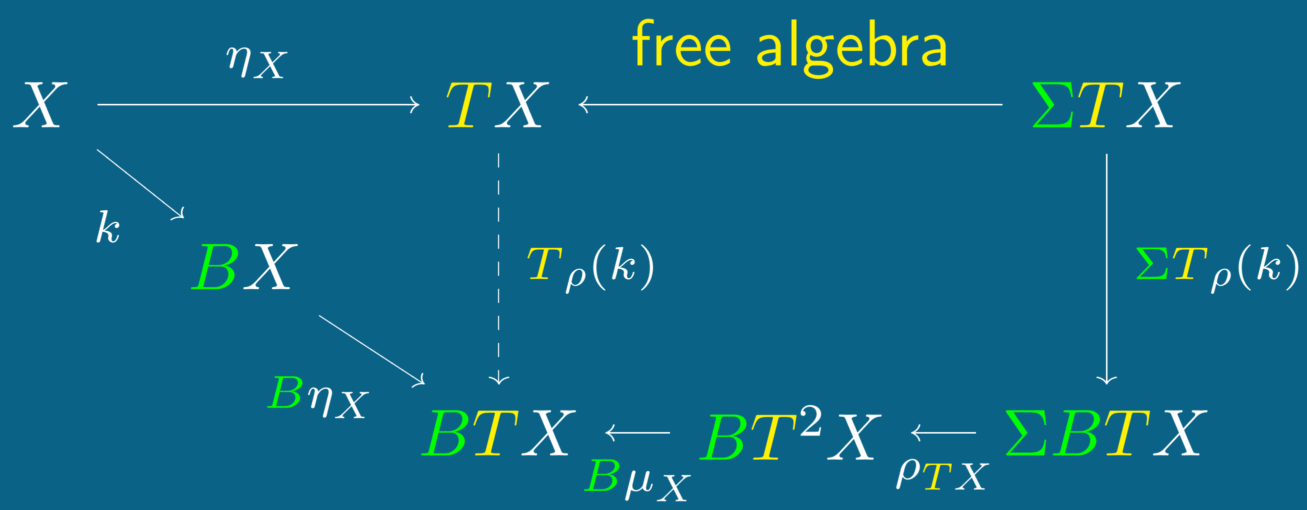

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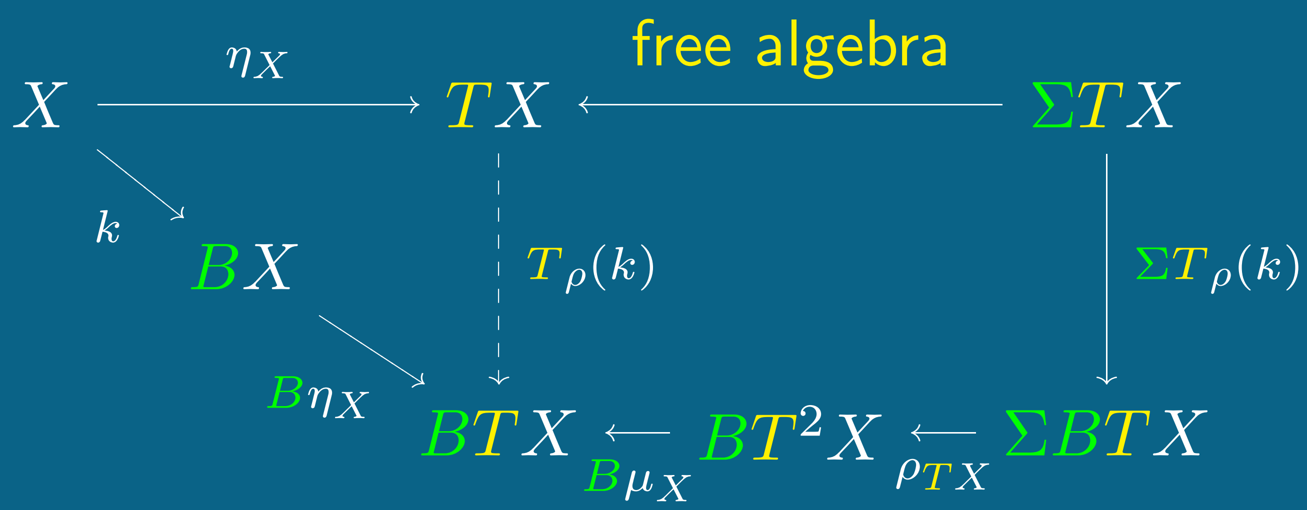
$x \overset{a}{\to} x'$ in k

