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On the Foundations of Final Coalgebra Semantics: non-well-founded sets, partial orders, metric spaces

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This work, a revised version of (Rutten and Turi, 1993), is part of a programme aiming at formulating a mathematical theory of structural operational semantics, complementing the established theory of domains and denotational semantics to form a coherent whole (Turi, 1996; Turi and Plotkin, 1997). The programme is based on a suitable interplay between the *induction principle* which pervades modern mathematics, and a dual, non-standard '*coinduction principle*' which underlies many of the recursive phenomena occurring in computer science.

The aim of the present survey is to show that the elementary categorical notion of a *final coalgebra* is a suitable foundation for such a coinduction principle. The properties of coalgebraic coinduction are studied both at an abstract categorical level and in some specific categories used in semantics, namely categories of non-well-founded sets, partial orders, and metric spaces.

1. Introduction

"The original stimulus for my own interest in the notion of a non-well-founded set came from a reading of the work of Robin Milner in connection with his development of a mathematical theory of concurrent processes. This topic in theoretical computer science is one of a number of such topics that are generating exciting new ideas and intuitions that are in need of suitable mathematical expression."

Peter Aczel, *Non-Well-Founded Sets*

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In set theory, mathematical induction is based on the notion of a *well-founded* relation, that is, a relation R such that, for every set x, there is *no* infinitely descending chain

$$\ldots R\, x_2\, R\, x_1\, R\, x_0 = x$$

The fact that standard mathematical constructions are inductive is mirrored by the common assumption that the axioms of set theory include the *foundation axiom* which postulates that the set-membership relation '$\in$' is well-founded: for every set x, there exists no infinitely descending chain

$$\ldots \in x_2 \in x_1 \in x_0 = x$$

In Milner's *Calculus of Communicating Systems* (CCS), the behaviour of a non-deterministic program P is given by the set

$$[\![P]\!] = \{P \xrightarrow{a_i} P_i\}$$

of *atomic transitions* $P \xrightarrow{a_i} P_i$ which the program can perform, producing an observable *action* a_i and becoming P_i. The meaning of a program P should abstract from the name of the programs involved in the transitions and focus on the actions which can be performed, together with the choices which can be made. It should then be the following 'coinductively' defined set.

$$[\![P]\!]^@ = \{<a, [\![P']\!]^@> \mid P \xrightarrow{a} P'\}$$

Now, in general, the transition relation is not well-founded, since, for instance, cyclic programs $P \xrightarrow{a} P$ are allowed. Therefore, the above meaning $[\![P]\!]^@$ can be a non-well-founded set.

Traditionally, this 'problem' has been overcome by imposing either an order or a metric on the transition relation and then defining $[\![P]\!]^@$ as a suitable *limit*. Instead, in (Aczel, 1988) a more radical approach is taken and the standard foundation axiom is replaced by the '*anti-foundation axiom*' which allows for non-well-founded sets. (See also the equivalent 'X_1-axiom' in (Forti and Honsell, 1983).)

One of the contributions of the present work is to show that the anti-foundation axiom is the dual, in a formal sense, of the foundation axiom. Indeed, we prove that foundation is equivalent to postulating that the universe of sets is an initial algebra for a suitable power-set endofunctor' and, dually, that anti-foundation is equivalent to postulating that the universe of sets is a final coalgebra (for the same power-set endofunctor).

From a categorical point of view one does not need to resort to non-standard foundations: as already clear in (Aczel, 1988), coinductive definitions can be founded on final coalgebras and these exist also in the standard category of ordinary sets (and in many other categories). What the anti-foundation axiom gives is the non-standard fact that the greatest (strict) fixed point

$$gfp(F) = F(gfp(F))$$

of an endofunctor F on the category $\underline{\mathrm{SET}}$ of classes (ie large sets) is a final F-coalgebra, provided F satisfies some mild conditions. This theorem (Aczel, 1988, "Special Final

Coalgebra Theorem") is the 'dual' of the standard fact (holding also without anti-foundation) that the least fixed points of most endofunctors on SET are initial algebras.

The special final coalgebra theorem is stated in terms of the "Solution Lemma" (Aczel, 1988). The final coalgebra presentation of anti-foundation introduced here makes the solution lemma (and its equivalence with anti-foundation) trivial. Correspondingly, the 'uniformity on maps' condition – which an endofunctor has to satisfy in order for the special final coalgebra theorem to hold – can be formulated here somewhat more transparently than in (Aczel, 1988).

Aczel's semantics of CCS exploits the special final coalgebra theorem by noticing that the intended operational model of CCS can be seen as a coalgebra of the endofunctor mapping a class X to the class $\mathcal{P}_S(Act \times X)$ having as elements (small) sets of pairs $<a, x>$, with $a \in Act$ and $x \in X$, where Act is the set of actions which can be performed by the programs. The carrier of this operational model is the set *Prog* of programs, and the coalgebra structure is the function $[\![\text{-}]\!] : Prog \to \mathcal{P}_S(Act \times Prog)$ defined for every program $P \in Prog$ as the set of pairs $<a, P'>$ such that P can perform an action a and become P':

$$[\![P]\!] = \{<a, P'> \mid P \xrightarrow{a} P'\}$$

The function $[\![\text{-}]\!]^{@}$ mapping a program to its abstract meaning can then be defined as the *coinductive extension* of this coalgebra structure, that is, as the unique coalgebra homomorphism from the intended operational model to the greatest fixed point of the '*behaviour endofunctor*'

$$BX = \mathcal{P}_S(Act \times X)$$

which, by the special final coalgebra theorem, is a final coalgebra:

$$\begin{tikzcd} Prog \arrow[r, dashed, "{[\![\text{-}]\!]^{@}}"] \arrow[d, "{[\![\text{-}]\!]}"'] & gfp(B) \arrow[d, equal] \\ B(Prog) \arrow[r, "{B([\![\text{-}]\!]^{@})}"'] & B(gfp(B)) \end{tikzcd}$$

That is, for every program P, $[\![P]\!]^{@} = \{<a, [\![P']\!]^{@}> \mid P \xrightarrow{a} P'\}$.

1.1. *Final Coalgebra Semantics*

The above semantics for CCS and its properties can be generalised to arbitrary behaviours, leading to what is called here the final coalgebra approach to semantics: given a coalgebraic operational model for a programming language, its coinductive extension – the *final coalgebra semantics* of the language – is a semantics which is *fully abstract* with respect to the behaviour, in the sense that two programs are identified if and only if they are behaviourally equivalent.

The basic idea is that a good, mathematical notion of behaviour should correspond to an endofunctor B, whose coalgebras should give the operational models for B. In particular, final coalgebras are canonical domains of interpretation. Indeed, as shown here, final coalgebras are 'strongly extensional' that is, two elements of the final B-coalgebra are equal if and only if they are 'B-bisimilar'. The latter is a coalgebraic generalization given in (Aczel and Mendler, 1989) of the notion of strong bisimilarity used in concurrency theory (Park, 1981); it subsumes other behavioural equivalences as, eg, applicative bisimulation and trace equivalence. (Cf (Fiore, 1996b; Rutten and Turi, 1994).) Thus, semantically, the final coalgebra of a behaviour endofunctor, if it exists, is an operational model where behaviourally equivalent elements are indistinguishable. In other words, the operational model given by finality is *internally fully abstract* with respect to B.

Since the coalgebra homomorphisms are morphisms between the carriers of the operational models which 'respect' the behaviour, the coinductive extension of an operational model identifies two elements if they are behaviourally equivalent in the above sense. Under the additional hypothesis that the behaviour B preserves 'weak pullbacks', coinduction can be 'pulled back' to coalgebraic bisimulation. and also the converse holds, that is, two elements are identified if and *only if* they are behaviourally equivalent.

This final coalgebra approach to semantics complements the standard initial algebra approach (Goguen et al., 1978), where, given a functorial notion of signature Σ for a language, the programs are described as the initial Σ-algebra, the most concrete denotational model; the other Σ-algebras are denotational models whose structure can always be inductively extended, by initiality, to a compositional interpretation of the programs. A suitable interplay between initial algebra and final coalgebra semantics is the basis of the categorical approach to structural operational semantics presented in (Turi and Plotkin, 1997). (See also (Turi, 1997; Turi, 1996).)

1.2. *Partial Orders*

Another way of looking at initial algebras and final coalgebras of endofunctors F is as *data types*: the initial F-algebra is the *inductive* data type corresponding to the 'type constructor' F, while the final F-coalgebra is the *coinductive* one. For instance, the type constructor $FX = 1 + X$ yields, in the category $\underline{\text{Set}}$ of sets and functions, the natural numbers N as inductive data type and the 'extended natural numbers' $N \cup \{\infty\}$ as coinductive one.

Studies on coinductive types in $\underline{\text{Set}}$ date back at least to (Arbib and Manes, 1980). A more recent view, put forward by Peter Freyd in (Freyd, 1991), is that data types should be defined in *algebraically compact categories*, that is, in categories where endofunctors have both initial algebras and final coalgebras which, moreover, do *coincide* in the sense that they are 'canonically isomorphic'. (See also (Freyd, 1990; Freyd, 1992).) This gives a useful mixed induction-coinduction principle. (Cf (Pitts, 1994a; Pitts, 1994b).)

One of the main examples of algebraically compact categories is the category $\underline{\text{Cppo}}_{\perp}$ of complete pointed partial orders and strict continuous functions: regarded as an 'order-enriched' category, it has as endofunctors the 'locally continuous' ones, which, as shown in (Smyth and Plotkin, 1982), make it algebraically compact indeed.

In the present study, an order-enriched version of the final coalgebra semantics approach is also introduced. A preliminary version of this work contained a definition of coalgebraic bisimulation in the order-enriched setting. Such a definition has been refined in (Fiore, 1996b), together with the theorem stating that final coalgebras of locally continuous endofunctor on $\underline{\mathrm{Cppo}}_{\perp}$ are strongly extensional with respect to this ordered coalgebraic bisimulation. This is generalized here to a full abstraction result which, in particular, can be applied to the final coalgebra semantics of non-terminating programs.

Notice that algebraic compactness is one of the axioms of Fiore and Plotkin's *axiomatic domain theory* (Fiore and Plotkin, 1992; Fiore and Plotkin, 1994; Fiore, 1996a) which aims at isolating the abstract properties which a category should satisfy for hosting interpretations of programming languages. In particular, the semantic domain of a language – in the present setting the final coalgebra of the behaviour – should 'live' in such a category, typically $\underline{\mathrm{Cppo}}_{\perp}$.

The category of sets, no matter whether ordinary or non-well-founded, is *not* algebraically compact (because algebraic compactness is inconsistent with cartesian closure). Still, a naive semantics in $\underline{\mathrm{Set}}$ is often a good starting point for more refined analyses, especially in settings where, as in concurrency, partiality does not play a central rôle.

1.3. *Metric Spaces*

With partial orders, one can assess not only the semantic equality or inequality of programs, but also whether a program is semantically 'better' than another. Instead of such 'qualitative' judgements, one might be interested in a 'quantitative' relation between programs, with the semantics giving the *distance* between each pair of programs. This leads to the use of metric spaces in semantics. (See (de Bakker and Rutten, 1992; de Bakker and de Vink, 1996) for overviews.)

The main mathematical tool available in (complete) metric spaces is "Banach's theorem" ensuring the existence of unique fixed points of 'contractive' endofunctions. Banach's theorem, especially in its higher-order form, can be used for dealing with coinduction. Here it is proved that the unique fixed point of a 'locally contractive' endofunctor as given by Banach's theorem is both a final coalgebra and an initial algebra of that endofunctor. This shows that the category $\underline{\mathrm{Cms}}$ of complete metric spaces and non-expansive functions, with as endofunctors the locally contractive ones, is, like $\underline{\mathrm{Cppo}}_{\perp}$, algebraically compact.

Structures where both the quantitative and qualitative information are present are the *quasimetric spaces* studied in (Smyth, 1988; Bonsangue et al., 1998). These are metric spaces where distances might be non-symmetric. Such an asymmetry requires a more subtle notion of limit than for metric spaces, for which contractive functions need not be continuous. But once these differences are taken into account, Banach's theorem holds also in this setting and the algebraic compactness theorem for $\underline{\mathrm{Cms}}$ carries over to the category $\underline{\mathrm{Cqms}}$ of complete quasimetric spaces and functions which are both non-expansive *and* continuous.

A further generalization is achieved in (Wagner, 1994) (see also (Flagg and Kopperman, 1997)), where structures parametric in a 'quantale' Ω are studied. In particular, generalized notions of Cauchy sequence and limit are given which, at a higher level, are used to show that the standard constructions of final coalgebras (alias initial algebras) in $\underline{\mathrm{Cms}}$ and in $\underline{\mathrm{Cppo}}_\perp$ are both instances of one construction parametric in Ω. The use of Ω stems from a foundational article by Lawvere (1973) presenting metric spaces, partial orders, and categories as instances of one and the same notion.

Contents

Section 2 contains the definitions of algebraic induction and coalgebraic coinduction. Also, examples of operational models as coalgebras of behaviour endofunctors are given. In Section 3, it is shown that initial algebras and final coalgebras can both be seen as adjoints to suitable forgetful functors. Following (Barr, 1993), this fact is used to show that the finite power-set functor has a final coalgebra.

In Section 4, the coalgebraic notion of bisimulation introduced in (Aczel and Mendler, 1989) is studied. Several properties are proved which generalize standard results on (ordinary) bisimulations. In particular, a general full abstraction theorem is proved stating that in final coalgebra semantics two programs are identified if and only if they are behaviourally equivalent. Next, it is shown how to obtain final coalgebras from weakly final ones using coalgebraic bisimulations. Finally, a new notion of bisimulation is introduced which is parametric in the possible transformations of states.

Section 5 is devoted to a coalgebraic presentation of Peter Aczel's theory of non-well-founded sets. The main novelty is that a categorical duality is proved between the anti-foundation axiom (giving non-well-founded sets) and the standard foundation axiom. Also, simplified presentations of the "solution lemma", of the "substitution lemma", and of the notion of a functor uniform on maps are given. This allows for a more transparent proof of the "special final coalgebra theorem", one of the most relevant results, from a semantical point of view, in (Aczel, 1988).

Section 6 is a survey on final coalgebras and their bisimulations in order-enriched categories; it is based on material in (Smyth and Plotkin, 1982) and (Fiore, 1996b). It also contains a full abstraction theorem with respect to the order-enriched version of coalgebraic bisimulation.

The last section is dedicated to final coalgebras in categories of metric spaces. It is shown that the categorical version of Banach's fixed point theorem introduced in (America and Rutten, 1989) yields a final coalgebra (canonically isomorphic to an initial algebra). Also, the above mentioned recent results on generalized notions of metric spaces, which reconcile the order-theoretic with the metric theoretic approach, are discussed.

2. From Algebraic Induction to Coalgebraic Coinduction

The most elementary form of induction, the one on the natural numbers

$$N = \{0,\, 1 = s(0),\, 2 = s^2(0),\, \ldots\}$$

is based on the following theorem.

Theorem 2.1. (Recursion Theorem) Given a set X, an element $e \in X$ and a function $g : X \to X$, there *exists* a *unique* function $f : N \to X$ from the set of natural numbers to the given set such that

$$f(0) = e \qquad \text{and} \qquad f(s(n)) = g(f(n))$$

for all numbers $n \in N$. □

The value e of the function f at (the least element) 0 (with respect to the order relation) is the 'base' of the induction and g defines the 'inductive step'.

The recursion theorem can be taken as the *definition* of natural numbers. That is, every set N with a distinguished element $0 \in N$ and a unary operation $s : N \to N$ such that the recursion theorem holds, is *isomorphic* to the natural numbers. (See, eg, (Mac Lane, 1986, Chapter 2).) As pointed out by Lawvere, the existence/uniqueness statement of the recursion theorem asserts the **universal property** characterizing the natural numbers: *initiality*. This property underlies induction, not only on the natural numbers, but in general.

Initiality is the most elementary universal property which an object of a category can enjoy: an object X is **initial** in a category if, for every object Y of the category, there *exists* a morphism $f : X \to Y$ from X to Y and, moreover, this morphism is *unique*.

