Algebraic & Coalgebraic Methods
in
Semantics

Lecture V

Natural Operational Semantics

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- *How do we define*:

$$\lambda : TD \Longrightarrow DT \text{ (distr. law) ?}$$

$$\begin{array}{ccc} D\text{-Coalg} & \xrightarrow{\widetilde{T}} & D\text{-Coalg} \\ \text{Forget} \big\downarrow & & \big\downarrow \text{Forget} \\ \mathcal{C} & \xrightarrow[T]{} & \mathcal{C} \end{array}$$

$$\begin{array}{ccc} T\text{-Alg} & \xrightarrow{\widetilde{D}} & T\text{-Alg} \\ \text{Forget} \big\downarrow & & \big\downarrow \text{Forget} \\ \mathcal{C} & \xrightarrow[D]{} & \mathcal{C} \end{array}$$

- $\Sigma \rightsquigarrow T \;= \langle T, \eta, \mu \rangle$ $\qquad$ Σ-Alg $\cong$ T-Alg
  - $\star$ *structural* induction
- $B \rightsquigarrow D \;= \langle D, \varepsilon, \delta \rangle$ $\qquad$ B-Coalg $\cong$ D-Coalg
  - $\star$ *behavioural* coinduction

$$
\begin{array}{ccc}
B\text{-Coalg} & \xrightarrow{\widetilde{T}} & B\text{-Coalg} \\
{\scriptstyle \text{Forget}}\downarrow & & \downarrow{\scriptstyle \text{Forget}} \\
\mathcal{C} & \xrightarrow[T]{} & \mathcal{C}
\end{array}
\qquad\qquad
\begin{array}{ccc}
\Sigma\text{-Alg} & \xrightarrow{\widetilde{D}} & \Sigma\text{-Alg} \\
{\scriptstyle \text{Forget}}\downarrow & & \downarrow{\scriptstyle \text{Forget}} \\
\mathcal{C} & \xrightarrow[D]{} & \mathcal{C}
\end{array}
$$

- $\star$ $X \xrightarrow{k} BX \qquad \mapsto \qquad TX \xrightarrow{\widetilde{T}(k)} BTX$
- $\star$ $\Sigma X \xrightarrow{h} X \qquad \mapsto \qquad \Sigma DX \xrightarrow{\widetilde{D}(k)} DX$

Structural Operational Semantics

Operational semantics by *structural induction*?

- *Plotkin*'s SOS (1981)
  - Predominant approach to operational semantics

Example: merge operator $x_1 \| x_2$

$$\frac{x_1 \stackrel{a}{\rightarrow} x_1'}{x_1 \| x_2 \stackrel{a}{\rightarrow} x_1' \| x_2} \qquad \frac{x_2 \stackrel{a}{\rightarrow} x_2'}{x_1 \| x_2 \stackrel{a}{\rightarrow} x_1 \| x_2'}$$

Structure: operator $\|$

Behaviour of $x_1 \| x_2$: defined in terms of the behaviour of the components x_1 and x_2

A simple Process Algebra

- $t ::= x \mid \text{nil} \mid a.t \mid t \| t$
  - ⋆ $\Sigma X = 1 + A \cdot X + X^2$

- *Rules* $\mathcal{R}$: $a.x \xrightarrow{a} x \qquad \dfrac{x_1 \xrightarrow{a} x_1'}{x_1 \| x_2 \xrightarrow{a} x_1' \| x_2} \qquad \dfrac{x_2 \xrightarrow{a} x_2'}{x_1 \| x_2 \xrightarrow{a} x_1 \| x_2'}$

- *Behaviour*: $\boxed{BX = (\mathcal{P}_{\mathrm{fi}} X)^A}$ (for each $a \in A$ only a finite choice)
  - ⋆ B-*coalgebras*: image finite transition systems
    - ∗ $X \xrightarrow{k} (\mathcal{P}_{\mathrm{fi}} X)^A \qquad x' \in k\,(x)\,(a) \iff x \xrightarrow{a} x'$

$$\boxed{[\![\mathcal{R}]\!]_X : \Sigma(X \times BX) \longrightarrow BTX}$$

Modelling the Rules thru' Σ and B

$t ::= x \mid \mathsf{nil} \mid a.t \mid t \| t \qquad \Sigma X = 1 + A \cdot X + X^2 \qquad BX = (\mathcal{P}_{\mathrm{fi}} X)^A$

$$[\![\mathcal{R}]\!]_X : \Sigma(X \times BX) \longrightarrow\!\!\triangleright\, BTX$$