- $\Sigma BTX \overset{\rho_{TX}}{\longrightarrow} BT^2X \overset{B\mu_X}{\longrightarrow} TX$



LHS: *conservative extension* (preservation of the behaviour of variables)

$$x' \in k(x)(a) \iff x' \in T_\rho(k)(x)(a)$$

$$x \overset{a}{\to} x' \text{ in } k \iff x \overset{a}{\to} x' \text{ in } T_\rho(k)$$

- Structural Recursion with accumulators:

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$$X \xrightarrow{\eta_X} TX \xleftarrow{\text{free algebra}} \Sigma TX$$

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$$
\begin{array}{ccc}
X & \xrightarrow{\eta_X} & TX & \xleftarrow{\text{free algebra}} & \Sigma T X \\
& \searrow^{k} & & & \\
& & BX & &
\end{array}
$$

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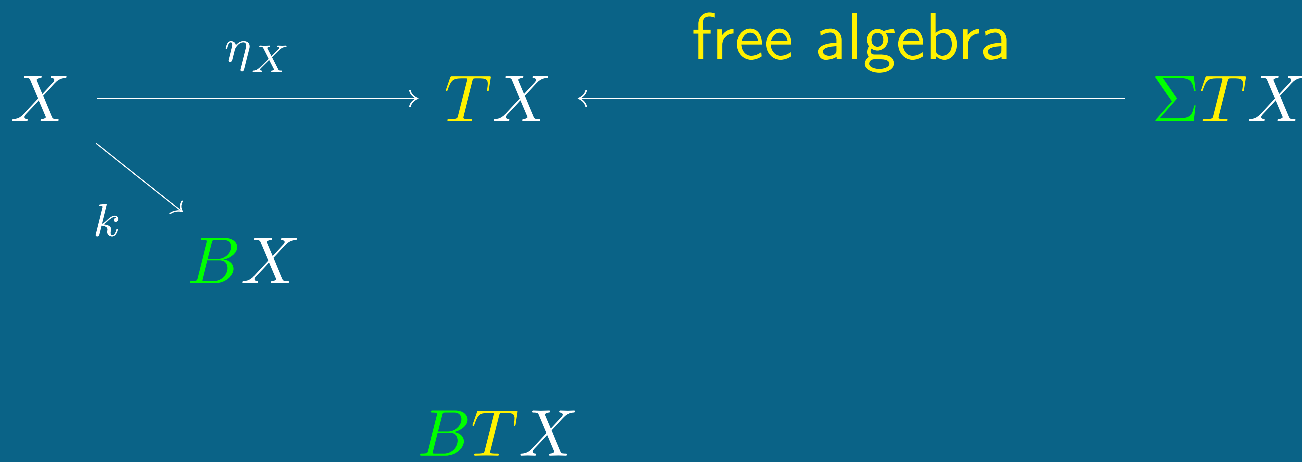

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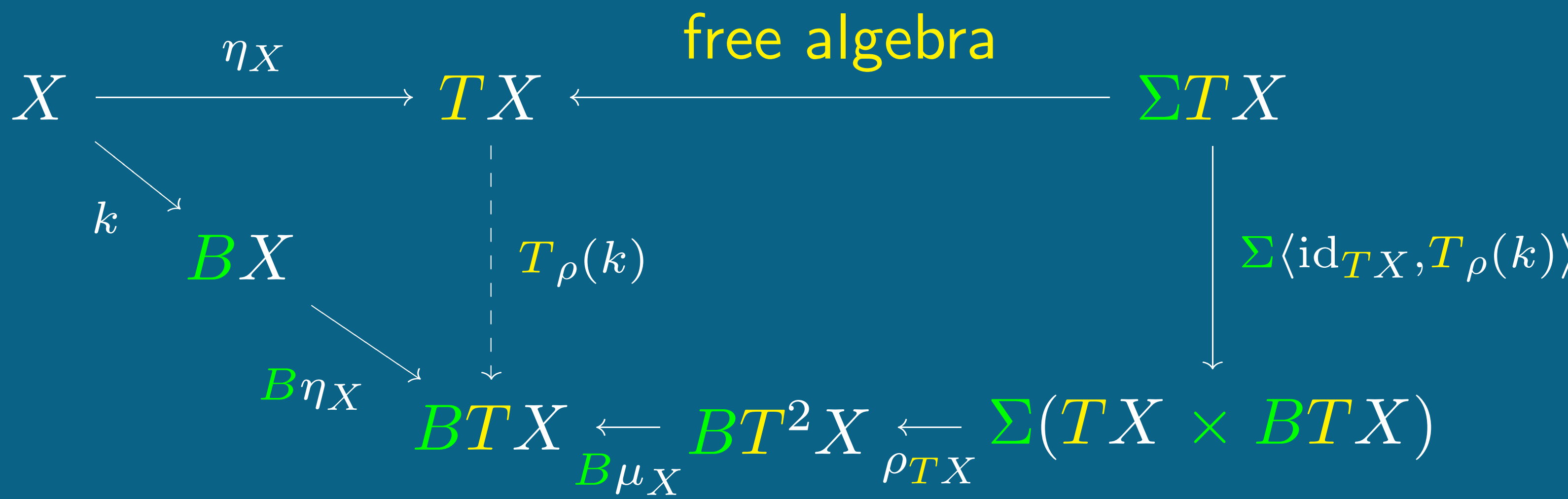