The basic way of understanding the natural numbers as an initial object is by regarding them as an object $\langle N, 0, s\rangle$ in the category having as objects triples $\langle X, e, t\rangle$, where X is a set with a distinguished element $e \in X$ and a function $t : X \to X$ on it. The morphisms $f : \langle X, e, t\rangle \to \langle X', e', t'\rangle$ of the category are functions $f : X \to X'$ such that

$$f(e) = e' \qquad \text{and} \qquad f(t(x)) = t'(f(x))$$

(It is easy to verify that the above objects and morphisms form a category with composition and identities as in <u>Set</u>.) Then the recursion theorem says exactly that the triple $\langle N, 0, s\rangle$ is initial in this category. (Notice that in the category <u>Set</u> the initial object is the trivial empty set.) Conversely, since initial objects, as all universals, are unique up to isomorphism, the initial object of this category defines the natural numbers up to isomorphism.

This specific form of induction can be generalized through the following series of abstractions.

Firstly, notice that the element $e \in X$ of a set X can be written as a function from the one-element set $1 = \{*\}$ to the set X; that is, one can identify a function $e : 1 \to X$ from the one-element set 1 to a set X with its value $e(*) \in X$ at the unique element $*$ of 1. Then the recursion theorem amounts to having an object $1 \xrightarrow{0} N \xrightarrow{s} N$ such that for every object $1 \xrightarrow{e} X \xrightarrow{g} X$, there exists a unique function $f : N \to X$ with

$$f \circ 0 = e \qquad \text{and} \qquad f \circ s = g \circ f$$

Diagrammatically, using dashed morphisms to denote morphisms given by universal properties, one has that the following diagram commutes.

$$\begin{tikzcd}
1 \arrow[r, "0"] \arrow[d, equal] & N \arrow[r, "s"] \arrow[d, dashed, "f"] & N \arrow[d, dashed, "f"] \\
1 \arrow[r, "e"'] & X \arrow[r, "g"'] & X
\end{tikzcd}$$

Secondly, every pair of functions with the same codomain (thus, eg, $e : 1 \to X$ and $g : X \to X$) can be made into a single morphism with as domain the disjoint union of the domains. This holds in general in every category with **coproducts**: given two objects X and Y in a category, their coproduct, if it exists, is an object $X + Y$ with two morphisms $\mathsf{inl}_X : X \to X + Y$ and $\mathsf{inl}_Y : Y \to X + Y$ which is *universal* in the sense that for every pair of morphisms $f : X \to Z$ and $f : Y \to Z$ there *exists* a *unique* morphism $[f, g] : X + Y \to Z$, making the following diagram commute.

$$\begin{tikzcd}
X \arrow[r, "\mathsf{inl}_X"] \arrow[dr, "f"'] & X + Y \arrow[d, dashed, "{[f,g]}"] & Y \arrow[l, "\mathsf{inr}_Y"'] \arrow[dl, "g"] \\
& Z &
\end{tikzcd}$$

(The dual of the coproduct $X+Y$ is the **product** $X \times Y$: its *projections* $\mathsf{fst}_X : X \times Y \to X$ and $\mathsf{snd}_Y : X \times Y \to Y$ are universal among all pairs of morphisms $f : Z \to X$ and $g : Z \to Y$.)

In <u>Set</u>, the disjoint union, together with the corresponding injection functions, is a coproduct. Hence, one can write $[e, g] : 1 + X \to X$ instead of $1 \xrightarrow{e} X \xrightarrow{g} X$. Correspondingly, the initiality of the natural numbers can be expressed by saying that for every function $h : 1 + X \to X$ there exists a unique morphism $f : N \to X$ such that the following diagram commutes.

$$\begin{tikzcd}
1 + N \arrow[r, "1+f"] \arrow[d, "{[0,s]}"'] & 1 + X \arrow[d, "h"] \\
N \arrow[r, dashed, "f"'] & X
\end{tikzcd}$$

The morphism $1 + f : 1 + N \to 1 + X$ is defined by universality: using id_X to denote the identity morphism on an object X,

$$1 + f = [\mathsf{inl}_1 \circ \mathsf{id}_1, \mathsf{inr}_X \circ f] = [\mathsf{inl}_1, \mathsf{inr}_X \circ f] : 1 + N \to 1 + X$$

Thus the operation $X \mapsto 1 + X$ on objects extends to an operation $f \mapsto 1 + f$ on morphisms: this defines a functor from $\underline{\mathrm{Set}}$ to $\underline{\mathrm{Set}}$, ie an **endofunctor** on $\underline{\mathrm{Set}}$

The third step of abstraction is now to move from the above (endo) functor $FX = 1+X$ on $\underline{\mathrm{Set}}$ to arbitrary endofunctors F on arbitrary categories $\underline{\mathrm{C}}$ and, correspondingly, to consider initial objects in categories of structures $h : FX \to X$ rather than $h : 1+X \to X$.

Given an endofunctor $F : \underline{\mathrm{C}} \to \underline{\mathrm{C}}$ on a category $\underline{\mathrm{C}}$ one can form the **category $\underline{\mathbf{C}}^F$ of F-algebras** having as objects pairs $\langle X, h\rangle$ with X an object and $h : FX \to X$ a morphism of $\underline{\mathrm{C}}$. A morphism $f : \langle X, h\rangle \to \langle X', h'\rangle$ between F-algebras is a morphism $f : X \to X'$ between their 'carriers' such that

$$\begin{tikzcd} FX \arrow[r, "Ff"] \arrow[d, "h"'] & FX' \arrow[d, "h'"] \\ X \arrow[r, "f"'] & X' \end{tikzcd}$$

commutes, that is, $f \circ h = h' \circ Ff$. Therefore, the natural numbers can also be understood as the initial algebra of the endofunctor $FX = 1 + X$ on $\underline{\mathrm{Set}}$.

The initial object in the category of algebras of an arbitrary endofunctor Σ, ie the **initial Σ-algebra**, does not always exists, but if it does, then its structure is an isomorphism:

Lemma 2.1. (Lambek's Lemma: initial algebras are isomorphisms) Let $\langle \overline{F}, \psi\rangle$ be the initial algebra of an arbitrary endofunctor Σ. Then the algebra structure $\psi : F\overline{F} \to \overline{F}$ is always an isomorphism

$$\psi : F\overline{F} \cong \overline{F} \qquad \text{(initial } F\text{-algebra)}$$

(To prove this notice that the initial algebra structure ψ is also an F-algebra morphism from $\langle F\overline{F}, F\psi\rangle$ to $\langle \overline{F}, \psi\rangle$.) $\square$

As mentioned in the introduction, initial algebras give a very useful *induction principle*. Indeed, every algebra structure $h : FX \to X$ of an arbitrary endofunctor F with initial algebra $F\overline{F} \cong \overline{F}$ can be inductively extended to a morphism $h^{\#} : \overline{F} \to X$ by taking the unique algebra morphism from the initial algebra to the algebra $\langle X, h\rangle$:

Inductive Extension

$$\begin{tikzcd} F\overline{F} \arrow[r, "Fh^{\#}"] \arrow[d, "\cong"'] & FX \arrow[d, "h"] \\ \overline{F} \arrow[r, dashed, "h^{\#}"'] & X \end{tikzcd}$$

Notice that initiality gives both the *existence* of inductive extensions and their *uniqueness*. The former accounts for the *definitional power* of induction, while the latter gives a *proof principle*, since one can prove two morphisms equal by showing that both fit as inductive extensions of an algebra structure.

2.1. *Coalgebras*

The dual of the notion of initiality is the notion of *finality*: an object X is **final** (or **terminal**) in a category when *from* every object of the category there is a unique morphism *to* X. And the dual of the notion of an algebra of an endofunctor F on a category C is the notion of an F-**coalgebra**, that is, a pair $\langle X, k\rangle$ with X an object and $k : X \to FX$ a morphism of $\underline{C}$. The coalgebras of an endofunctor $F : \underline{C} \to \underline{C}$ are the objects of the **category $\underline{\mathbf{C}}_F$ of F-coalgebras**, having as morphisms $f : \langle X, k\rangle \to \langle X', k'\rangle$ those morphisms $f : X \to X'$ between the carriers such that

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "k"'] & X' \arrow[d, "k'"] \\ FX \arrow[r, "Ff"'] & FX' \end{tikzcd}$$

commutes, ie $Ff \circ k = k' \circ f$.

The final object of a category, as initial objects, coproducts, pullbacks, and all other universals, is unique up to isomorphisms. Therefore, one often speaks of *the* final coalgebra instead of *a* final coalgebra.

2.2. *Operational models as coalgebras of behaviours*

As mentioned in the Introduction, the semantic relevance of coalgebras is that they can be used to represent operational models. Let us write B (behaviour) for endofunctors whose coalgebras have a computational interest.

The main example of such a functorial notion of behaviour is the endofunctor

$$BX = \mathcal{P}(Act \times X)$$

mapping a set X to the set of pairs $<a, x>$, with a in Act and x in X. Its coalgebras $\langle X, k\rangle$ can be seen as **labelled transition systems** (Plotkin, 1981), that is, triples $\langle X, Act, \to\rangle$, where X is a set of 'states', Act is a set of 'labels', and $\to \subseteq X \times Act \times X$ is a 'transition' relation. The bijection between coalgebras $\langle X, k\rangle$ of the above behaviour and labelled transition systems $\langle X, Act, \to\rangle$ is given by the following correspondence.

$$x \xrightarrow{a} x' \iff <a, x'> \in k(x) \tag{1}$$

Labelled transition systems are operational models for non-deterministic (and concurrent) languages with atomic actions. The transition $x \xrightarrow{a} x'$ in the model tells that the state x can perform an action a and reach the state x'. The natural morphisms of transition systems are functions $f : X \to X'$ between their sets of states which preserve the transition relation. That is, if $x \xrightarrow{a} x'$, then $f(x) \xrightarrow{a} f(x')$. Coalgebraically, this accounts for only one half of the equality $Bf \circ k = k' \circ f$ which has to hold for a coalgebra homomorphism, namely the inclusion

$$Bf \circ k \subseteq k' \circ f$$

The reverse inclusion amounts to the condition that if $f(x) \xrightarrow{a} y$ for some y in X', then there exists a state x' in X such that $x \xrightarrow{a} x'$ and $f(x') = y$. Morphisms of transition systems which verify this extra condition are exactly those whose graph is a strong bisimulation. They are also known in the literature as the 'zig-zag morphisms' (van Benthem, 1984) (or as the 'saturating morphisms' (Arnold and Dicky, 1989)), which, as shown in (Joyal et al., 1993), are the '$\underline{\mathrm{P}}$-open morphisms', for a suitable subcategory $\underline{\mathrm{P}}$ of the category of transition systems. (Cf next section.)

There are several interesting variations of the above behaviour obtaining by varying the choice of power-set endofunctor. For instance, by taking the restriction of $\mathcal{P}$ to the *finite* power-set $\mathcal{P}_{\mathrm{fi}}$, one has the behaviour

$$BX = \mathcal{P}_{\mathrm{fi}}(Act \times X)$$

whose coalgebras are the finitely branching transition systems, with states each able to chose among a finite set of transitions. Similarly, the endofunctor $\mathcal{P}_S$ on the category $\underline{\mathrm{SET}}$ of classes (ie large sets) which maps a class to the class of its sub*sets* (thus small sets) gives the behaviour

$$BX = \mathcal{P}_S(Act \times X)$$

whose coalgebras are the 'locally small transition systems', that is, transition systems possibly having a large set (ie a class) of states, each able to choose among a (small) set of transitions.

The above notions of behaviour all account for non-deterministic transition systems. For the simpler case of deterministic systems, where each state can perform at most one transition one can consider the behaviour endofunctor

$$BX = 1 + Act \times X$$

A coalgebra structure $k : X \to 1 + Act \times X$ maps a state x either to the only element '$*$' of the singleton set $1 = \{*\}$ (ie x cannot perform any transition) or to a pair $<a, x'>$ (ie $x \xrightarrow{a} x'$). One can check that the set $Act^\infty = Act^* \cup Act^\omega$ of finite and infinite words over the alphabet Act is the carrier of a final coalgebra for this behaviour.

The above behaviours, whose coalgebras correspond to various forms of labelled transition systems, are suitable for modelling imperative and concurrent languages. Instead, for modelling applicative languages, one usually needs behaviours involving some form of function space functor. An example is the endofunctor

$$BX = 1 + X^Y$$

The 'exponent' X^Y is the set of functions from Y to X. In order to avoid the usual 'mixed variance' problems, Y is here treated as a parameter. By putting $Y = X$ one obtains that the corresponding coalgebras are the *quasi-applicative transition systems* defined in (Abramsky, 1990). The 'exception' 1 in the above behaviour can be used to encode non-termination. For example, for X and Y both equal to the set Λ of closed *λ-terms*, one can define a coalgebra structure

$$ev : \Lambda \to 1 + \Lambda^{\Lambda}$$

by putting, for every λ-term $M \in \Lambda$,

$$ev(M) = P \mapsto N[P/x]$$

if M converges to 'principal weak head normal form' $\lambda x . N$, and

$$ev(M) = *$$

otherwise.

Examples of operational models as coalgebras in categories of partial orders and of metric spaces are given in Sections 6 and 7. Many more functorial notions of behaviour are currently under investigation.

2.3. *Coinduction*

Let F be an endofunctor which has a **final coalgebra** (ie the final object in the corresponding category of coalgebras) and let $\widehat{F}$ denote the carrier of this final coalgebra. The structure φ of a final coalgebra is, l like that of an initial algebra, an isomorphism (Lemma 2.1), because the notion of isomorphism is 'self-dual'. Thus:

Observation 2.1. (Final coalgebras are isomorphisms) Let $\langle \widehat{F}, \varphi \rangle$ be the final coalgebra of an arbitrary endofunctor F. Then the coalgebra structure $\varphi : F\widehat{F} \to \widehat{F}$ is always an isomorphism

$$\varphi : \widehat{F} \cong F\widehat{F} \qquad\qquad \text{(final } F\text{-coalgebra)}$$

□

Notice, that by cardinality reasons, this implies that the (unrestricted) power-set endofunctor $\mathcal{P}$ cannot have a final coalgebra, and the same holds for the behaviour $BX = \mathcal{P}(Act \times X)$. However, final coalgebras for the variations considered above of these endofunctors do exists. As illustrated by Corollary 3.1, the finite power-set endofunctor $\mathcal{P}_{\mathrm{fi}} : \underline{\mathrm{Set}} \to \underline{\mathrm{Set}}$ has a final coalgebra and similar arguments show that also the power-set endofunctor $\mathcal{P}_S : \underline{\mathrm{SET}} \to \underline{\mathrm{SET}}$ on the category of classes, and the behaviour endofunctors $BX = \mathcal{P}_{\mathrm{fi}}(Act \times X)$ and $BX = \mathcal{P}_S(Act \times X)$ have final coalgebras.

Any coalgebra structure $k : X \to FX$ can be 'coinductively' extended to a morphism $k^{@} : X \to \widehat{F}$ by taking the unique coalgebra morphism from the coalgebra $\langle X, k \rangle$ to the final coalgebra:

Coinductive Extension

$$\begin{tikzcd} X \arrow[r, dashed, "k^{@}"] \arrow[d, "k"'] & \widehat{F} \arrow[d, "\cong"] \\ FX \arrow[r, "Fk^{@}"'] & F\widehat{F} \end{tikzcd}$$

For example, by taking for F the behaviour $BX = \mathcal{P}_S(Act \times X)$ and for $k : X \to FX$ the coalgebra structure $[\![\text{-}]\!] : Prog \to B(Prog)$ corresponding to the intended operational model for CCS as mentioned in the introduction, then the coinductive extension $[\![\text{-}]\!]^{@} : Prog \to \widehat{B}$ is a final coalgebra semantics: for every program P of CCS,

$$[\![P]\!]^{@} = \varphi^{-1}\{\, <a, [\![P']\!]^{@}> \mid P \xrightarrow{a} P'\} \tag{2}$$

where φ^{-1} is the inverse of the final coalgebra isomorphism $\varphi : \widehat{B} \cong B\widehat{B}$ given by (the dual of) Lambek's lemma (Observation 2.1). If one postulates the anti-foundation axiom, then the (strict) greatest fixed point $gfp[B] = B(gfp[B])$ of the above B is a final coalgebra, and the equation (2) is strict, ie the isomorphism φ cuts down to an identity:

$$[\![P]\!]^{@} = \{\, <a, [\![P']\!]^{@}> \mid P \xrightarrow{a} P'\}$$

3. Adjunctions subsume Induction and Coinduction

Induction and coinduction can be made parametric in a suitable sense by considering the more general notions of 'free algebras' and 'cofree coalgebras' instead of initial algebras and final coalgebras only. This gives a natural way of understanding initial algebra and final coalgebra constructions as 'canonical' categorical constructions, namely as *adjoint* to suitable elementary functors, which allows one to solve the problem of establishing the existence of initial algebras and final coalgebras by means of general categorical theorems as the "Special Adjoint Functor Theorem". Following (Barr, 1993), this is done here, in particular, for the coalgebras of the (finite) power-set functor. Moreover, the more general induction and coinduction principles available from free algebras and final coalgebras respectively are shown to be founded on the ubiquitous categorical notion of an *adjunction*.