$[\![\mathcal{R}]\!]_X : 1 + A \cdot (X \times (\mathcal{P}_{\mathrm{fi}} X)^A) + (X \times (\mathcal{P}_{\mathrm{fi}} X)^A X)^2 \longrightarrow\!\!\triangleright\, (\mathcal{P}_{\mathrm{fi}} TX)^A$

- $[\![\mathsf{nil}]\!]_X \; : 1 \longrightarrow\!\!\triangleright\, (\mathcal{P}_{\mathrm{fi}} TX)^A$
- $[\![a.]\!]_X \; : A \cdot (X \times (\mathcal{P}_{\mathrm{fi}} X)^A) \longrightarrow\!\!\triangleright\, (\mathcal{P}_{\mathrm{fi}} TX)^A$
- $[\![\;\|\;]\!]_X \; : (X \times (\mathcal{P}_{\mathrm{fi}} X)^A)^2 \longrightarrow\!\!\triangleright\, (\mathcal{P}_{\mathrm{fi}} TX)^A$

$[\![\mathsf{nil}]\!]_X : 1 \longrightarrow\!\triangleright (\mathcal{P}_{\mathrm{fi}} T X)^A$

- *no rule* $\qquad [\![\mathsf{nil}]\!]_X = a \mapsto \emptyset$

$[\![a.]\!]_X : A{\cdot}(X \times (\mathcal{P}_{\mathrm{fi}} X)^A) \longrightarrow\!\triangleright (\mathcal{P}_{\mathrm{fi}} T X)^A$

- $a.x \stackrel{a}{\to} x \qquad [\![c.]\!]_X(x,\beta) = a \mapsto \begin{cases} \{x\} & \text{if } a = c \\ \emptyset & \text{otherwise} \end{cases}$

$[\![\ \|\ ]\!]_X : (X \times (\mathcal{P}_{\mathrm{fi}} X)^A)^2 \longrightarrow\!\triangleright (\mathcal{P}_{\mathrm{fi}} T X)^A$

- $\dfrac{x_1 \stackrel{a}{\to} x_1'}{x_1 \| x_2 \stackrel{a}{\to} x_1' \| x_2} \qquad \dfrac{x_2 \stackrel{a}{\to} x_2'}{x_1 \| x_2 \stackrel{a}{\to} x_1 \| x_2'}$

$$(x_1,\beta_1)[\![\ \|\ ]\!]_X(x_2,\beta_2) = a \mapsto \begin{array}{c} \{x_1' \| x_2 \mid x_1' \in \beta_1(a)\} \\ \cup \\ \{x_1 \| x_2' \mid x_2' \in \beta_2(a)\} \end{array}$$

Naturality

$$[\![\mathcal{R}]\!]_X : 1 + A\cdot(X \times (\mathcal{P}_{\mathrm{fi}} X)^A) + (X \times (\mathcal{P}_{\mathrm{fi}} X)^A)^2 \longrightarrow\!\!\!\triangleright (\mathcal{P}_{\mathrm{fi}} T X)^A$$

- *natural* in X
  - ⋆ $f : X \longrightarrow\!\!\!\triangleright Y \;\equiv$ *renaming* (possibly identifying variables)
    - $*$ first *renaming* and then *applying* $\mathcal{R}$

      $\equiv$

      first *applying* $\mathcal{R}$ and then *renaming*

GSOS

Which rules can we model as a transformation

$$\Sigma(X \times (\mathcal{P}_{\mathrm{fi}} X)^A) \longrightarrow (\mathcal{P}_{\mathrm{fi}} T X)^A \quad ?$$

$$\frac{\{x_i \stackrel{a}{\to} y_{ij}^a\}_{1\le j\le m_i^a}^{1\le i\le n,\, a\in A_i} \quad \{x_i \stackrel{b}{\not\to}\}_{b\in B_i}^{1\le i\le n}}{\sigma(x_1,\ldots,x_n) \stackrel{c}{\to} t}$$

What about *naturality* in X?