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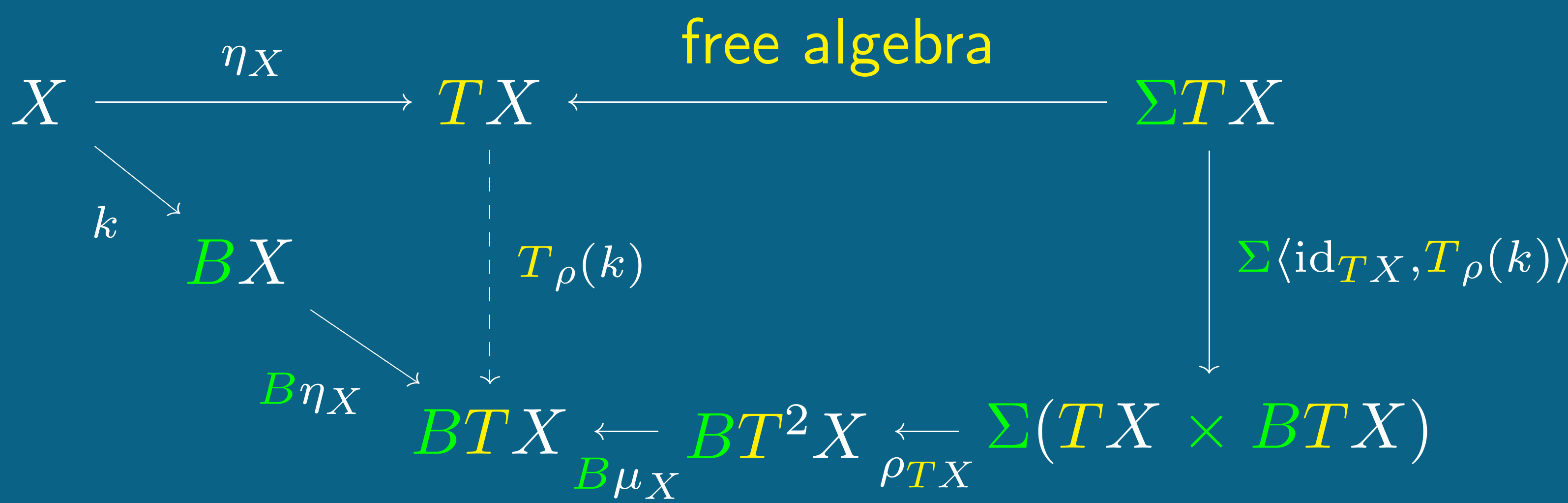
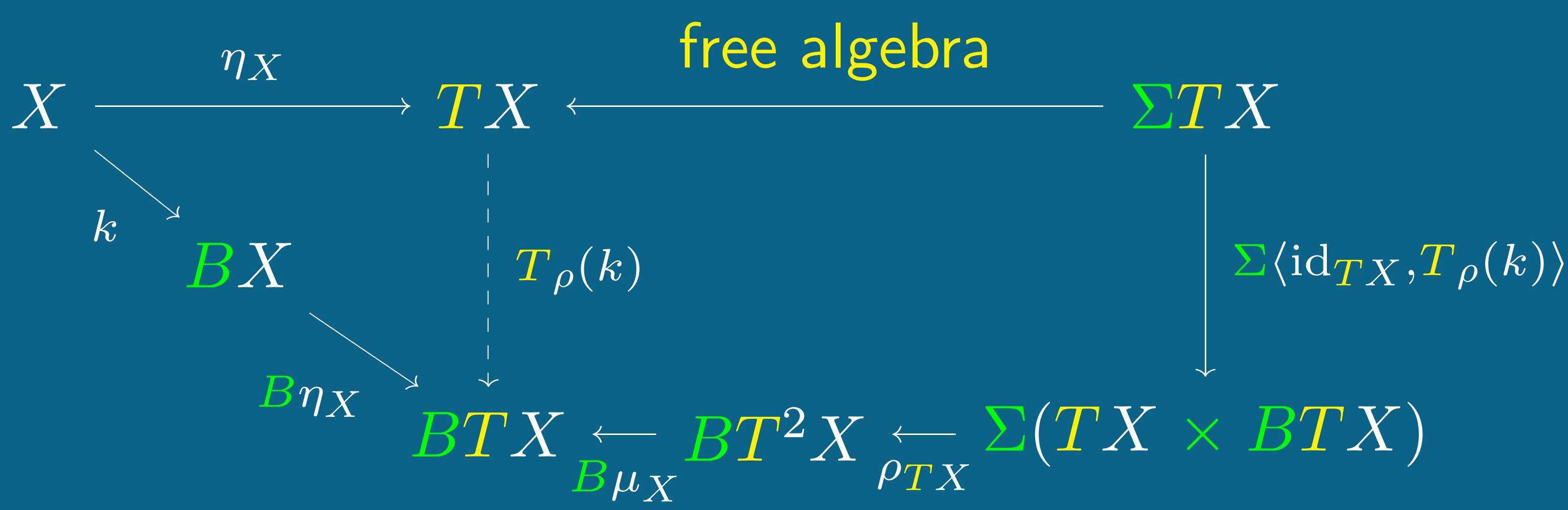