3.1. *Free algebras of terms*

In categories with binary coproducts, the free algebra of an endofunctor F over an object X is the initial algebra of the endofunctor $X + F$, mapping an object Y to the coproduct $X + FY$. Let us illustrate this by considering the free algebras over sets generated by a signature Σ, that is a set of function symbols σ and a corresponding 'arity' function.

The carrier of a free algebra over a set X generated by a signature Σ is the set TX of terms with variables x in X generated by Σ. Categorically, a signature can be seen as an operator

$$X \mapsto \coprod_{\sigma \in \Sigma} X^{\mathrm{ar}(\sigma)}$$

mapping a set X to the coproduct, indexed by the symbols in Σ, of powers $X^{\mathrm{ar}(\sigma)}$ of X. For instance, for the signature $\{0, s\}$ corresponding to the natural numbers, which consists of a constant (ie 0-ary symbol) and a unary symbol, one has $X \mapsto X^0 + X^1 = 1 + X$.

For every signature Σ this operator extends to an endofunctor $\Sigma : \underline{\mathrm{Set}} \to \underline{\mathrm{Set}}$ in the obvious way. The initial Σ-algebra exists and its carrier $\overline{\Sigma}$ is the set $T0$ of closed terms generated by Σ.

More generally, for every set X, the endofunctor

$$(X + \Sigma) : \underline{\mathrm{Set}} \to \underline{\mathrm{Set}}$$

with action $Y \mapsto X + \Sigma Y$, has an initial algebra with as carrier the set TX of terms with variables $x \in X$. Since, by Lambek's lemma, initial algebras are isomorphisms

$$X + \Sigma TX \cong TX$$

the set TX is a coproduct and its algebra structure is the copair of the injections

$$\mathsf{inl}_X : X \to TX \qquad \mathsf{inr}_X : \Sigma TX \to TX$$

The left injection is the usual insertion of variables $x \in X$ into the terms $t \in TX$, which is usually left implicit. Formally, x is simply an element of the set X and it is only after applying inl_X to it that one obtains a variable. This variable-making function is usually written as

$$\eta_X = \mathsf{inl}_X : X \to TX$$

The other injection $\mathsf{inr}_X : \Sigma TX \to TX$ is the operation which permits to construct a new term given any n-ary operator σ and terms $t_1, \ldots, t_n$; also the right injection is usually left implicitly and one writes simply $\sigma(t_1, \ldots, t_n)$ for the resulting term.

As $T0$, also TX, being an initial algebra, comes with an induction principle which, since TX is a coproduct, can be rephrased as follows. For every Σ-algebra structure $h : \Sigma Z \to Z$ and every 'valuation' function $f : X \to Z$ of the variables in X as elements of the algebra $\langle Z, h\rangle$, there exists a unique function $f^\sharp : TX \to Z$ making

$$\begin{tikzcd}
X \arrow[r, "\eta_X = \mathsf{inl}_X"] \arrow[dr, "f"'] & TX \arrow[d, dashed, "f^\sharp"] & \Sigma TX \arrow[l, "\mathsf{inr}_X"'] \arrow[d, "\Sigma f^\sharp"] \\
 & Z & \Sigma Z \arrow[l, "h"]
\end{tikzcd}$$

commute. Omitting the injections,

$$f^\sharp(x) = f(x) \quad \text{and} \quad f^\sharp(\sigma(t_1, \ldots, t_n)) = h(\sigma(f^\sharp(t_1), \ldots, f^\sharp(t_n)))$$

This **inductive extension of** h **along the** valuation **function** f is, formally, the inductive extension $[f, h]^\#$ of the $(X+\Sigma)$-algebra structure on Z given by the copair

$$\begin{tikzcd}
X \arrow[r, tail] \arrow[dr, "f"'] & X+\Sigma Z \arrow[d, dashed, "{[f,h]}"] & \Sigma Z \arrow[l, tail] \arrow[dl, "h"] \\
 & Z &
\end{tikzcd}$$

For instance, this induction principle can be used to show that the operator T inductively extends to a functor $T : \underline{\mathrm{Set}} \to \underline{\mathrm{Set}}$. Indeed, to define its action Tf on a function $f : X \to Y$, take the inductive extension of $\mathsf{inr}_Y : \Sigma TY \to TY$ along the composite $\mathsf{inl}_Y \circ f$:

$$\begin{tikzcd} X \arrow[r, "\eta_X = \mathsf{inl}_X"] \arrow[d, "f"'] & TX \arrow[d, dashed, "Tf = (\eta_Y \circ f)^\sharp"] & \Sigma TX \arrow[l, "\mathsf{inr}_X"'] \arrow[d, "\Sigma Tf"] \\ Y \arrow[r, "\eta_Y = \mathsf{inl}_Y"'] & TY & \Sigma TY \arrow[l, "\mathsf{inr}_Y"] \end{tikzcd} \tag{3}$$

To prove that this definition is functorial, ie $T(\mathsf{id}_X) = \mathsf{id}_{TX}$ and $T(g \circ f) = Tg \circ Tf$, for $g : Y \to Z$, one exploits the uniqueness of inductive extensions: the function id_{TX} fits as $(\eta_X \circ \mathsf{id}_X)^\sharp = (\eta_X)^\sharp$ and $Tg \circ Tf$ fits as $(\eta_Z \circ g \circ f)^\sharp$.

Notice that a function $f : X \to Y$ can be seen as a '*renaming*' of variables and then the function $Tf : TX \to TY$ is the inductive extension of such a renaming from simple variables to complex terms with variables.

3.2. *Adjunctions*

The existence of an inductive extension $f^\sharp : TX \to Z$ of every algebra structure $h : \Sigma Z \to Z$ along any function $f : X \to Z$ shows that there is a bijection between Σ-algebra morphisms of type $\langle TX, \mathsf{inr}_X\rangle \to \langle Z, h\rangle$ and functions of type $X \to Z$:

$$\begin{array}{ccc} f^\sharp & \langle TX, \mathsf{inr}_X\rangle \longrightarrow \langle Z, h\rangle & \text{in } \underline{\mathrm{Set}}^F \\ \updownarrow & \overline{\overline{\qquad\qquad\qquad\qquad}} & \\ f & X \longrightarrow Z & \text{in } \underline{\mathrm{Set}} \end{array}$$

Moreover, this bijection is *natural* in X and $\langle Z, h\rangle$. This tells that the two (operations extending to) functors $X \mapsto \langle TX, \mathsf{inr}_X\rangle$ and $\langle Z, h\rangle \mapsto Z$ are *adjoint.* (See, eg, (Mac Lane, 1971, page 78) for the definition of adjoint functors.) The former is the functor from $\underline{\mathrm{Set}}$ to $\underline{\mathrm{Set}}^\Sigma$ mapping a set to the free Σ-algebra over it; the latter is the **forgetful functor**

$$U^\Sigma : \underline{\mathrm{Set}}^\Sigma \to \underline{\mathrm{Set}}$$

mapping algebras to their carriers. The existence of free algebras is then equivalent to the existence of a **left adjoint** for U^Σ. Correspondingly, the above inductive extension $f^\sharp$ is called the **left adjunct** of f.

For every category of F-algebras such a forgetful functor

$$U^F : \underline{\mathrm{C}}^F \to \underline{\mathrm{C}}$$

exists. If it has a left adjoint

$$F^\star : \underline{\mathrm{C}} \to \underline{\mathrm{C}}^F$$

then, for every object X in $\underline{\mathrm{C}}$, $F^\star X$ is the *free F-algebra* over X. (If $F = \Sigma$, then

$F^\star X = \langle TX, \mathsf{inr}_X \rangle$.) For every morphism $f : X \to Z = U^F\langle Z, h\rangle$ in $\underline{C}$, the left adjunct $f^\sharp : F^\star X \to \langle Z, h\rangle$ is the inductive extension of h along f.

Dually, for every endofunctor $F : \underline{C} \to \underline{C}$, if the forgetful functor

$$U_F : \underline{C}_F \to \underline{C}$$

mapping a coalgebra to its carrier has a **right adjoint**

$$F_\star : \underline{C} \to \underline{C}_F$$

in the sense that there is a natural bijection

$$\begin{array}{ccc} f & U_F\langle Z, k\rangle = Z \longrightarrow X & \text{in } \underline{C} \\ \Updownarrow & \overline{\overline{\qquad\qquad\qquad\qquad}} & \\ f^\flat & \langle Z, k\rangle \longrightarrow F_\star X & \text{in } \underline{C}_F \end{array}$$

then $F_\star X$ is the **cofree coalgebra** of F over X. The **right adjunct** $f^\flat$ of a morphism f in $\underline{C}$ as given by the above bijection is the **coinductive extension** of the coalgebra structure $k : Z \to FZ$ **along the morphism** $f : Z \to X$.

One can check that, in categories with binary products, a cofree coalgebra of an endofunctor F over an object X is a final $(X \times F)$-coalgebra. Therefore, letting $|F_\star X|$ denote the carrier $U_F(F_\star(X))$ of the cofree coalgebra $F_\star X$,

$$|F_\star X| \cong X \times F\,|F_\star X| \tag{4}$$

This allows one to write the coinductive extension $f^\flat : \langle Z, k\rangle \to |F_\star X|$ of a coalgebra structure $k : Z \to FZ$ along a morphism $f : Z \to X$ as follows:

$$\begin{tikzcd} & Z \arrow[r, "k"] \arrow[dl, "f"'] \arrow[d, dashed, "f^\flat"] & FZ \arrow[d, "Ff^\flat"] \\ X & {|F_\star X|} \arrow[l, "\mathsf{fst}_X"] \arrow[r, "\mathsf{snd}_X"'] & F\,|F_\star X| \end{tikzcd}$$

where for $\mathsf{fst}_X : |F_\star X| \cong X \times F\,|F_\star X| \to X$ and $\mathsf{snd}_X : |F_\star X| \cong X \times F\,|F_\star X| \to F\,|F_\star X|$ are, respectively the first and second projection of the product (4). An example of such a coinductive extension is given in §4.4.

One important property of right (left) adjoints is that they preserve limits (colimits). (See, eg, (Mac Lane, 1971, Theorem V.5.1).) Therefore, if $\underline{C}$ has a final (initial) object 1 (0), then $F_\star 1$ $(F^\star 0)$ is a final coalgebra (initial algebra). In particular, $\underline{\mathrm{Set}}$ has a final object, namely the singleton set, therefore the following proposition implies that the finite power-set endofunctor $\mathcal{P}_{\mathrm{fi}} : \underline{\mathrm{Set}} \to \underline{\mathrm{Set}}$ on $\underline{\mathrm{Set}}$ has a final coalgebra.

Theorem 3.1. The forgetful functor from the category of $\mathcal{P}_{\mathrm{fi}}$-coalgebras to $\underline{\mathrm{Set}}$ has a right adjoint.

Proof. It is an instance of the proof of (Barr, 1993, Theorem 1.2), which is stated in terms of 'accessible' endofunctors. The core of the proof is the fact that the $\mathcal{P}_{\text{fi}}$-coalgebras with carrier of cardinality not greater than ω form a small 'generating set' $\mathcal{G}$, that is, every $\mathcal{P}_{\text{fi}}$-coalgebra can be obtained by quotienting the disjoint union of a suitable set of $\mathcal{P}_{\text{fi}}$-coalgebras in $\mathcal{G}$. The existence of such a small generating set allows one to apply the "Special Adjoint Functor Theorem". (For more details see (Turi, 1996, §13).) □

Corollary 3.1. The finite power-set endofunctor has a final coalgebra.

4. Coalgebraic Bisimulations

There are several notions of *behavioural equivalence* for a transition system; the most general one corresponds to a relation on its states called *(strong) bisimulation*. The final coalgebra of the behaviour corresponding to transition systems 'classifies' bisimilar states in the sense that two states are bisimilar if and only if they have the same final coalgebra semantics, ie the same abstract global behaviour. In other words, coinduction can be 'pulled back' to bisimulation. As a corollary, the final coalgebra is 'internally fully abstract'.

In this section, all this is generalized to arbitrary endofunctors preserving 'weak pullbacks'.

Given a labelled transition system $\langle X, Act, \to \rangle$, two states $x, y \in X$ are **(strongly) bisimilar** if there exists a relation R on the sets of states X such that xRy (ie $<x,y> \in R$) and

1 if $x \xrightarrow{a} x'$ then $y \xrightarrow{a} y'$ for some y' such that $x' R\, y'$

2 and, conversely, if $y \xrightarrow{a} y'$ then $x \xrightarrow{a} x'$ for some x' such that $x' R\, y'$

More generally, one can consider two transition systems with the same set Act of labels and relations R between their carriers X and Y. Then, a relation R is a **(strong) bisimulation** between the two transition systems if, for every $x \in X$ and $y \in Y$ such that xRy, the above clauses 1 and 2 hold.

In terms of the coalgebraic representation (1) of transition systems, a relation R between the carriers X and Y of two coalgebras $\langle X, k \rangle$ and $\langle Y, \ell \rangle$ is a (strong) bisimulation between the two coalgebras when, for all x in X and y in Y such that xRy,

— if $<a, x'> \in k(x)$ then $<a, y'> \in \ell(y)$ for some y' such that $x' R\, y'$

— and, conversely, if $<a, y'> \in \ell(y)$ then $<a, x'> \in k(x)$ for some x' such that $x' R\, y'$

Notice that bisimulations are themselves coalgebras. Indeed, from the above conditions, one can define a coalgebra structure

$$\widetilde{R} : R \to \mathcal{P}(Act \times R)$$

on the relation R by putting, using the infix notation,

$$<a, <x', y'>> \in x\widetilde{R}y \iff <a, x'> \in k(x),\ <a, y'> \in \ell(y), \text{ and } x' R\, y'$$

In the sequel, the above notion of bisimulation is called **ordinary bisimulation**, in order to distinguish it from the following more general notion of 'coalgebraic bisimulation'.

Consider the two '**legs**' $r_1 : R \to X$ and $r_2 : R \to Y$ obtained by composing the insertion $R \hookrightarrow X \times Y$ of the relation R into the cartesian product $X \times Y$ with the first and second projection, respectively. If the relation R is an (ordinary) bisimulation, then its legs r_1 and r_2 lift to coalgebra morphisms; that is, the two squares in

$$\begin{tikzcd} & R \arrow[dl, "r_1"'] \arrow[d, "\widetilde{R}"] \arrow[dr, "r_2"] & \\ X \arrow[d, "k"'] & BR \arrow[dl, "Br_1"'] \arrow[dr, "Br_2"] & Y \arrow[d, "\ell"] \\ BX & & BY \end{tikzcd} \tag{5}$$

commute. The converse is also true; namely, if a relation lifts to a coalgebra of the above behaviour endofunctor $BX = \mathcal{P}(Act \times X)$ in a way that its legs also lift to corresponding coalgebra morphisms as in the above diagram, then this relation is a bisimulation. Indeed, the first condition is obvious, while the second and the third follow from the commutativity of the left and the right diagram, respectively. Notice that there might be more structures $\widetilde{R}$ making the above diagram commute, corresponding to the several ways in which, in general, a relation can be a bisimulation.

In general, for an arbitrary endofunctor B on a category $\underline{C}$, a **(coalgebraic) bisimulation** (or a B**-bisimulation**) between two B-coalgebras $\langle X, k\rangle$ and $\langle Y, \ell\rangle$ is a **span**

$$\begin{tikzcd} & R \arrow[dl, "r_1"'] \arrow[dr, "r_2"] & \\ X & & Y \end{tikzcd}$$

of morphisms in $\underline{C}$ such that there exists a B-coalgebra structure $\widetilde{R} : R \to BR$ which makes the diagram (5) commute. Notice that the stress is put on the fact that the legs of the span R *lift* to coalgebra morphisms, rather than on the actual (possibly not unique) coalgebraic structure of R. Therefore, let us *forget* about the coalgebraic structure of R and write

$$\begin{tikzcd} & R \arrow[dl, "r_1"'] \arrow[dr, "r_2"] \arrow[d, phantom, "\sim" near start] & \\ \langle X, k\rangle & & \langle Y, \ell\rangle \end{tikzcd}$$

to express that $\langle R, r_1, r_2\rangle$ is a span between the carriers X and Y which lifts to a bisimulation between the coalgebras $\langle X, k\rangle$ and $\langle Y, \ell\rangle$.