- ⋆ x_i and y_{ij}^a all distinct
- ⋆ x_i and y_{ij}^a the only variables occurring in t

GSOS rules [*Bloom, Istrail, Meyer 88*]!

Theorem

(A finite)

There is a correspondence between transformations of type

$$\Sigma(X \times (\mathcal{P}_{\mathrm{fi}} X)^A) \longrightarrow (\mathcal{P}_{\mathrm{fi}} T X)^A$$

natural in X

and

image finite sets of GSOS rules for Σ

(over a fixed denumerably infinite set of variables V).

- This correspondence is 1-1 up to equivalence of sets of rules.

From Rules to Distributive Laws

$\rho_X : \Sigma(X \times BX) \longrightarrow\!\!\triangleright\, BTX$

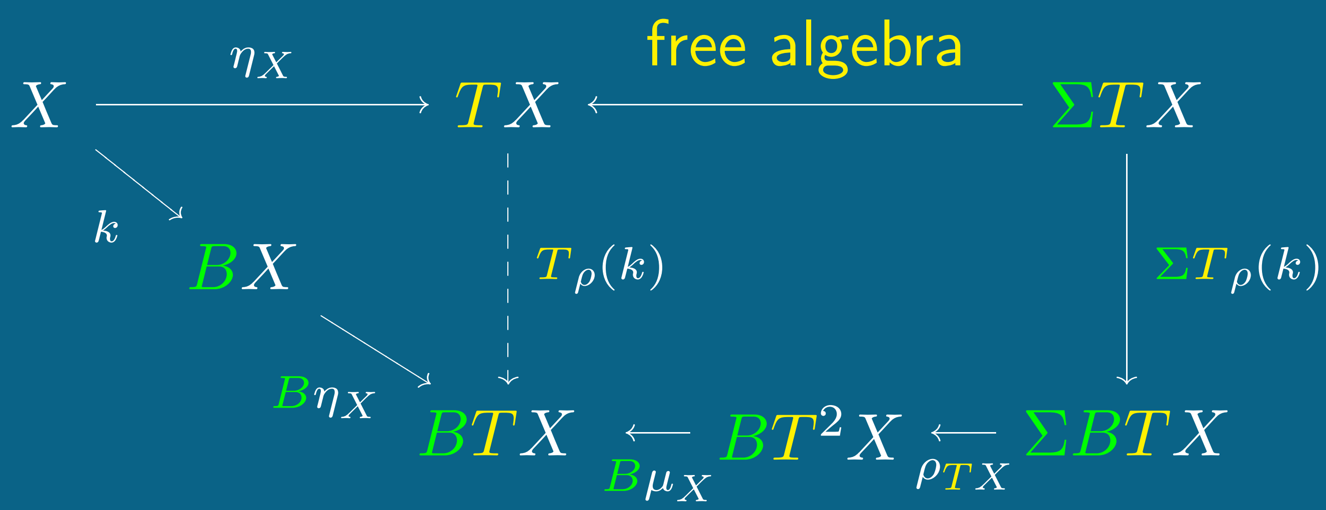
$$\begin{array}{ccc} B\text{-Coalg} & \xrightarrow{T_\rho} & B\text{-Coalg} \\ {\scriptstyle \text{Forget}}\downarrow & & \downarrow{\scriptstyle \text{Forget}} \\ \mathcal{C} & \xrightarrow[T]{} & \mathcal{C} \end{array}$$

- $X \stackrel{k}{\longrightarrow\!\!\triangleright} BX \quad \mapsto \quad TX \stackrel{T_\rho(k)}{\longrightarrow\!\!\triangleright} BTX$

If $\rho_X : \Sigma BX \longrightarrow\!\!\triangleright\, BTX$

- Derive a Σ-algebra structure for BTX!
  - ⋆ $\Sigma BTX \stackrel{\rho_{TX}}{\longrightarrow\!\!\triangleright} BT^2X \stackrel{B\mu_X}{\longrightarrow\!\!\triangleright} BTX$