$$t_1 \| t_2 \xrightarrow{a} t$$

- Structural Recursion with accumulators:



$$t_1 \| t_2 \stackrel{a}{\to} t \iff \begin{cases} t_1 \stackrel{a}{\to} t_1' \\ \\ \end{cases}$$

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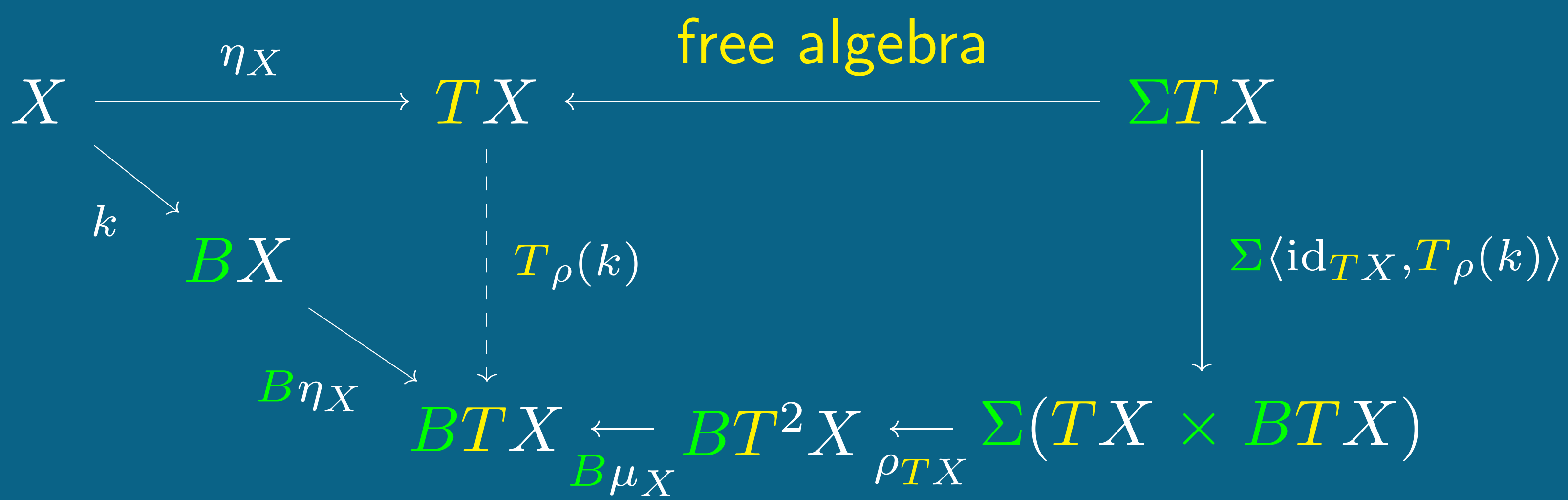

$$t_1 \| t_2 \stackrel{a}{\rightarrow} t \iff \begin{cases} t_1 \stackrel{a}{\rightarrow} t_1' \quad \& \quad t = t_1' \| t_2 \\ \\ \\ \end{cases}$$