In the sequel, with a slight abuse of notation, we often leave the legs r_1 and r_2 implicit and simply write R for the whole span $\langle R, r_1, r_2\rangle$ Also, whenever possible, we use the infix notation xRy to express that there exists an element z of R such that $r_1(z) = x$ and $r_2(z) = y$.

Proposition 4.1. (Coinductive extensions identify bisimilar elements) For every bisimulation $\langle R, r_1, r_2\rangle$ between two coalgebras $\langle X, k\rangle$ and $\langle Y, \ell\rangle$ the following diagram commutes.

$$
\begin{tikzcd}
 & \underset{\sim}{R} \arrow[dl, "r_1"'] \arrow[dr, "r_2"] & \\
\langle X, k\rangle \arrow[dr, "k^{@}"'] & & \langle Y, \ell\rangle \arrow[dl, "\ell^{@}"] \\
 & \langle \widehat{B}, \varphi\rangle &
\end{tikzcd}
$$

This is a trivial consequence of the fact that both composites in the diagram are coalgebra morphisms to the final coalgebra, hence they must be the same. □

Semantically, this implies that if two programs are bisimilar, then they have the same final coalgebra semantics.

Corollary 4.1. (Strong Extensionality) Every bisimulation on the final coalgebra has equal legs, that is, the following diagram commutes.

$$
\begin{tikzcd}
 & \underset{\sim}{R} \arrow[dl, "r_1"'] \arrow[dr, "r_2"] & \\
\langle \widehat{B}, \varphi\rangle \arrow[dr, equal] & & \langle \widehat{B}, \varphi\rangle \arrow[dl, equal] \\
 & \langle \widehat{B}, \varphi\rangle &
\end{tikzcd}
$$

□

In other words, in a final coalgebra one cannot distinguish between bisimilar elements which, semantically, amounts to the fact that final coalgebras of behaviours are **internally fully abstract**.

4.1. *Pullback bisimulations*

A canonical way of defining spans is by 'pullbacks'. Given two morphisms $f : X \to Z$ and $g : Y \to Z$ with a common codomain Z, their **pullback** is the universal among the spans between X and Y that form commuting squares with f and g; that is, a span $\langle P, p : P \to X, q : P \to Y\rangle$ such that $f \circ p = g \circ q$ and for every span $\langle P', p' : P' \to X, q' : P' \to Y\rangle$ such that $f \circ p' = g \circ q'$ there exists a unique morphism $u : P' \to P$ such that

$p' = p \circ u$ and $q' = q \circ u$. Diagrammatically:

$$\begin{tikzcd}[column sep=small]
& P' \arrow[d, dashed, "u"] \arrow[ddl, bend right, "p'"'] \arrow[ddr, bend left, "q'"] & \\
& P \arrow[dl, "p"] \arrow[dr, "q"'] \arrow[dd, phantom, very near start, "\lrcorner"] & \\
X \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \\
& Z &
\end{tikzcd} \tag{6}$$

(Notice the notation for the pullback square.) In $\underline{\mathrm{Set}}$, the pullback of two functions f and g is the relation $R = \{<x, y> \mid fx = gy\}$ with the evident projections as legs.

In a category $\underline{\mathrm{C}}$ with pullbacks, one can define the **internal equality** $\mathrm{EQ}(X)$ of an object X as the 'kernel pair' of the identity on X, that is, as the following pullback.

$$\begin{tikzcd}[column sep=small]
& \mathrm{EQ}(X) \arrow[dl, "e_1"'] \arrow[dr, "e_2"] \arrow[dd, phantom, very near start, "\lrcorner"] & \\
X \arrow[dr, equal] & & X \arrow[dl, equal] \\
& X &
\end{tikzcd}$$

Proposition 4.2. For every coalgebra, the internal equality lifts to a coalgebraic bisimulation.

Proof. Firstly, notice that the two legs e_1 and e_2 of the equality are the same. Next, consider the 'diagonal' $\mathsf{d}_X : X \to \mathrm{EQ}(X)$

$$\begin{tikzcd}[column sep=small]
& X \arrow[d, dashed, "\mathsf{d}_X"] \arrow[ddl, bend right, "\mathsf{id}_X"'] \arrow[ddr, bend left, "\mathsf{id}_X"] & \\
& \mathrm{EQ}(X) \arrow[dl, "e_1"] \arrow[dr, "e_2"'] \arrow[dd, phantom, very near start, "\lrcorner"] & \\
X \arrow[dr, equal] & & X \arrow[dl, equal] \\
& X &
\end{tikzcd}$$

given by the universal property of $\mathrm{EQ}(X)$. (In $\underline{\mathrm{Set}}$, the value of the diagonal d_X at an element x of X is the pair $<x, x>$.) For any endofunctor B and any B-coalgebra $\langle X, k \rangle$

since the composite $e_i \circ \mathsf{d}_X$ is the identity on X, the diagram

$$\begin{tikzcd}
\mathrm{EQ}(X) \arrow[r, "e_i"] \arrow[d, "e_i"'] & X \arrow[r, "k"] & BX \arrow[r, "B\mathsf{d}_X"] & B(\mathrm{EQ}(X)) \arrow[d, "Be_i"] \\
X \arrow[rrr, "k"'] & & & BX
\end{tikzcd}$$

commutes; hence, the composite $B\mathsf{d}_X \circ k \circ e_i$ lifts the equality $\mathrm{EQ}(X)$ to a bisimulation on the coalgebra $\langle X, k\rangle$:

$$\begin{tikzcd}
& \mathrm{EQ}(X) \arrow[dl, "e_1"'] \arrow[dr, "e_2"] \arrow[d, phantom, "\sim"] & \\
\langle X, k\rangle & & \langle X, k\rangle
\end{tikzcd}$$

□

Consider the category $\underline{\mathbf{Bisim}}(B)$ having as objects spans lifting to bisimulations of an endofunctor B and as morphisms triples of morphisms $< r, f, g >$ making everything in sight in

$$\begin{tikzcd}
& R \arrow[dl, "r_1"'] \arrow[dr, "r_2"] \arrow[d, phantom, "\sim"] \arrow[drrr, "r"] & & & & \\
\langle X, k\rangle \arrow[drrr, "f"'] & & \langle Y, \ell\rangle \arrow[drrr, "g"] & & R' \arrow[dl, "r'_1"'] \arrow[dr, "r'_2"] \arrow[d, phantom, "\sim"] & \\
& & & \langle X', k'\rangle & & \langle Y', \ell'\rangle
\end{tikzcd} \tag{7}$$

commute – where f and g are morphisms in $\underline{\mathrm{C}}_B$, while r is a morphism in $\underline{\mathrm{C}}$. Then the equality $\mathrm{EQ}(\widehat{B})$ on (the carrier of) a final coalgebra is a final object in this category. This is an immediate consequence of the fact that $\mathrm{EQ}(\widehat{B})$ is a pullback (in $\underline{\mathrm{C}}$):

$$\begin{tikzcd}
& R \arrow[dl, "r_1"'] \arrow[dr, "r_2"] \arrow[d, phantom, "\sim"] \arrow[drrr, dashed] & & & & \\
\langle X, k\rangle \arrow[drrr, "k^@"'] & & \langle Y, \ell\rangle \arrow[drrr, "\ell^@"] & & \mathrm{EQ}(\widehat{B}) \arrow[dl, "e_1"'] \arrow[dr, "e_2"] \arrow[d, phantom, "\sim"] & \\
& & & \langle \widehat{B}, \varphi\rangle & & \langle \widehat{B}, \varphi\rangle
\end{tikzcd} \tag{8}$$

That is, from any span R lifting to a bisimulation there is a mediating morphism to the equality $\mathrm{EQ}(\widehat{B})$ on a final coalgebra because the two legs of R can be coinductively prolonged to form a suitable cone on (the carrier of) a final coalgebra. Therefore:

Proposition 4.3. The internal equality of the final coalgebra lifts to the final bisimulation. □

This gives an alternative way of understanding the fact that coinductive extensions identify bisimilar elements.

4.2. *Coinduction pulled back to bisimulation*

If the pullback of the coinductive extensions of two coalgebras lifts to a bisimulation, as, eg, when the functor B under consideration 'preserves weak pullbacks', then this pullback is the final bisimulation between the two coalgebras. In other words, coinduction can be 'pulled back' to bisimulation. Together with the above property that coinductive extensions identify bisimilar elements, this gives that two programs have the same final coalgebra semantics if and only if they are bisimilar.

Let B be an endofunctor on a category $\underline{C}$ with pullbacks. Recall that pullbacks, like all universals, are determined by two conditions: *uniqueness* and *existence*. When only the existence part is known to hold one speaks of a **weak pullback** (and of a **weak universal** in general). Now, not all pullbacks lift to B-bisimulations, but a sufficient condition is that the endofunctor B **preserves weak pullbacks**, that is, the image under B of a weak pullback is still a weak pullback.

Proposition 4.4. If an endofunctor B preserves weak pullbacks then pullbacks lift to B-bisimulations. (Cf (Aczel and Mendler, 1989, Proposition 6.2).)

Proof. If the image under B of a weak pullback is still a weak pullback, then every pullback in $\underline{C}$ of morphisms which are coalgebra homomorphisms lifts to a B-bisimulation. Indeed, since pullbacks are also weak pullbacks, for all $f : \langle X, k\rangle \to \langle Z, j\rangle$ and $g : \langle Y, \ell\rangle \to \langle Z, j\rangle$ in $\underline{C}_B$, the *existence* of a (possibly not unique) suitable coalgebra structure $\widetilde{R} : R \to BR$ for the pullback R of f and g in $\underline{C}$ is ensured by the weakly universal property of the weak pullback BR:

$$
\begin{tikzcd}
 & R \arrow[rrr, dotted, "\widetilde{R}"] \arrow[dl, "r_1"'] \arrow[dr, "r_2"] & & & BR \arrow[dl, "Br_1"'] \arrow[dr, "Br_2"] & \\
X \arrow[rrr, bend left, "k"] \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \arrow[rrr, bend right, "\ell"'] & BX \arrow[dr, "Bf"'] & & BY \arrow[dl, "Bg"] \\
 & Z \arrow[rrr, "j"'] & & & BZ &
\end{tikzcd}
$$

(The coalgebra structures k and ℓ turn the legs of R into a cone over the diagram for which BR is a weak pullback.) $\square$

Let us check that the behaviour endofunctor $BX = \mathcal{P}(Act \times X)$ preserves weak pullbacks and hence, by the above argument, pullbacks lift to (ordinary) bisimulations.

The problem of showing that the functor B preserves weak pullbacks can be reduced to the problem of showing that B maps (ordinary) pullbacks to weak pullbacks. Indeed, the following holds.

Observation 4.1. In $\underline{\text{Set}}$, weak pullbacks embed pullbacks. That is, the diagram

$$
\begin{tikzcd}
W \arrow[r, "w_2"] \arrow[d, "w_1"'] & Y \arrow[d, "g"] \\
X \arrow[r, "f"'] & Z
\end{tikzcd}
$$

is a weak pullback diagram if and only if there exists an injection $m : R \rightarrowtail W$ of the pullback $R = \{<x, y>|\ fx = gy\}$ of f and g into W such that

$$\begin{tikzcd}
R \arrow[dr, tail, "m"] \arrow[drr, bend left, "r_2"] \arrow[ddr, bend right, "r_1"'] & & \\
& W \arrow[r, "w_2"] \arrow[d, "w_1"] & Y \arrow[d, "g"] \\
& X \arrow[r, "f"'] & Z
\end{tikzcd}$$

commutes. $\square$

Proposition 4.5. Pullbacks lift to ordinary bisimulations. From the above observation, if

$$BR = \{<a, x, y>|\ a \in \mathsf{Act},\ fx = gy\}$$

is a weak pullback for $Bf : BX \to BZ$ and $Bg : BY \to BZ$, the set BW inherits the weak universality of BR by means of the mediating morphism $Bm : BR \to BW$.
In turn, in order to prove that BR is a weak pullback for Bf and Bg it suffices to prove that the (ordinary) pullback R' of Bf and Bg factorizes through it in the sense that there exists a function $h : R' \to BR$ such that $r'_i = Br_i \circ h$:

$$\begin{tikzcd}
R' \arrow[dr, dotted, "h"] \arrow[drr, bend left, "r'_2"] \arrow[ddr, bend right, "r'_1"'] & & \\
& BR \arrow[r, "Br_2"] \arrow[d, "Br_1"] & BY \arrow[d, "Bg"] \\
& BX \arrow[r, "Bf"'] & BZ
\end{tikzcd}$$

Indeed, then every other cone (f', g') over the co-span $\langle Bf, Bg\rangle$ factorizes through the pullback as follows.

$$\begin{tikzcd}
\cdot \arrow[dr, dashed] \arrow[ddrrr, bend left, "g'"] \arrow[dddrr, bend right, "f'"'] & & & \\
& R' \arrow[dr, dotted, "h"] \arrow[drr, bend left, "r'_2"] \arrow[ddr, bend right, "r'_1"'] & & \\
& & BR \arrow[r, "Br_2"] \arrow[d, "Br_1"] & BY \arrow[d, "Bg"] \\
& & BX \arrow[r, "Bf"'] & BZ
\end{tikzcd}$$

Let us now try and define such a function $h : R' \to BR$ from the pullback R' of Bf and Bg to the image under B of the pullback R of f and g. By definition of pullbacks in Set, the set R' is consist of those pairs

$$<\{<a_i, x_i>\}_{i \in I}, \{<a_j, y_j>\}_{j \in J}>$$

such that the index sets I and J are finite and

$$Bf\{<a_i, x_i>\}_{i \in I} = Bg\{<a_j, y_j>\}_{j \in J}$$

The latter holds if and only if for every $i \in I$ there exists a $j \in J$ such that

$$<a_i, fx_i> = <a_j, gy_j> \qquad \text{(ie } a_i = a_j,\ fx_i = gy_j\text{)}$$

and, conversely, for every $j \in J$ there exists an $i \in I$ such that $<a_i, fx_i>=<a_j, gy_j>$. But then one can define $h : R' \to BR$ as mapping every pair

$$\{<a_i, x_i>\}_{i\in I}\ R'\ \{<a_j, y_j>\}_{j\in J}$$

to the set

$$\{<a_i, x_i, y_j> \mid a_i = a_j,\ fx_i = gy_j\} \in BR$$

This gives the desired factorization. Notice that the mediating function h is not unique.
□

We have seen the category $\underline{\text{Bisim}}(B)$ of all B-bisimulations. For every two B-coalgebras $\langle X, k\rangle$ and $\langle Y, \ell\rangle$, one can consider the evident subcategory $\underline{\textbf{Bisim}}(\langle X, k\rangle, \langle Y, \ell\rangle)$ of B-bisimulations between $\langle X, k\rangle$ and $\langle Y, \ell\rangle$.

Proposition 4.6. Let B be an endofunctor on a category $\underline{\text{C}}$ with pullbacks and let the final B-coalgebra exist. If pullbacks lift to B-bisimulations, then the pullback (in $\underline{\text{C}}$) of the coinductive extensions of two B-coalgebras $\langle X, k\rangle$ and $\langle Y, \ell\rangle$ lifts to the final object in $\underline{\text{Bisim}}(\langle X, k\rangle, \langle Y, \ell\rangle)$.

$$\begin{tikzcd}
& \cdot \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\sim" very near start] & \\
\langle X, k\rangle \arrow[dr, "k^{@}"'] & & \langle Y, \ell\rangle \arrow[dl, "\ell^{@}"] \\
& \langle \widehat{B}, \varphi\rangle &
\end{tikzcd}$$

□

Write $\overset{k,\ell}{\sim}$ for the span obtained above by 'pulling back' the coinductive extensions of the coalgebra structures k and ℓ. Then, in $\underline{\text{Set}}$, if the span $\overset{k,\ell}{\sim}$ lifts to a bisimulation,

$$x \overset{k,\ell}{\sim} y \iff k^{@}(x) = \ell^{@}(y)$$

for any two elements $x \in X$ and $y \in Y$. (The implication from left to right follows the property that coinductive extensions always identify bisimilar elements.) Semantically:

Theorem 4.1. (Full Abstraction) Given a coalgebraic operational model $[\![\text{-}]\!] : \mathit{Prog} \to B(\mathit{Prog})$ of a language with programs $P \in \mathit{Prog}$ and behaviour B such that pullbacks lift to B-bisimulations (as, eg, in Proposition 4.4), two programs $P, P' \in \mathit{Prog}$ are B-bisimilar if and only if they have the same final coalgebra semantics. That is, for behaviours on 'concrete' categories like $\underline{\text{Set}}$,

$$P \overset{[\![\text{-}]\!]}{\sim} P' \iff [\![P]\!]^{@} = [\![P']\!]^{@}$$

□

4.3. *From weakly final to final coalgebras*

Let us look at a concrete description in terms of trees and bisimulation of the final coalgebra for the finite power-set endofunctor $\mathcal{P}_{\mathrm{fi}}$ on $\underline{\mathrm{Set}}$. Similar characterization hold also for the power-set endofunctor $\mathcal{P}_S$ on $\underline{\mathrm{SET}}$ and for the behaviours $BX = \mathcal{P}_{\mathrm{fi}}(Act \times X)$ and $BX = \mathcal{P}_S(Act \times X)$.