- $\Sigma B T X \xrightarrow{\rho_{TX}} B T^2 X \xrightarrow{B\mu_X} T X$

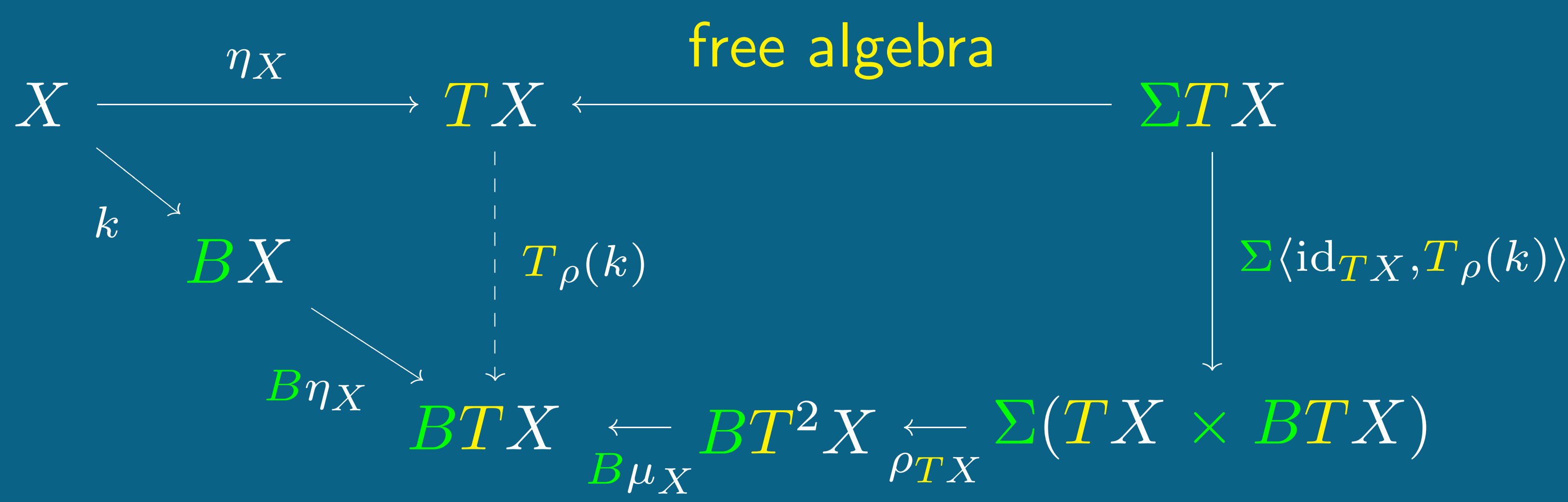


LHS: *conservative extension* (preservation of the behaviour of variables)

$$x' \in k(x)(a) \iff x' \in T_\rho(k)(x)(a)$$

$$x \overset{a}{\to} x' \text{ in } k \iff x \overset{a}{\to} x' \text{ in } T_\rho(k)$$

- Structural Recursion with accumulators:



$$t_1 \| t_2 \overset{a}{\to} t \iff \begin{cases} t_1 \overset{a}{\to} t_1' \quad \& \quad t = t_1' \| t_2 \\ \qquad\qquad \text{or} \\ t_2 \overset{a}{\to} t_2' \quad \& \quad t = t_1 \| t_2' \end{cases}$$

- ρ-bialgebras $\equiv$ Alex Simpson's GSOS models [LICS 95]

initial ρ-bialgebra:

$$\begin{array}{ccccc} T0 & & \xleftrightarrow{\text{initial algebra}} & & \Sigma T0 \\ \Big\downarrow T_\rho(0) & & & & \Big\downarrow \Sigma\langle \mathrm{id}_{T0}, T_\rho(0)\rangle \\ BT0 & \xleftarrow[B\mu_0]{} & BT^2 0 & \xleftarrow[\rho_{T0}]{} & \Sigma(T0 \times BT0) \end{array}$$