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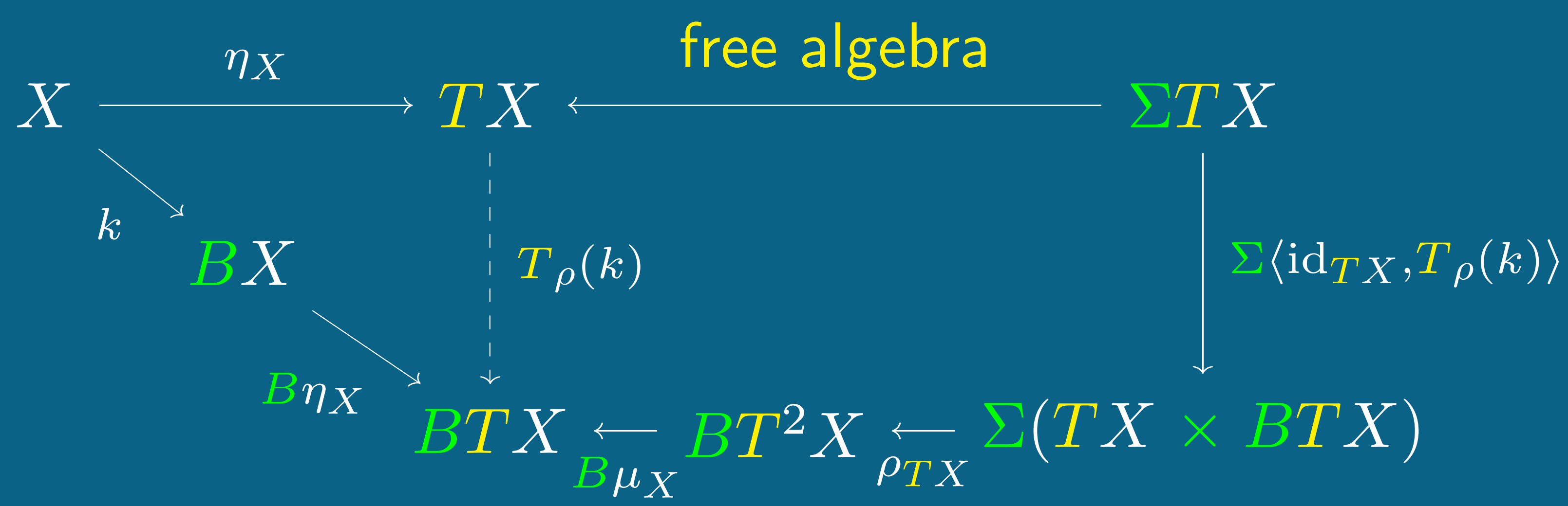

$$t_1 \| t_2 \overset{a}{\to} t \iff \begin{cases} t_1 \overset{a}{\to} t_1' \quad \& \quad t = t_1' \| t_2 \\ \text{or} \\ \end{cases}$$

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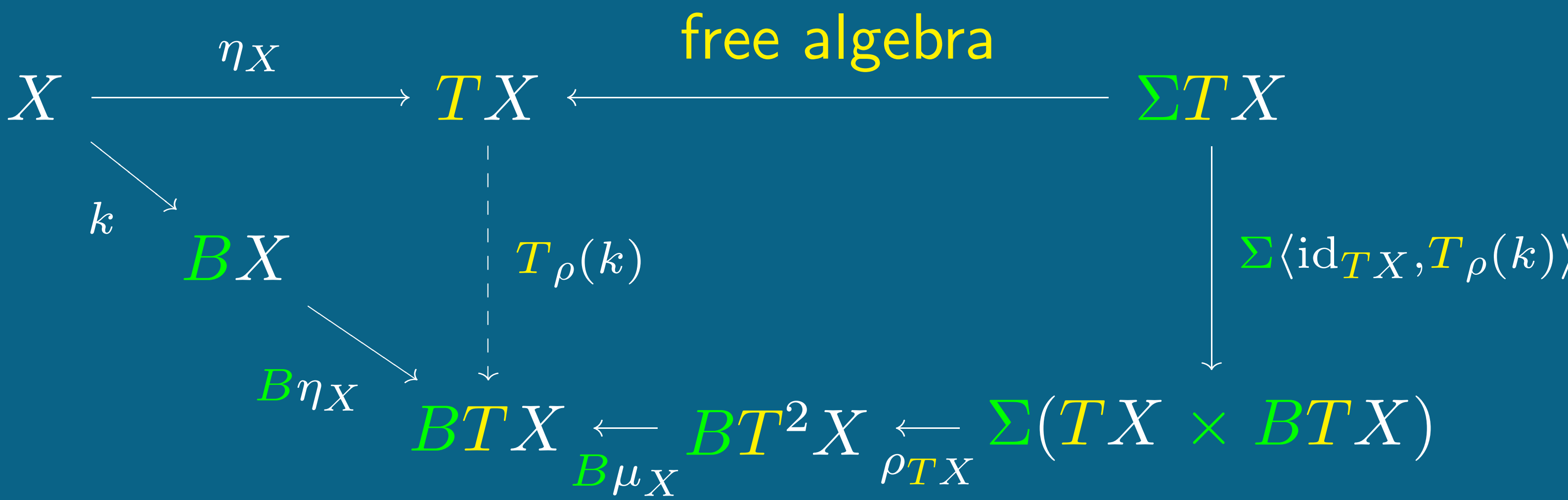