Recall that the coalgebras of the finite power-set endofunctor $\mathcal{P}_{\mathrm{fi}}$ are in a one-to-one correspondence with the finitely branching, directed graphs:

$$x \longrightarrow x' \iff x' \in k(x)$$

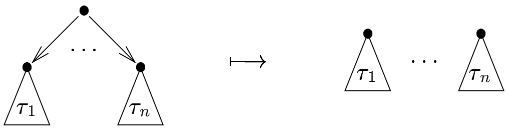
Next, notice that the value of the coinductive extension of a coalgebra structure $k : X \to \mathcal{P}_{\mathrm{fi}}(X)$ at an element x of X is, omitting the final coalgebra isomorphism $\varphi : \widehat{\mathcal{P}_{\mathrm{fi}}} \cong \mathcal{P}_{\mathrm{fi}}\widehat{\mathcal{P}_{\mathrm{fi}}}$, is given by the equation

$$k^{@}(x) = \{k^{@}(x_i) \mid x_i \in k(x)\}$$

which can be seen as the recursive definition of a tree:

$$k^{@}(x) = \begin{tikzcd}[column sep=small] & \bullet \arrow[dl] \arrow[dr] & \\ k^{@}(x_1) & \cdots & k^{@}(x_n) \end{tikzcd} \qquad (\text{for } k(x) = \{x_1, \ldots, x_n\})$$

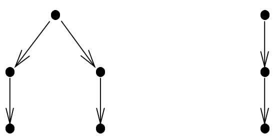
This is a rooted tree, finitely branching, and possibly of infinite depth. Neither nodes nor arcs are labelled. The set $\mathcal{T}$ of these rooted finitely branching trees can be seen as (the carrier of) a coalgebra of the finite power-set functor: every tree $\tau \in \mathcal{T}$ is mapped to the (finite) set $\{\tau_1, \ldots, \tau_n\}$ of children of its root:



This coalgebra is not a final but a *weakly final* coalgebra, that is, it is a coalgebra which ensures the existence but not the uniqueness of coinductive extensions. For example, the coalgebra structure $k : X = \{x, x_1, x_2, x_1', x_2'\} \to \mathcal{P}_{\mathrm{fi}}(X)$

$$k(x) = \{x_1, x_2\} \quad k(x_1) = \{x_1'\} \quad k(x_2) = \{x_2'\} \quad k(x_1') = 0 = k(x_2')$$

can be extended to both the following trees.



Proposition 4.7. The final coalgebra of the finite power-set functor is the set of rooted finitely branching trees quotiented by the corresponding (final) coalgebraic bisimulation. □

In order to prove the above proposition, consider the quotient of a $\mathcal{P}_{\mathrm{fi}}$-coalgebra $\langle X, k\rangle$ modulo its final bisimulation R_k (which, by the above considerations, exists). Categorically, this amounts to taking the *coequalizer* $q : X \to X/R_k$ of the two legs $r_1, r_2 : R_k \to X$ of the bisimulation R_k:

$$\begin{tikzcd} R_k \arrow[r, shift left, "r_1"] \arrow[r, shift right, "r_2"'] \arrow[d, dotted] & X \arrow[r, "q"] \arrow[d, "k"] & X/R_k \arrow[d, dashed] \\ \mathcal{P}_{\mathrm{fi}}(R_k) \arrow[r, shift left, "\mathcal{P}_{\mathrm{fi}}(r_1)"] \arrow[r, shift right, "\mathcal{P}_{\mathrm{fi}}(r_2)"'] & \mathcal{P}_{\mathrm{fi}}X \arrow[r, "\mathcal{P}_{\mathrm{fi}}(q)"'] & \mathcal{P}_{\mathrm{fi}}(X/R_k) \end{tikzcd}$$

Notice this lifts to a coequalizer in the category of coalgebras. The coalgebra structure for X/R_k is given by the universal property of the coequalizer. Indeed, since the legs of the bisimulation R_k are coalgebra morphisms, the composite function $\mathcal{P}_{\mathrm{fi}}(q) \circ k : X \to \mathcal{P}_{\mathrm{fi}}(X/R_k)$ equates the two legs of R_k. The corresponding unique mediating function from X/R_k to $\mathcal{P}_{\mathrm{fi}}(X/R_k)$ is the desired structure. Write $\langle X, k\rangle/R_k$ for this quotient coalgebra.

Lemma 4.1. From every coalgebra there is *at most one morphism* to the quotient coalgebra $\langle X, k\rangle/R_k$.

Proof. Consider two coalgebra morphisms $f, g : \langle Y, \ell\rangle \to \langle X, k\rangle/R_k$. Since, as shown in the previous section, the internal equality always lifts to a coalgebraic bisimulation

$$\begin{tikzcd} & \mathrm{EQ}(Y) \arrow[dl, "e_1"'] \arrow[dr, "e_2"] \arrow[d, phantom, "\sim"] & \\ \langle Y, \ell\rangle & & \langle Y, \ell\rangle \end{tikzcd}$$

one has that the equality on Y with as legs the composites $f \circ e_1, g \circ e_2 : \mathrm{EQ}(Y) \to X/R_k$ lifts to a bisimulation on the quotient coalgebra $\langle X, k\rangle/R_k$:

$$\begin{tikzcd} & & \mathrm{EQ}(Y) \arrow[dl, "e_1"'] \arrow[dr, "e_2"] \arrow[d, phantom, "\sim"] & & \\ & \langle Y, \ell\rangle \arrow[dl, "f"'] & & \langle Y, \ell\rangle \arrow[dr, "g"] & \\ \langle X, k\rangle/R_k & & & & \langle X, k\rangle/R_k \end{tikzcd}$$

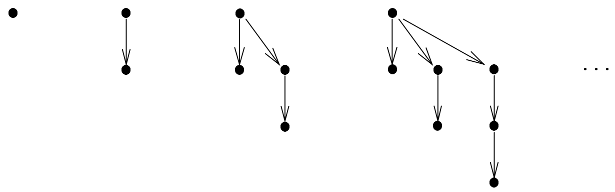
Therefore, for every $y \in Y$, $f(y)$ is bisimilar to $g(y)$. Since, by construction, the quotient $\langle X, k\rangle/R_k$ is *strongly extensional*, that is, bisimulation is the equality, one has that $f(y)$ is equal to $g(y)$ for every $y \in Y$, hence $f = g$ and the lemma is proved. (Cf (Aczel, 1988, Theorem 2.19).) $\square$

Therefore, the quotient modulo bisimulation of a weakly final $\mathcal{P}_{\mathrm{fi}}$-coalgebra is necessarily final: the existence of a morphism from every coalgebra is guaranteed by being the quotient of a weakly final coalgebra, the uniqueness is guaranteed by the above property of quotients modulo bisimulation. In particular, the weakly final coalgebra of rooted finitely branching trees can be thus quotiented by bisimulation to yield the final coalgebra of the finite power-set functor. This concludes the proof of the above proposition.

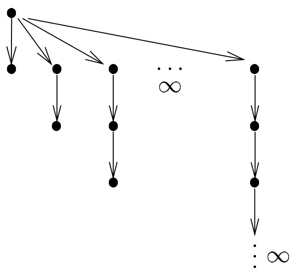
Notice that the finite power-set functor is *not* $\boldsymbol{\omega}^{\mathsf{op}}$-continuous, that is, the limit of the following chain is not a fixed point for the finite power-set functor $\mathcal{P}_{\mathrm{fi}}$.

$$1 \xleftarrow{\;1\;} \mathcal{P}_{\mathrm{fi}}1 \xleftarrow{\;\mathcal{P}_{\mathrm{fi}}1\;} \mathcal{P}_{\mathrm{fi}}{}^2 1 \xleftarrow{\;\mathcal{P}_{\mathrm{fi}}{}^2 1\;} \cdots$$

Indeed: Each object $\mathcal{P}_{\mathrm{fi}}{}^n 1$ of the chain is the set of finitely branching trees with depth at most n, quotiented by bisimulation. Correspondingly, the following sequence of trees belong to the above chain.



The problem is then that the limit has to contain the following tree with *infinitely* many branches,



while the final coalgebra, as shown above, contains only *finitely* branching trees.

The above lemma can be used to prove the following generalization of Proposition 4.7.

Proposition 4.8. The quotient modulo B-bisimulation of a weakly final B-coalgebra, if it exists, is a final B-coalgebra. □

4.4. *Bisimulations along morphisms*

The fact that coinductive extensions can be pulled back to bisimulations can be generalized to coinductive extensions *along morphisms*. This leads to a new, more general notion of ordinary bisimulation in which not only the actions but also some (properties of the) states can be observed.

Let $B : \underline{\mathrm{C}} \to \underline{\mathrm{C}}$ be an endofunctor such that the forgetful functor $U_B : \underline{\mathrm{C}}_B \to \underline{\mathrm{C}}$ has a right adjoint. Recall that final coalgebras $\widehat{B} \cong B\widehat{B}$ are a special case of cofree coalgebras $B_\star X$, namely $B_\star 1$, and that, correspondingly, the coinduction principle of final coalgebras generalizes to the arbitrary cofree coalgebras: for every coalgebra structure $k : Z \to BZ$

and morphism $f : Z \to X$ one has a unique coalgebra morphism $f^\flat : \langle Z, k\rangle \to B_\star X$, namely the coinductive extension of k along f. (Cf §3.2.)

Next, consider a B-bisimulation

$$\begin{tikzcd} & R \arrow[dl, "r_1"'] \arrow[dr, "r_2"] \arrow[d, phantom, "\sim"] & \\ \langle X, k\rangle & & \langle Y, \ell\rangle \end{tikzcd}$$

and two morphisms $f : X \to Z$ and $g : Y \to Z$ in $\underline{\mathrm{C}}$ such that the diagram

$$\begin{tikzcd} & R \arrow[dl, "r_1"'] \arrow[dr, "r_2"] & \\ X \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \\ & Z & \end{tikzcd} \tag{9}$$

commutes. Then also the diagram

$$\begin{tikzcd} & R \arrow[dl, "r_1"'] \arrow[dr, "r_2"] \arrow[d, phantom, "\sim"] & \\ \langle X, k\rangle \arrow[dr, "f^\flat"'] & & \langle Y, \ell\rangle \arrow[dl, "g^\flat"] \\ & B_\star Z & \end{tikzcd}$$

commutes, because both composites $f^\flat \circ r_1$ and $g^\flat \circ r_2$ fit as the unique coinductive extension of the (no matter which!) coalgebra structure on R along the composite $f \circ r_1 = g \circ r_2 : R \to Z$.

If pullbacks lift to B-bisimulation, then the pullback (in the base category) of the coinductive extensions $f^\flat$ and $g^\flat$ of k and ℓ along f and g is the final object in the evident category of bisimulations making (9) commute.

Example 4.1. Consider the simple behaviour $BX = 1 + Act \times X$ (see §2.2) and, correspondingly, ordinary bisimulation for deterministic transition systems. Let the set Act of actions be trivial, that is, let Act be made of only one action a. Let $\langle X, k\rangle$ and $\langle Y, \ell\rangle$ be the same coalgebra having as carrier the set $\mathcal{Z}$ of integers and as structure $\ell : \mathcal{Z} \to B(\mathcal{Z})$ the one corresponding to the following (deterministic) transition system: 0 is inert, a positive integer n performs a transition to its predecessor $n-1$, and a negative integer $-n$ performs a transition to its successor $-n+1$:

$$0 \downarrow * \qquad n \xrightarrow{a} n-1 \qquad -n \xrightarrow{a} -n+1$$

Finally, let Z be the three-elements set $\{0, \clubsuit, \diamondsuit\}$. Thus:

$$X = \mathcal{Z} = Y \qquad Z = \{0, \clubsuit, \diamondsuit\} \qquad Act = \{a\}$$

Now, different bisimulations are possible according to the choice of the functions f, g :

$\mathcal{Z} \to \{0, \clubsuit, \diamondsuit\}$. Let us fix the function $g : \mathcal{Z} \to \{0, \clubsuit, \diamondsuit\}$ to be the one mapping odd numbers to $\clubsuit$ and even numbers to $\diamondsuit$. If f is equal to g, then every number is bisimilar to itself and to its opposite. For instance,

$$f^\flat(-3) \;=\; \clubsuit \xrightarrow{a} \diamondsuit \xrightarrow{a} \clubsuit \xrightarrow{a} 0 \;=\; g^\flat(3)$$

and thus -3 is bisimilar to 3 (wrt g).
The above amounts to assume that one can observe in both transition systems whether a number is odd or even. If, instead, in the first transition system one can observe this only for positive numbers, thus, eg, $f(-n) = 0$ and $f(n) = g(n)$, then one has that a positive number n is bisimilar to both $-n$ and n (wrt f and g) but its opposite $-n$ is not bisimilar to any number in the second transition system.
Finally, if one cannot observe at all in the first transition system whether a number is odd or even (ie $f(z) = 0$ for all $z \in \mathcal{Z}$) then only the two 0's are bisimilar.
(Notice that the morphisms f and g can be regarded as *abstract interpretations* of the states.)

Example 4.2. Another example is when one has a distinguished subset $Obs(X) \subset X$ of states which are 'observable'. This can be expressed by taking $Z = Obs(X) \cup \{\perp\}$ and $f : X \to Obs(X) \cup \{\perp\}$ to be

$$f(x) \;=\; \begin{cases} x & \text{if } x \in Obs(X) \\ \perp & \text{otherwise} \end{cases}$$

Notes. The notion of an ordinary bisimulation stems from the work of Park (1981) and Milner (1980) on concurrency. Coalgebraic bisimulations for relations between sets were introduced in (Aczel and Mendler, 1989). (See also the dual algebraic congruences in (Manes, 1976, page 167).) The more general use of spans stems from (Joyal et al., 1993), where, however, 'open morphisms' are used instead of coalgebraic homomorphisms. The idea is that, given a category $\underline{\mathrm{M}}$ of operational models, and a subcategory $\underline{\mathrm{P}} \hookrightarrow \underline{\mathrm{M}}$ of computations (or 'path objects'), two models X and Y (ie two objects of $\underline{\mathrm{M}}$) are $\underline{\mathrm{P}}$-*bisimilar* if there is a span of $\underline{P}$-*open morphisms* in $\underline{\mathrm{M}}$. When $\underline{\mathrm{M}}$ is the category of transition systems and $\underline{\mathrm{P}}$ is a suitable category of 'abstract' finite transition sequences, then a morphism $f : X \to Y$ of transition systems is $\underline{\mathrm{P}}$-open if and only if it is a coalgebra morphism $f : \langle X, k\rangle \to \langle Y, \ell\rangle$, for k and ℓ the coalgebra structures corresponding, via (1), to the transition systems on X and Y, respectively. It would be interesting to investigate whether this correspondence between open morphisms and coalgebra homomorphisms holds also for different notions of path objects and endofunctors.

For a large class of functors over cpos, a generalized notion of bisimulation is considered in (Pitts, 1994a) which, more subtly, amounts to first lifting the behaviour endofunctors to a category of relations and then studying the coalgebras therein. A systematic way of defining these liftings for 'simple' functors is described in (Hermida and Jacobs, 1995).

5. Non-Well-Founded Sets

This section is devoted to a coalgebraic presentation of Peter Aczel's theory of "non-well-founded sets" (Aczel, 1988). A categorical duality is proved between the 'anti-foundation axiom' giving non-well-founded sets and the 'foundation axiom': it is shown that the former is equivalent to postulating that 'the universe $V = \mathcal{P}_S V$ is a final coalgebra', while the latter is equivalent to '$V = \mathcal{P}_S V$ is an initial algebra'. (The endofunctor $\mathcal{P}_S$ maps a class to the class of its (small) subsets.)