- ⋆ In the transition system corresponding to $T_\rho(0)$:

$$t_1 \parallel b.t_2 \xleftarrow{a} a.t_1 \parallel b.t_2 \xrightarrow{b} a.t_1 \parallel t_2$$

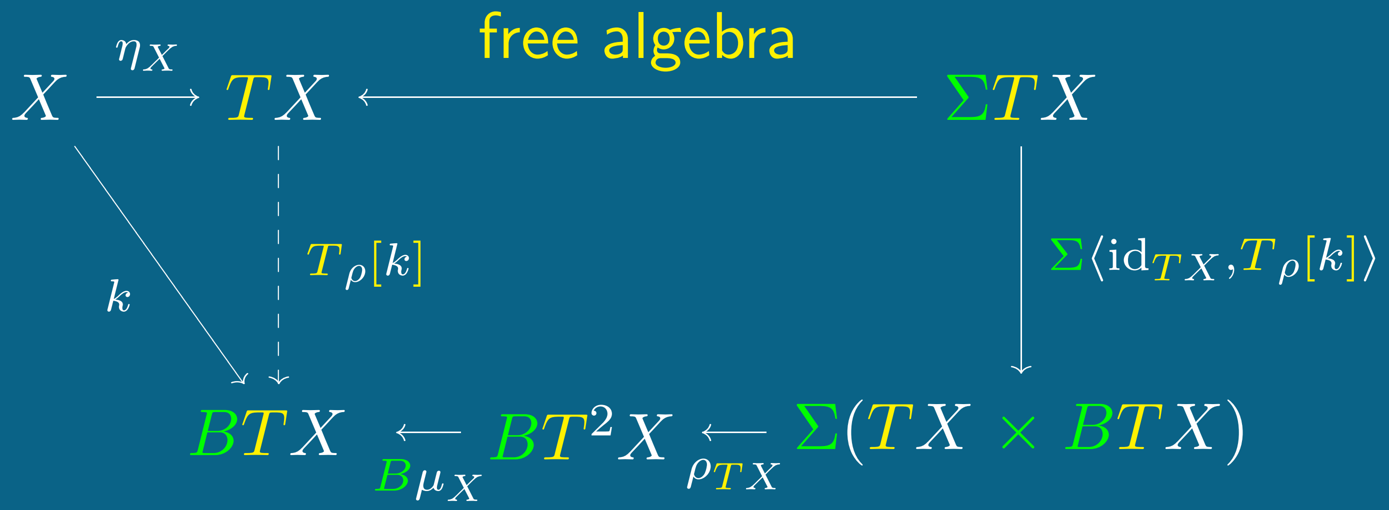
- ⋆ Superunique semantics!

Guarded Recursive Programs

$$x = a.(x \| y) \qquad X \stackrel{\text{def}}{=} \{x, y\} \qquad k : X \longrightarrow (\mathcal{P}_{\text{fi}} T X)^A$$

$$y = b.y \qquad\qquad k(x)(a) = \{x \| y\}$$

$$k(y)(b) = \{y\}$$



- More details in §5 of my thesis and in my CTCS 97 paper

Dual Case: "Denotational" Rules

What is the dual of the operational rules

$$\Sigma(X \times BX) \longrightarrow\!\!\triangleright\, BTX?$$

$$\boxed{\Sigma\, DX \longrightarrow\!\!\triangleright\, B\,(X + \Sigma X)}$$

- These are the tree rules (*Fokkink & van Glabbeek 96*)
- See also §10 of my thesis

Conclusions

- Rule Formats (De Simone, GSOS, tyft/tyxt, tree)
  - ⋆ bisimulation is a conguence
  - ⋆ *specific, entangled in syntax*
- *Natural Rules*
  - ⋆ Operational & Denotational!
    - ∗ superunique semantics
    - ∗ coalgebraic bisimulation is a congruence
  - ⋆ *abstract, modular, conceptual*
  - ⋆ *mathematical account of well-behaved operational semantics*

For more examples (involving modularity and categories other than $\mathbf{Set}$) see:

- my CTCS 97 paper

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Lecture IV
Beginning of this Lecture