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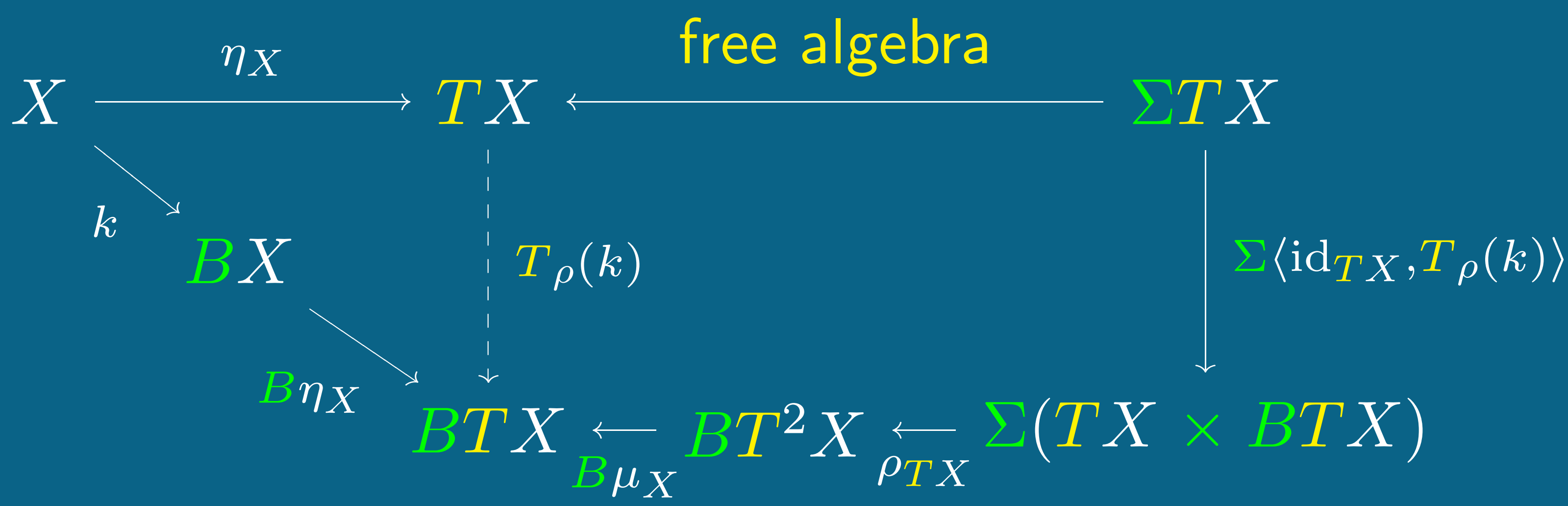

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- ρ-bialgebras

- ρ-bialgebras $\equiv$ Alex Simpson's GSOS models [LICS 95]

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  initial ρ-bialgebra:

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initial ρ-bialgebra:

$$\begin{array}{ccccc} T0 & & \xleftrightarrow{\text{initial algebra}} & & \Sigma T0 \\ {\scriptstyle T_\rho(0)}\downarrow & & & & \downarrow{\scriptstyle \Sigma\langle \mathrm{id}_{T0}, T_\rho(0)\rangle} \\ BT0 & \xleftarrow[B\mu_0]{} & BT^2 0 & \xleftarrow[\rho_{T0}]{} & \Sigma(T0 \times BT0) \end{array}$$

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$\star$ In the transition system corresponding to $T_\rho(0)$:

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⋆ Superunique semantics!