The semantic motivation for the use of anti-foundation is that it permits to prove the "Special Final Coalgebra Theorem" (Aczel, 1988) which states that, under mild assumptions, the greatest fixed point of an endofunctor on (possibly non-well-founded) sets is a final coalgebra.

The special final coalgebra theorem is stated in terms of the "Solution Lemma". The final coalgebra presentation of anti-foundation adopted here renders this lemma (and its equivalence with anti-foundation) trivial. Correspondingly, the 'uniformity on maps' condition which an endofunctor has to satisfy in order for the special final coalgebra theorem to hold can be formulated in a more transparent way than in (Aczel, 1988).

5.1. *Basic Set Theory*

One way of understanding the abstract notion of a *set* is as a collection x such that its elements have "no internal structure whatsoever" and x itself has "no internal structure except for equality and inequality of pairs of elements". (Cf (Lawvere, 1976, page 119).) Axiomatically, this corresponds to taking the membership relation '$\in$' as the only primitive notion of set theory and to postulating the following 'extensionality axiom', the first axiom of set theory.

Extensionality: Two sets are equal iff they have the same elements.

Next, for every property P in a (first-order) language with membership and equality only, one would like the collection $\{x \mid \mathsf{P}(x)\}$ of sets which have the property P to be a set. However, Russell's paradoxical set $\{x \mid x \notin x\}$ shows that this 'strong comprehension axiom' cannot be stated in its full generality. One needs to consider properties *relative* to the elements of an already defined set. This leads to the 'comprehension axiom', the second axiom of set theory.

Comprehension: For every property P and every set v, the collection

$$\{x \mid \mathsf{P}(x) \wedge x \in v\}$$

is a set.

As comprehension can be applied only to members of already defined sets, it is necessary to postulate the existence of some sets, either primitive or derived by applying some basic operators:

Empty Set: There exists a set 0 with no elements.

Pairing, Union, Power-Set: $\{x, y\}$, $\bigcup x$, $\mathcal{P}(x)$ are all sets, for x, y sets.

As usual, $\bigcup x$ and $\mathcal{P}(x)$ stand for the collection of all members of members of x and the collection of all subsets of x, respectively. In turn, the subset relation '$\subseteq$' can be derived from the membership relation:

$$x \subseteq y \iff \forall v \, (v \in x \Rightarrow v \in y)$$

By means of the union operator one can define an operator s acting as successor as follows: $s(x) = x \cup \{x\}$. The existence of an infinite set can be stated by postulating the existence of a set containing the natural numbers. That is:

Infinity: There exists a set containing 0 and closed under the successor operator s.

(The axioms above, as well as those given in the sequel, are written for convenience in natural language but note that they can also be expressed in the language of set theory – see, eg, (Levy, 1979).)

Further useful notions can be derived from the above axioms, like, for instance, the notion of *ordered pair*:

$$<x, y> = \{x, \{x, y\}\}$$

A formal definition of function can then be given as a collection f of ordered pairs such that for every x there exists a unique y with $< x, y > \in f$. Two more axioms about functions are then usually added:

Replacement: The image of a set under a function is a set.

Choice: Every surjective function has a 'right inverse'.

A *right inverse* for a function $f : a \to b$ is a function $g : b \to a$ such that $f \circ g$ is the identity on b. The above axiom of choice is equivalent to postulate that for every set a there exists a *choice function*, that is, a function f such that, for every $x \in a$, $f(x) \in x$.

The above axioms (extensionality, comprehension, empty set, pairing, union, power set, infinity, replacement, choice) are the basic axioms of set theory; let us call the theory associated with (ie, the collection of all sentences derivable from) them **basic set theory** and the corresponding category of sets and functions **Set**. (Basic set theory is usually called *ZFC*$^-$ in the literature – see, eg, (Levy, 1979).)

Classes. Even though the collection $\{x \mid \mathsf{P}(x)\}$ of all sets x having a given property P might not be a set, it can still be of interest for set theory. Such 'specifiable' collections are called **classes**. Clearly, a set is a class, but the converse is not true, in which case one speaks of a **proper class**. (Also the terminology '*large set*', vs '*small set*', is used.) In the sequel, lower case letters are used for (small) sets and capital letters for classes.

The equality between classes is determined by their *small* elements. That is, two classes $X = \{x \mid \mathsf{P}(x)\}$ and $Y = \{x \mid \mathsf{P}'(x)\}$ are equal if and only if P and P' hold for the same (small) sets.

An example of a proper class is the **universe of sets**, namely the collection of all sets:

$$V = \{x \mid x = x\}.$$

(Since the property $x = x$ trivially holds for all sets, the class V is the collection of all sets indeed.) Notice that different properties may specify the same class. For instance, any property other than '$x = x$' which holds for all sets can be used to specify the universe.

Next, let **SET** be the category of classes and (class) functions corresponding to basic set theory. The claim is that the universe V can be seen as the carrier of both an algebra and a coalgebra structure of the endofunctor

$$\mathcal{P}_S : \underline{\mathrm{SET}} \to \underline{\mathrm{SET}} \qquad\qquad X \mapsto \{x \mid x \text{ is a set } \wedge\ x \subseteq X\}$$

which maps a class to the class of its (small) sub*sets*. By taking subsets rather than subclasses one overcomes the cardinality problem which prevents the unrestricted power-set from having a fixed-point.

Proposition 5.1. The universe V is a strict fixed point $V = \mathcal{P}_S V$ of the endofunctor $\mathcal{P}_S : \underline{\mathrm{SET}} \to \underline{\mathrm{SET}}$.

Proof. The universe V is, by definition, the largest class, hence, since $\mathcal{P}_S V$ is itself a class, $\mathcal{P}_S V \subseteq V$. For the converse it is sufficient to prove that every set x is a subset of V. That is, for every $y \in x$, y is also in V. This is immediate from the fact that y is a set. □

Therefore, the identity on V can be seen both as a $\mathcal{P}_S$-algebra and as a $\mathcal{P}_S$-coalgebra structure for V.

5.2. *Well-Founded Sets and Foundation*

From the axioms of basic set theory alone it is not possible to draw a canonical picture of what the universe looks like, a picture independent of the specific interpretation one might give to the theory. This was felt as a problem already in the early developments of set theory. The solution was found in the '*foundation axiom*', which was then added to basic set theory. This axiom restricts the universe to the 'smallest' of all possible ones. Then the picture arises of a universe in which sets are constructed by iterative applications of the power-set operator starting from the empty set. Every set has a *rank*, namely the stage at which it appears in such a '*cumulative hierarchy*'.

In this section it is proved that the foundation axiom is equivalent to postulating that the universe $V = \mathcal{P}_S V$ is the initial algebra of the power-set endofunctor $\mathcal{P}_S$ on $\underline{\mathrm{SET}}$.

A set x is **well-founded** with respect to the membership relation '$\in$' if either it is empty or has a least element with respect to $\in$. In other words, there is no infinitely descending chain of elements starting from x. Correspondingly, let the class

$$W = \{x \mid x \text{ is well-founded with respect to the relation } \in\}$$

be the **universe of well-founded sets**.

The 'foundations axiom' amounts to postulating that all sets in the universe V are well-founded, that is,

Foundation Axiom: $\qquad V = W$

Now, notice that the class $\mathcal{P}_S W$ of (small) subsets of well-founded sets is the same as W, because the elements of a well-founded set are themselves well-founded. Thus

$$\mathcal{P}_S W = W$$

and the identity on W can be seen as a $\mathcal{P}_S$-algebra structure.

Proposition 5.2. The universe of well-founded sets is an initial $\mathcal{P}_S$-algebra.

Proof. For every $\mathcal{P}_S$-algebra structure $h : \mathcal{P}_S X \to X$ there exists a unique function $h^\# : W \to X$ such that the following diagram commutes.

$$\begin{tikzcd}
\mathcal{P}_S W \arrow[r, "\mathcal{P}_S(h^\#)"] \arrow[d, equal] & \mathcal{P}_S X \arrow[d, "h"] \\
W \arrow[r, dashed, "h^\#"'] & X
\end{tikzcd}$$

That is,

$$\begin{array}{lcl} h^\#(0) & = & h(0) \\ h^\#\{x_i\}_I & = & h\{h^\#(x_i)\}_I \end{array}$$

The proof is by straightforward induction on the (well-founded!) membership relation $\in$. $\square$

An immediate consequence of the initiality of W is the existence of a 'rank' function, mapping every well-founded set to a suitable 'ordinal'. An **ordinal** is a well-founded set which is *totally* ordered by the membership relation and which is 'transitive'. (A **transitive set** is a set x such that every element $y \in x$ is also a subset $y \subseteq x$.) Correspondingly, one can form the class $\mathcal{O}$ of all ordinals, which is a subclass of W.

If α and β are two ordinals such that $\beta \in \alpha$, one usually writes $\beta < \alpha$. The first ordinals are: 0, $s(0)$, $s^2(0)$, etc. The first limit ordinal is $\omega = \bigcup_{n\in N} s^n(0)$, which, by the infinity axiom, is indeed a set. In general, because every ordinal is totally ordered by $\in$, the union $\bigcup\{\alpha_i\}_I$ of a set $\{\alpha_i\}_I$ of ordinals is the least upper bound of the α_i's. As a consequence, the union operator is a $\mathcal{P}_S$-algebra structure on the class $\mathcal{O}$ of ordinals:

$$\bigcup : \mathcal{P}_S(\mathcal{O}) \to \mathcal{O} \qquad\qquad \{\alpha_i\}_I \mapsto \bigcup\{\alpha_i\}_I$$

The inductive extension $\mathsf{rank} = \bigcup^\# : W \to \mathcal{O}$ of this algebra structure on $\mathcal{O}$ is the function assigning a 'rank' to every well-founded set. This can be thought of as the stage at which a well-founded set is constructed in an idealized construction starting from the empty set and then iteratively applying the power-set functor $\mathcal{P}_S$:

$$\begin{array}{lcl} \mathsf{rank}(0) & = & 0 \\ \mathsf{rank}\{x_i\}_I & = & \bigcup\{\mathsf{rank}(x_i)\}_I \end{array}$$

Another consequence of the initiality of W is that $W = \mathcal{P}_S W$ is the least (pre-) fixed point for $\mathcal{P}_S$:

$$W = \mathit{lfp}[\mathcal{P}_S]$$

That is, for every class X such that $\mathcal{P}_S X \subseteq X$, one has that $W \subseteq X$. Indeed, regarding the inclusion of $\mathcal{P}_S X$ into X as a function $\kappa : \mathcal{P}_S X \hookrightarrow X$, one has that its inductive extension $\kappa^{\#} : W \to X$ is of the following form.

$$\begin{array}{lcl} \kappa^{\#}(0) & = & 0 \\ \kappa^{\#}\{x_i\}_I & = & \kappa\{\kappa^{\#}(x_i)\}_I \end{array}$$

Then, to see that $\kappa^{\#}$ is the inclusion of W into X, it suffices to notice that the power-set functor

$\mathcal{P}_S$ **'preserves inclusion functions'**

that is, if $\iota : X \hookrightarrow Y$ is the inclusion of a subclass X of Y into Y, then the function $\mathcal{P}_S(\iota) : \mathcal{P}_S X \to \mathcal{P}_S Y$ is the inclusion of $\mathcal{P}_S X$ into $\mathcal{P}_S Y$.

Usually, initial algebras are unique up to isomorphism, but in this setting one has a stronger result:

$$\mathcal{P}_S X = X \text{ is an initial } \mathcal{P}_S\text{-algebra} \iff X = W \tag{10}$$

That is, any other initial algebra which is a (strict) fixed point of $\mathcal{P}_S$ is not only isomorphic but equal to W. In order to prove this, ie the non-trivial implication from left to right, one can use very much the same argument as the one used above to prove that W is the least fixed point of $\mathcal{P}_S$.

Therefore, by taking the X in (10) to be the universe V, one has that the foundation axiom '$V = W$' is equivalent to postulating that the universe V is an initial algebra of the power-set functor:

Theorem 5.1. (Foundation is Initiality)

$$V = W \iff \mathcal{P}_S V = V \text{ is an initial } \mathcal{P}_S\text{-algebra.}$$

□

5.3. *Anti-Foundation and Finality*

As shown in the introduction, not all sets occurring in the mathematical practice are well-founded. In order to ensure the existence of *non-well-founded sets*, one can postulate the 'anti-foundation axiom', which here is shown to be the dual of the initial algebra formulation of 'foundation':

Foundation:	$\mathcal{P}_S V = V$ is an initial $\mathcal{P}_S$-algebra.
Anti-Foundation:	$V = \mathcal{P}_S V$ is a final $\mathcal{P}_S$-coalgebra.

That is, anti-foundation postulates that the universe is the 'largest' possible one, while foundation postulates that it is the 'smallest'.

Theorem 5.2. (Final $\mathcal{P}_S$-Coalgebra Theorem.) The endofunctor

$$\mathcal{P}_S : \underline{\mathrm{SET}} \to \underline{\mathrm{SET}} \qquad\qquad X \mapsto \{x \mid x \text{ is a set } \wedge\ x \subseteq X\}$$

on the category $\underline{\mathrm{SET}}$ of classes (ie large sets) which are definable within basic set theory, has a final coalgebra. □

The above theorem stating the existence of a final $\mathcal{P}_S$-coalgebra is an instance of the "final coalgebra theorem" in (Aczel and Mendler, 1989). It can also be seen as an instance of Theorem 3.1, replacing the finite power-set $\mathcal{P}_{\text{fi}}$ by $\mathcal{P}_S$; correspondingly, the core of the proof is the fact that the coalgebras of the restriction $\mathcal{P} : \underline{\text{Set}} \to \underline{\text{Set}}$ of the endofunctor $\mathcal{P}_S$ to the category $\underline{\text{Set}}$ of small sets form a 'generating class' for the $\mathcal{P}_S$-coalgebras; that is, every $\mathcal{P}_S$-coalgebra can be obtained by quotienting the disjoint union of a suitable class of $\mathcal{P}$-coalgebras.

By considering the unlabelled version of the correspondence (1) given in Section 2 one has that there is a one-to-one correspondence between directed (small) graphs $\langle X, \to \rangle$ and coalgebras $k : X \to \mathcal{P}X$: for every $x, x' \in X$,

$$x \longrightarrow x' \iff x' \in k(x)$$

Similarly, the coalgebras of the power-set functor $\mathcal{P}_S$ are the same as the directed '**locally small**' graphs, that is, the (possibly large) directed graphs such that the collection of children of every node is a (small) set.

Peter Aczel's original formulation of the anti-foundation axiom is in terms of small graphs and 'decorations'. A **decoration** for (the graph corresponding to) a $\mathcal{P}_S$-coalgebra $\langle X, k \rangle$ is a coalgebra morphism from $\langle X, k \rangle$ to $V = \mathcal{P}_S V$

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "k"'] & V \arrow[d, equal] \\ \mathcal{P}_S X \arrow[r, "\mathcal{P}_S f"'] & \mathcal{P}_S V \end{tikzcd}$$

That is, a function f from X to the universe V such that, for every $x \in X$,

$$f(x) = \{f(x') \mid x' \in k(x)\}$$

In terms of graphs, this corresponds to a function mapping every node to a set in the following way.

$$f(x) = \{f(x') \mid x \longrightarrow x'\}$$

Therefore, by definition of final coalgebra, the coalgebra $V = \mathcal{P}_S V$ is final if and only if every (directed) locally small graph has a unique decoration. Now, the claim is that 'locally small' can be replaced by 'small' in the above equivalence. That is:

Lemma 5.1. Every *locally small* graph has a unique decoration if (and only if) every *small* graph has a unique decoration.

Proof. (By contradiction.) Assume that every small graph has a unique decoration and that there are two distinct decorations f and g of (a coalgebra $\langle X, k \rangle$ corresponding to) a locally small graph. Then there is a node $x \in X$ such that

$$f(x) \neq g(x)$$

Now, the subgraph of $\langle X, k \rangle$ accessible from x is not only locally small but also (totally) small, that is, there are only set-many nodes accessible from x, because every node has only set-many children. But then f and g are both decorations for this small subgraph, which, by hypothesis, implies that

$$f(x) = g(x)$$

(The same argument can be used to prove that the class of small $\mathcal{P}_S$-coalgebras forms a *generating class* for the $\mathcal{P}_S$-coalgebras.) $\square$

As a consequence, the postulate '$V = \mathcal{P}_S V$ is a final $\mathcal{P}_S$-coalgebra' is equivalent to Peter Aczel's original formulation of anti-foundation (which is equivalent to Forti and Honsell's "X_1-axiom" (Forti and Honsell, 1983)):

Anti-Foundation Axiom: Every directed small graph has a unique decoration.

That is:

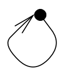
Theorem 5.3. (Anti-Foundation is Finality.) Every directed small graph has a unique decoration if and only if $V = \mathcal{P}_S V$ is a final $\mathcal{P}_S$-coalgebra. $\square$

Notice that no axiom is needed in order to obtain a unique decoration for a *well-founded* graph: One can check that the class WG of well-founded directed small graphs is a (strict) fixed point for the power-set functor $\mathcal{P}_S$, and, moreover, that $\mathcal{P}_S(\mathsf{WG}) = \mathsf{WG}$ is an initial $\mathcal{P}_S$-algebra. Therefore WG is isomorphic to the universe of well-founded sets W and the image under this isomorphism of a well-founded graph is its unique decoration. (Cf "*Mostowski's collapsing lemma*" in (Aczel, 1988).)

When anti-foundation is postulated also non-well-founded graph have a unique decoration, but the converse is not true anymore. That is, there exist (non-well-founded) sets which 'decorate' different graphs. An example is the archetypal non-well-founded set, namely the *self-singleton* set

$$\Omega = \{\Omega\}$$

which is a member (and the only member) of itself. If anti-foundation is assumed, then both the root of the graph with one node and one arc



and the root of the graph consisting in one infinite path

$$\bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \cdots$$

are necessarily mapped to Ω by the corresponding unique decorations.

Notice that, besides applications in the semantics of programming languages (eg, (Aczel, 1988; Mukai, 1991; Rutten and Turi, 1993; Aczel, 1994; Baldamus, 1994; Honsell and Lenisa, 1995; Hartonas, 1997)), non-well-founded sets have been extensively used in *Situation Theory* (eg, (Barwise and Etchemendy, 1987)), where they are better known as **hypersets**. (Correspondingly, models of the universe of non-well-founded sets are also called **hyperuniverses**.)

Reasoning about non-well-founded sets: bisimulation. By the extensionality axiom, the equality between two sets is determined by the membership relation. One of the consequences of *foundation* is that, since then the membership relation is well-founded, one can use *induction* to reason about (the equality between) sets. Categorically, this induction principle follows from the fact that foundation postulates that the universe is an initial algebra. Dually, anti-foundation, by postulating that the universe is a final coalgebra, gives a *coinduction* principle for reasoning about (possibly non-well-founded) sets.

Now, as shown in Section 4, if an endofunctor *preserves weak pullbacks* then coinduction (with respect to its final coalgebra) can be 'pulled back' to the corresponding coalgebraic notion of *bisimulation*. In particular, the power-set functor $\mathcal{P}_S$ does preserve weak pullbacks; the proof is essentially the same as the one given in Section 4 for the behaviour $BX = \mathcal{P}(Act \times X)$.

By instantiating the general definition of coalgebraic bisimulation (Section 4) to the $\mathcal{P}_S$-coalgebras one has that a $\mathcal{P}_S$-bisimulation for a coalgebra $\langle X, k\rangle$ is a span R such that, for all x_1, x_2 in X, if $x_1 R x_2$ then

— if $x_1 \longrightarrow x_1'$ then $x_2 \longrightarrow x_2'$ for some x_2' such that $x_1' R\, x_2'$

— and, conversely, if $x_2 \longrightarrow x_2'$ then $x_1 \longrightarrow x_1'$ for some x_1' such that $x_1' R\, x_2'$.

(Here the notation $x \longrightarrow x'$ stands for 'there is an arc from x to x' in the graph corresponding to the coalgebra $\langle X, k\rangle$'.)

In particular, a $\mathcal{P}_S$-bisimulation R for the universe V is such that, for every set x and y, if xRy then for every set x' in x there exists a set y' in y such that $x'R\, y'$ and, conversely, for every set y' in y there exists a set x' in x such that $x'R\, y'$. Therefore, by strong extensionality,

$$x = y \iff \exists R\, (xR\, y \wedge (\forall x' \in x,\ \exists y' \in y,\ x'R\, y') \wedge (\forall y' \in y,\ \exists x' \in x,\ x'R\, y'))$$

5.4. *Systems of Set-Equations as Coalgebras*

The self-singleton non-well-founded set $\Omega = \{\Omega\}$ can be seen as the unique solution of the 'set-equation'

$$x = \{x\}$$

In general, all non-well-founded sets arise from systems of set-equations with, on the left hand side, variables $x \in X$, and, on the right hand side, *well-founded* sets, possibly containing variables from X. This is the content of the "Solution Lemma", which is proved below in an elementary way thanks to the coalgebraic account of anti-foundation (and the initial algebra presentation of well-founded sets).

The definition of the universe of well founded sets W can be made parametric: for every (possibly large) set X, the **expanded universe of well-founded sets** WX is the class of all well-founded sets with variable $x \in X$. That is, every set in WX is either empty, or an element of X, or it has a least element with respect to the membership relation $\in$. For $X = 0$ this yields the standard universe $W0$ of well-founded sets. Thus, in the sequel, W stands for an operator mapping a (large) set to the corresponding expanded universe of well-founded sets, rather than for the simple universe of well-founded sets.

The fact that $W0$ is the least (strict) fixed point of the power-set functor $\mathcal{P}_S$ and that $\mathcal{P}_S W0 = W0$ is an initial $\mathcal{P}_S$-algebra generalizes as follows: the class WX is the least (strict) fixed point of the endofunctor $X+\mathcal{P}_S(\text{-})$ on SET and

$$X + \mathcal{P}_S WX = WX \tag{11}$$

is an initial algebra for this endofunctor. The universal property given by initiality can be used to extend the operator W to a functor, like in diagram (3) of §3.1. That is, for every function $f : X \to Y$, the function $Wf : WX \to WY$ is the inductive extension of the algebra structure $\mathsf{inr}_Y : \mathcal{P}_S WY \to WY$ along the composite $\eta_Y \circ f : X \to WY$, where the left injection $\eta_Y = \mathsf{inl}_Y : Y \to WY$ is the usual insertion-of-variables function.

Observation 5.1. By the above definition and the one of free algebras in §3.2, the forgetful functor from the category of $\mathcal{P}_S$-algebras to SET has a left adjoint, namely the functor mapping a class X to the free algebra (11). □

(In general, every free $\mathcal{P}_S$-algebra over a possibly large set X can be used to model the universe of Zermelo-Fraenkel set theory expanded with elements of X as *atoms*. This fact can be seen as an instance of a more general result in (Joyal and Moerdijk, 1995), namely Theorem II.5.5, which is stated in terms of free "*Zermelo-Fraenkel algebras* and intuitionistic set theory.)

Now, the idea is that a system of 'set-equations' like, eg,

$$\begin{array}{rcl} x & = & \{x, \{y\}\} \\ y & = & \{y, 0\} \end{array} \tag{12}$$

can be seen as a function k mapping the variables $x, y, \ldots \in X$ of the system to elements of $\mathcal{P}_S WX$, ie sets of well-founded sets possibly with variables in X. For instance, the above system corresponds to a function $k : \{x, y\} = X \to \mathcal{P}_S WX$ mapping x to $\{x, \{y\}\}$ and y to $\{y, 0\}$. Therefore, in general, a **system of set-equations in** X is a coalgebra $\langle X, k\rangle$ of the composite endofunctor $\mathcal{P}_S W$ on SET.

In order to solve a system of set-equations $\langle X, k\rangle$ one can (postulate anti-foundation and) use the finality of the universe $V = \mathcal{P}_S V$. For this, one first needs to extend the $\mathcal{P}_S W$-coalgebra structure $k : X \to \mathcal{P}_S WX$ to a $\mathcal{P}_S$-coalgebra structure as follows. Since $WX = X + \mathcal{P}_S WX$ is a coproduct, one can form the copair $[k, \mathsf{id}] : WX \to \mathcal{P}_S WX$ of k and the identity id on $\mathcal{P}_S WX$. This is a $\mathcal{P}_S$-coalgebra structure behaving as k on $x \in X$ and as the identity on $v \in \mathcal{P}_S WX$. Its coinductive extension $\overline{k} = [k, \mathsf{id}]^{@} : WX \to V$ with respect to the final $\mathcal{P}_S$-coalgebra $V = \mathcal{P}_S V$ is then the (unique) **solution of the**

system $k : X \to \mathcal{P}_S WX$ of set-equations:

$$
\begin{tikzcd}
X \arrow[r, "\eta_X"] \arrow[dr, "k"'] & WX \arrow[r, dashed, "\overline{k} = {[k, \mathsf{id}]}^{@}"] \arrow[d, "{[k, \mathsf{id}]}"] & V \arrow[d, equal] \\
& \mathcal{P}_S WX \arrow[r, "\mathcal{P}_S \overline{k}"'] & \mathcal{P}_S V
\end{tikzcd}
$$

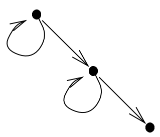
Omitting, as usual, the injections, and letting v and v' range over objects of type $\mathcal{P}_S W$, one has that

$$\overline{k}(x) = \{\overline{k}(v) \mid v \in hx\} \qquad \text{and} \qquad \overline{k}(v) = \{\overline{k}(v') \mid v' \in v\}$$

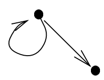
For example, the unique solution of equation $k(x) = \{x\}$ is the self-singleton (non-well-founded) set $\overline{k}(x) = \{\overline{k}(x)\}$, that is, $\overline{k}(x) = \Omega$. Similarly, the solution of the above system (12) is

$$
\begin{aligned}
\overline{k}(x) &= \{\overline{k}(x), \{\overline{k}(y)\}\} \\
\overline{k}(y) &= \{\overline{k}(y), 0\}
\end{aligned}
$$

In terms of graphs, the sets $\overline{k}(x)$ and $\overline{k}(y)$ correspond to



and



respectively.

The Solution Lemma. The above property that every system of set-equations has a unique solution, is called the **solution lemma** in (Aczel, 1988). (See also (Barwise and Etchemendy, 1987, Chapter 3).) It is obtained assuming the anti-foundation axiom. Conversely, postulating the solution lemma, one can prove that $V = \mathcal{P}_S V$ is the final $\mathcal{P}_S$-coalgebra. Indeed, for every $\mathcal{P}_S$-coalgebra $\langle X, k \rangle$, one obtains

$$
\begin{tikzcd}
X \arrow[r, "\eta_X"] \arrow[d, "k"'] & WX \arrow[r, dashed, "\overline{\mathcal{P}_S(\eta_X) \circ k}"] \arrow[d, "{[\mathcal{P}_S(\eta_X) \circ k, \mathsf{id}]}"] & V \arrow[d, equal] \\
\mathcal{P}_S X \arrow[r, "\mathcal{P}_S(\eta_X)"'] & \mathcal{P}_S WX \arrow[r, "\mathcal{P}_S(\overline{\mathcal{P}_S(\eta_X) \circ k})"'] & \mathcal{P}_S V
\end{tikzcd}
$$

The desired coinductive extension of the coalgebra structure $k : X \to \mathcal{P}_S X$ is given by the composite coalgebra morphism

$$k^{@} = \overline{\mathcal{P}_S(\eta_X) \circ k} \circ \eta_X : X \to V$$

Therefore:

Proposition 5.3. The solution lemma is equivalent to anti-foundation. $\square$

Notice that, assuming anti-foundation, the upper rectangle in the following diagram commutes, because all other sub-diagrams commute.

$$\begin{tikzcd}
 & \mathcal{P}_S WX \arrow[r, "\mathcal{P}_S\overline{k}"] \arrow[d, "\mathsf{inr}_X"'] \arrow[dd, equal, bend left] & \mathcal{P}_S V \arrow[d, equal] \\
X \arrow[r, "\eta_X"] \arrow[dr, "k"'] & WX \arrow[r, "\overline{k}"] \arrow[d, "{[k, \mathsf{id}]}"'] & V \arrow[d, equal] \\
 & \mathcal{P}_S WX \arrow[r, "\mathcal{P}_S\overline{k}"'] & \mathcal{P}_S V
\end{tikzcd}$$

Therefore, the solution $\overline{k} : WX \to V$ of a system of set equations $\langle X, k \rangle$ is not only a $\mathcal{P}_S$-coalgebra morphism but also a $\mathcal{P}_S$-algebra morphism from $\langle WX, \mathsf{inr}_X \rangle$ to $\mathcal{P}_S V = V$. The algebra $\langle WX, \mathsf{inr}_X \rangle$ is a *free* $\mathcal{P}_S$*-algebra* over X.

The Substitution Lemma from Freeness. In the present approach, the proof of the solution lemma is trivial. The original proof, instead, makes use of a **substitution lemma** (Aczel, 1988). This lemma asserts that, for every function $f : X \to V$, there exists a unique extension $f^\sharp : WX \to V$ of f to $WX = X + \mathcal{P}_S WX$ such that, omitting the injections,

$$f^\sharp(x) = f(x) \qquad \text{and} \qquad f^\sharp(v) = \{f^\sharp(v') \mid v' \in v\}$$

Now, also this becomes trivial here, because of the initial algebra presentation of the expanded universe of well-founded sets WX. Indeed, the desired function $f^\sharp : WX \to V$ is the inductive extension of the $\mathcal{P}_S$-algebra structure $\mathcal{P}_S V = V$ along $f : X \to V$. That is:

$$\begin{tikzcd}
X \arrow[r, "\eta_X"] \arrow[dr, "f"'] & WX \arrow[d, dashed, "f^\sharp"] & \mathcal{P}_S WX \arrow[l, "\mathsf{inr}_X"'] \arrow[d, "\mathcal{P}_S f^\sharp"] \\
 & V \arrow[r, phantom, "="] & \mathcal{P}_S V
\end{tikzcd}$$

Notice that, in contrast with (Aczel, 1988), anti-foundation is *not* used here.

5.5. *From Greatest Fixed Points to Final Coalgebras*

The greatest (strict) fixed point $V = \mathcal{P}_S V$ of the power-set functor $\mathcal{P}_S$ can be seen as the final coalgebra of the restriction of the functor $\mathcal{P}_S$ to the subcategory $\underline{\mathbf{SET}}_\subset$ of inclusion functions. Anti-foundation postulates that this final coalgebra lifts to a final coalgebra in $\underline{\mathrm{SET}}$. If an endofunctor is 'uniform on maps', then, assuming anti-foundation, its final coalgebra in the subcategory $\underline{\mathrm{SET}}_\subset$ also lifts to a final coalgebra in $\underline{\mathrm{SET}}$. This is the content of the "Special Final Coalgebra Theorem".

In this section, a new formalization of the notion of uniformity on maps in terms of natural transformations is given. The proof of the theorem is then rephrased in terms of this definition.

Let F be an endofunctor on $\underline{\mathrm{SET}}$. A post-fixed point $X \subseteq FX$ for F can be seen as an inclusion function $X \hookrightarrow FX$, hence as an F-coalgebra structure on X. If the endofunctor F preserves inclusion functions, ie F applied to $X \hookrightarrow Y$ is an inclusion $FX \hookrightarrow FY$, then one can restrict F to the subcategory $\underline{\mathrm{SET}}_\subset$ of classes and inclusion functions. The post-fixed points of F are then its coalgebras in this subcategory. In particular, the final F-coalgebra in $\underline{\mathrm{SET}}_\subset$, if it exists, is the greatest (post-)fixed point

$$gfp[F] = F(gfp[F])$$

of F. The claim is that if F is 'uniform on maps' then, assuming anti-foundation, $gfp[F] = F(gfp[F])$ is also a final coalgebra.

Intuitively, an endofunctor on $\underline{\mathrm{SET}}$ is uniform on maps if it is completely determined by its action on objects (ie classes). Most of endofunctors are thus uniform on maps. For instance, consider the endofunctor $X \mapsto A \times X$ mapping a class X to its product with a fixed class A. Given a function $f : X \to Y$, the value of $A \times f$ at an element $<a, x>$ of $A \times X$ is the pair $<a, f(x)> \in A \times Y$ which is obtained by applying f to the $x \in X$ in $A \times X$. This suggests that the class X should be regarded as a class of variables and that, in general, the action of a functor F uniform on maps on a function f should simply be the substitution of the variables x occurring in FX by $f(x)$.