Guarded Recursive Programs

Guarded Recursive Programs

x

Guarded Recursive Programs

$$x \quad = \quad a.(x \| y)$$

Guarded Recursive Programs

$$x = a.(x \| y)$$

$$y$$

Guarded Recursive Programs

$$\begin{aligned} x &= a.(x \| y) \\ y &= b.y \end{aligned}$$

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$$\begin{aligned} x &= a.(x \| y) \\ y &= b.y \end{aligned} \qquad X \stackrel{\text{def}}{=} \{x, y\} \qquad k : X \longrightarrow\!\!\triangleright (\mathcal{P}_{\text{fi}} T X)^A$$

Guarded Recursive Programs

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Guarded Recursive Programs

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Guarded Recursive Programs

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$$X \xrightarrow{\eta_X} TX \xleftarrow{\quad\text{free algebra}\quad} \Sigma T X$$

Guarded Recursive Programs

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$$BTX$$

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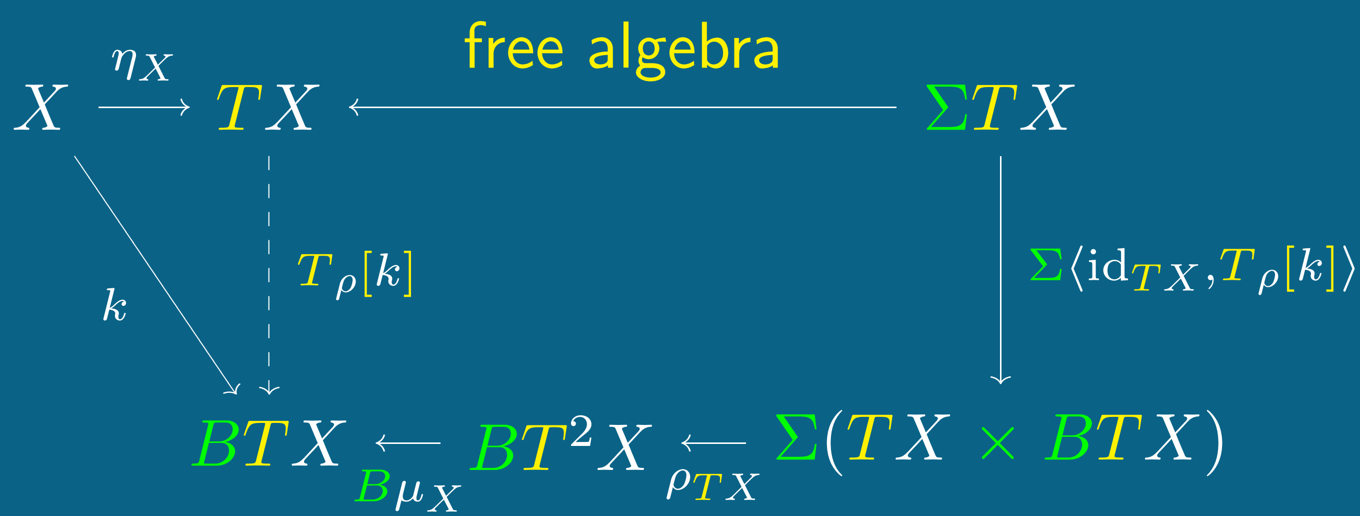

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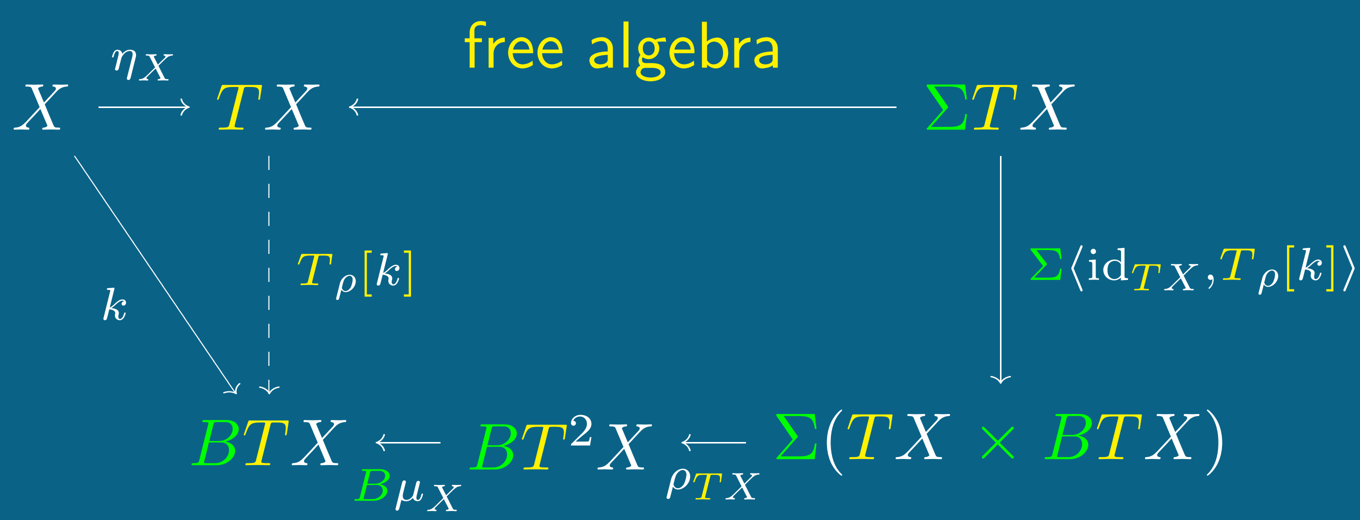

- More details in §5 of my thesis

Guarded Recursive Programs

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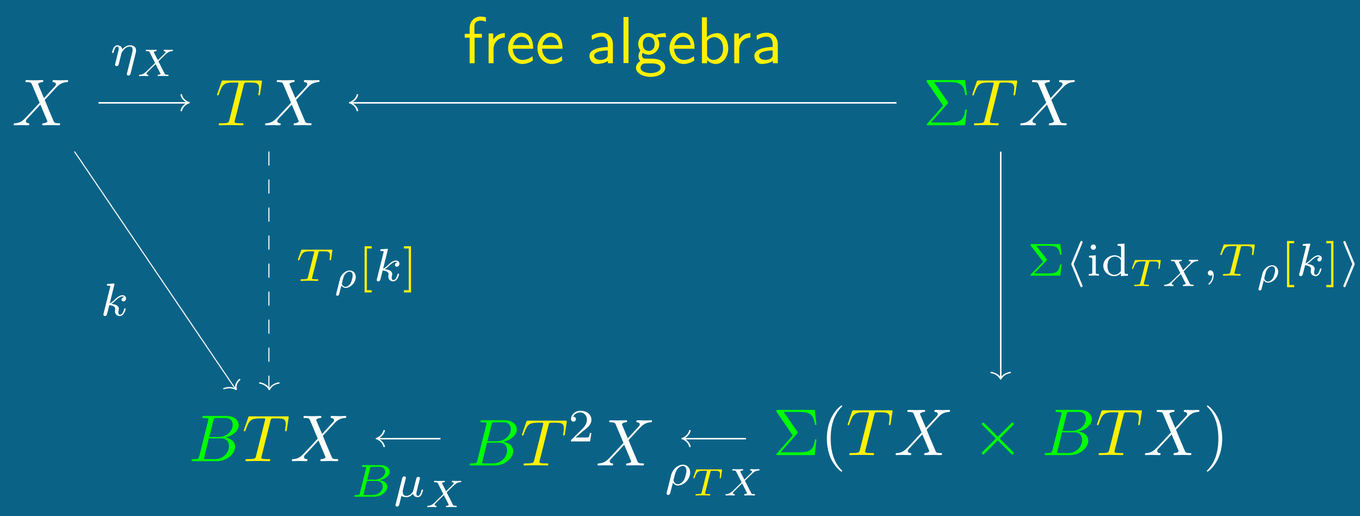

- More details in §5 of my thesis and in my CTCS 97 paper

Dual Case: “Denotational” Rules

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What is the dual of the operational rules

Dual Case: “Denotational” Rules

What is the dual of the operational rules

$$\Sigma(X \times BX) \longrightarrow\!\!\triangleright BTX?$$

Dual Case: "Denotational" Rules

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- See also §10 of my thesis

Conclusions

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- Rule Formats

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- Rule Formats (De Simone, GSOS, tyft/tyxt, tree)

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    - ∗ coalgebraic bisimulation is a congruence
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  - ⋆ *mathematical account of well-behaved operational semantics*

For more examples (involving modularity and categories other than $\mathbf{Set}$) see:

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Back to: Lecture III
Lecture IV
Beginning of this Lecture