Formally, this can be expressed by means of the expanded universe of well-founded sets $WX = X + \mathcal{P}_S WX$. What one needs is a natural transformation

$$\rho : F \Rightarrow \mathcal{P}_S W$$

which, for every X, 'embeds' FX into $\mathcal{P}_S WX$ – the class of sets of (well-founded) sets having $x \in X$ as variables.

Naturality amounts to having, for every function $f : X \to Y$, the following diagram commute.

$$\begin{tikzcd} FX \arrow[r, "Ff"] \arrow[d, "\rho_X"'] & FY \arrow[d, "\rho_Y"] \\ \mathcal{P}_S WX \arrow[r, "\mathcal{P}_S Wf"'] & \mathcal{P}_S WY \end{tikzcd}$$

It should be an 'embedding' in the sense that, for every X and for every $v \in FX$, by 'forgetting' the distinction between variables and sets in $\rho_X(v) \in \mathcal{P}_S WX$ one should get back the original set v. This operation of forgetting the distinction between variables and sets in objects of type $\mathcal{P}_S W$ can be made formal as follows.

Consider the inductive extension $\varepsilon_V : WV \to V$ of the $\mathcal{P}_S$-algebra structure $\mathcal{P}_S V = V$ along the identity on V:

$$\begin{tikzcd}
V \arrow[r, "\eta_V"] \arrow[dr, equal] & WV \arrow[d, dashed, "\varepsilon_V"] & \mathcal{P}_S WV \arrow[l, "\mathsf{inr}_V"'] \arrow[d, "\mathcal{P}_S(\varepsilon_V)"] \\
 & V \arrow[r, equal] & \mathcal{P}_S V
\end{tikzcd}$$

Omitting, as usual, the injections, one has that, for every $v \in WV$, $\varepsilon_V(v) = v$ if v is a variable and $\varepsilon_V(v) = \{\varepsilon_V(v_i)\}_I$ if $v = \{v_i\}_I$.

Definition 5.1. (Uniformity on Maps) An endofunctor $F : \underline{\mathrm{SET}} \to \underline{\mathrm{SET}}$ is *uniform on maps* if there exists a natural transformation

$$\rho : F \Rightarrow \mathcal{P}_S W$$

such that

$$\begin{tikzcd}
FV \arrow[r, hook] \arrow[d, "\rho_V"'] & V \arrow[d, equal] \\
\mathcal{P}_S WV \arrow[r, "\mathcal{P}_S(\varepsilon_V)"'] & \mathcal{P}_S V
\end{tikzcd}$$

commutes. □

Before setting out to prove the special final coalgebra theorem, notice that, by Observation 5.1 and the definition of adjunction, there is a bijection (natural in X and $\langle Y, h\rangle$) between functions $f : X \to Y$ and $\mathcal{P}_S$-algebra morphisms $g : \langle WX, \mathsf{inr}_X\rangle \to \langle Y, h\rangle$. This bijection maps f to its *left adjunct*

$$f^\sharp = \varepsilon_{\langle Y,h\rangle} \circ Wf$$

and g to its *right adjunct*

$$g^\flat = Ug \circ \eta_X = g \circ \eta_X \tag{13}$$

where $\varepsilon_{\langle Y,h\rangle} : WY \to Y$ (the 'counit' of the adjunction) is the inductive extension of the right injection $\mathsf{inr}_Y : \mathcal{P}_S WY \to WY$ along the identity on Y.

$$\begin{tikzcd}
Y \arrow[r, "\eta_Y"] \arrow[dr, equal] & WY \arrow[d, dashed, "\varepsilon_{\langle Y,h\rangle}"] & \mathcal{P}_S WY \arrow[l, "\mathsf{inr}_Y"'] \arrow[d, "\mathcal{P}_S(\varepsilon_{\langle Y,h\rangle})"] \\
 & Y & \mathcal{P}_S Y \arrow[l, "h"]
\end{tikzcd}$$

(Thus, in particular, the above function $\varepsilon_V : WV \to V$ is the value of the counit at the algebra $\mathcal{P}_S V = V$. Formally, $\varepsilon_V = U\varepsilon_{(\mathcal{P}_S V = V)} = \varepsilon_{(\mathcal{P}_S V = V)}$, where U is the forgetful functor mapping algebras to their carriers.)

Theorem 5.4. (The Special Final Coalgebra Theorem.) Let F be an endofunctor on $\underline{\mathrm{SET}}$ which cuts down to an endofunctor on the subcategory $\underline{\mathrm{SET}}_\subset$ of inclusion functions.
If F is uniform on maps, then, assuming anti-foundation, its final coalgebra

$$gfp[F] = F(gfp[F])$$

in $\underline{\mathrm{SET}}_\subset$ lifts to a final F-coalgebra in $\underline{\mathrm{SET}}$.

Proof. Consider an F-coalgebra structure

$$k : X \to FX$$

By uniformity on maps, there exists a function $\rho_X : FX \to \mathcal{P}_S WX$, hence k can be made into a system of set-equations in X by composing it with ρ_X. Take its solution $\overline{\rho_X \circ k} : WX \to V$ and define a function f from X to V as the right adjunct (13) of this solution with respect to the above adjunction; that is,

$$f = (\overline{\rho_X \circ k})^\flat = \overline{\rho_X \circ k} \circ \eta_X : X \to V$$

Diagrammatically:

$$\begin{tikzcd}
X \arrow[r, "\eta_X"'] \arrow[rr, bend left, "f = (\overline{\rho_X \circ k})^\flat"] \arrow[d, "k"'] & WX \arrow[r, dashed, "\overline{\rho_X \circ k}"'] \arrow[d, "{[\rho_X \circ k, \mathsf{id}]}"'] & V \arrow[d, equal] \\
FX \arrow[r, "\rho_X"'] & \mathcal{P}_S WX \arrow[r, "\mathcal{P}_S(\overline{\rho_X \circ k})"'] & \mathcal{P}_S V
\end{tikzcd}$$

The claim is that, under the above hypotheses, f is an F-coalgebra morphism from $\langle X, k\rangle$ to $gfp[F] = F(gfp[F])$, that is, the diagram

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "k"'] & gfp[F] \arrow[d, equal] \\
FX \arrow[r, "Ff"'] & F(gfp[F])
\end{tikzcd}$$

commutes. More precisely: Let Y be the image under f of X. The function $f : X \to V$ can be factorized, like every function in $\underline{\mathrm{SET}}$, as

$$X \overset{f}{\twoheadrightarrow} Y \hookrightarrow V$$

The claim is then that the class Y is a post-fixed point for F, ie $Y \subseteq FY$, and f is a coalgebra morphism from $\langle X, k\rangle$ to $Y \hookrightarrow FY$, ie

$$\begin{tikzcd}
X \arrow[r, two heads, "f"] \arrow[d, "k"'] & Y \arrow[d, hook] \\
FX \arrow[r, "Ff"'] & FY
\end{tikzcd}$$

commutes.

If the above holds, since F cuts down to an endofunctor on the subcategory $\underline{\mathrm{SET}}_{\subset}$ of inclusions, the composition of f the inclusion $Y \hookrightarrow gfp[F]$ of Y into the greatest fixed point of F is an F-coalgebra morphism:

$$\begin{tikzcd}
X \arrow[r, two heads, "f"] \arrow[d, "k"'] & Y \arrow[d, hook] \arrow[r, hook] & gfp[F] \arrow[d, equal] \\
FX \arrow[r, "Ff"'] & FY \arrow[r, hook] & F(gfp[F])
\end{tikzcd}$$

In order to prove the above claim, notice that everything in sight in the following diagram commutes.

$$\begin{tikzcd}
X \arrow[r, "\eta_X"'] \arrow[d, "k"'] \arrow[rrr, bend left, "f"] & WX \arrow[rr, "f^\sharp"] \arrow[dr, "Wf"] \arrow[dd, "{[\rho_X \circ k, \mathsf{id}]}"'] & & V \arrow[dd, equal] \\
& & WV \arrow[ur, "\varepsilon_V"'] & \\
FX \arrow[r, "\rho_X"] \arrow[dr, "Ff"'] & \mathcal{P}_S WX \arrow[rr, "\mathcal{P}_S(f^\sharp)"] \arrow[dr, "\mathcal{P}_S Wf"] & & \mathcal{P}_S V \\
& FV \arrow[r, "\rho_V"'] \arrow[uuurr, hook, bend right] & \mathcal{P}_S WV \arrow[ur, "\mathcal{P}_S \varepsilon_V"'] &
\end{tikzcd}$$

In particular, the outer diagram does commute, hence:

$$\begin{tikzcd}
X \arrow[r, two heads, "f"] \arrow[dd, "k"'] & Y \arrow[dr, hook] & \\
& & V \\
FX \arrow[r, "Ff"'] & FY \arrow[ur, hook] &
\end{tikzcd}$$

Therefore, for all $x \in X$,

$$f(x) = (Ff \circ k)(x)$$

which implies that the image Y of X under f is included in the image of FX under Ff, hence

$$Y \subseteq FY$$

and f is a coalgebra morphism from $\langle X, k\rangle$ to $Y \hookrightarrow FY$.

Therefore, for every F-coalgebra $\langle X, k\rangle$, there exists a coalgebra morphism to $gfp[F] = F(gfp[F])$. Moreover, this morphism is unique. Indeed, the above arguments also show that every coalgebra morphism from $\langle X, k\rangle$ to $gfp[F] = F(gfp[F])$ fits as the right adjunct $(\overline{\rho_X \circ k})^\flat$ of the unique solution of a system of set-equations, hence it is unique. □

Notes. An alternative (but more restrictive) form of the special final coalgebra theorem in the standard category of ordinary sets is presented in (Paulson, 1995).

The special final coalgebra theorem is the 'dual' of the standard fact that least (strict) fixed points of most endofunctors on <u>SET</u> are initial algebras. (Cf (Aczel, 1988, Theorem 7.6).) It gives an elementary way of finding final coalgebras, at the price of assuming anti-foundation. For instance, under foundation, the endofunctor $BX = Act \times X$ has the empty set 0 as the unique strict fixed point, while, under anti-foundation, the empty set is the least fixed point and the set Act^ω of infinite words over the alphabet Act is the greatest fixed point of B: the special final coalgebra theorem tells then that $Act^\omega = Act \times Act^\omega$ is a final B-coalgebra.

Notice that one can prove the (non-strict!) fixed point $Act^\omega \cong Act \times Act^\omega$ is a final B-coalgebra in <u>Set</u>, *independently* of the use of anti-foundation. In general, as shown in (Aczel and Mendler, 1989), endofunctors to which the special final coalgebra theorem applies always have a final coalgebra in the category of ordinary (possibly large) sets. Thus, unless one is really interested in strict fixed points $\widehat{B} = B\widehat{B}$ rather than fixed points up to isomorphism $\widehat{B} \cong B\widehat{B}$, the interest can be shifted from non-well-founded sets and greatest fixed points to ordinary sets and final coalgebras.

6. Partial Orders

When interpreting the programs of a language, one would like, in general, not only to establish whether two programs P, P' are behaviourally equivalent, but also when one program P' is (behaviourally) 'better' than another program P. Write $P \lesssim P'$ for this relation, with the intended meaning that P' can simulate everything P does, but not necessarily vice versa. This 'simulation' relation '$\lesssim$' should be reflexive (ie $P \lesssim P$) and transitive (ie, if $P \lesssim P'$ and $P' \lesssim P''$, then $P \lesssim P''$), that is, it should be a *preorder* for the programs.

The main contribution of this section is a full abstraction theorem with respect to this behavioural preorder. This is obtained by working in an 'order-enriched' setting where the morphism sets are partially ordered and composition yields order-preserving maps.

6.1. *Order-Enriched Categories and Final Coalgebras*

A preorder $\leq$ for a set X is a *partial order* if it is anti-symmetric in the sense that if $x \leq y$ and $y \leq x$ then $x = y$, for all $x, y \in X$. The category <u>**Poset**</u> is the category of partially ordered sets (posets) $\langle X, \leq\rangle$ and monotone functions $f : \langle X, \leq\rangle \to \langle X', \leq'\rangle$, ie order-preserving functions $f : X \to X'$. To ease the notation, let us write X for a poset

$\langle X, \leq \rangle$ and $\leq_X$ for its order; also, whenever possible, let us drop the subscript and write simply $\leq$ for $\leq_X$.

For every two posets X, Y, the set $\underline{\text{Poset}}(X, Y)$ of monotone functions $f : X \to Y$ from X to Y is partially ordered pointwise:

$$f \leq g \iff f(x) \leq_Y g(x) \text{ for all } x \in X$$

That is, the 'hom-sets' of Poset are objects of Poset itself. Moreover, for every two hom-sets $\underline{\text{Poset}}(X, Y)$, and $\underline{\text{Poset}}(Y, Z)$, the composition function

$$\circ : \underline{\text{Poset}}(Y, Z) \times \underline{\text{Poset}}(X, Y) \to \underline{\text{Poset}}(X, Y)$$

is monotone

$$g \leq g',\, f \leq f' \;\Rightarrow\; g \circ f \leq g' \circ f'$$

That is, composition is a morphism of Poset.

Loosely speaking, a category $\underline{C}$ whose hom-sets are objects of a category $\underline{V}$ and whose composition, for every X, Y, Z in $\underline{C}$, is a morphism

$$\circ : \underline{C}(Y, Z) \times \underline{C}(X, Y) \to \underline{C}(X, Z)$$

of $\underline{V}$ is called a **V-enriched category** or, shortly, a **V-category**; when $\underline{C} = \underline{V}$ then $\underline{C}$ is also called **self-enriched**. Notice the assumption that $\underline{V}$ has binary products $\times$. In general, this assumption can be relaxed: it is sufficient to assume that $\underline{V}$ has a 'monoidal' structure. (For more on enriched categories see (Lawvere, 1973; Kelly, 1982; Casley et al., 1991).)

Thus Poset is a self-enriched category. Another example is the category **Cpo** of *complete* partial orders (*cpos*), ie partial orders closed under least upper bounds of ω-chains, and *continuous* functions, ie (monotone) functions which preserve these least upper bounds.

A functor between two $\underline{V}$-enriched categories is a **V-functor** if it is 'locally' a morphism of $\underline{V}$, that is, the action of F on each hom-set is a morphism of $\underline{V}$. Thus, a **Poset-functor** F is a *locally monotone* functor, that is,

$$f \leq g \;\Rightarrow\; Ff \leq Fg$$

and a **Cpo-functor** is a *locally continuous* functor F, that is, a functor such that

$$F(\textstyle\bigsqcup_I f_i) = \textstyle\bigsqcup_I F(f_i)$$

for every ω-chain $\{f_i\}_I$ of continuous functions. Every Cpo-functor is also a Poset-functor.

Initial algebras (and final coalgebras) of Cpo-endofunctors can be obtained by means of the (dual of the) following colimit construction.

Lemma 6.1. (Basic Lemma (Smyth and Plotkin, 1982)) Let F be an endofunctor on a category $\underline{C}$ with an initial object 0. Let Δ be the diagram

$$0 \xrightarrow{0_{F0}} F0 \xrightarrow{F0_{F0}} F^2 0 \xrightarrow{F^2 0_{F0}} \cdots \tag{14}$$

obtained by the iterative application of the functor F to the unique morphism $0_{F0} : 0 \to$

$F0$ given by initiality. If Δ has a colimit and F preserves it

$$F(\mathrm{Colim}(\Delta)) \cong \mathrm{Colim}(F\Delta)$$

then $\mathrm{Colim}(\Delta)$ is the carrier of an initial algebra for F. Its initial algebra structure is obtained by first noticing that the colimiting cocone for Δ (without the first morphism) is a cocone for $F\Delta$ and then taking the unique mediating morphism from $F(\mathrm{Colim}(\Delta))$ to $\mathrm{Colim}(\Delta)$. □

The dual of this lemma is obtained by reversing the morphisms and replacing initial by final, colimit by limit, and algebra by coalgebra.

In (Smyth and Plotkin, 1982) it is shown that every $\underline{\mathrm{Cpo}}$-functor on a $\underline{\mathrm{Cpo}}$-category satisfies the hypotheses of the basic lemma, hence it has an initial algebra. The proof involves the use of the category $\underline{\mathrm{Cpo}}^E$ of 'embedding-projection pairs', ie pairs of morphisms in opposite directions satisfying suitable conditions. The use of this auxiliary category gives also a tool for treating mixed variant-contravariant functors F over $\underline{\mathrm{Cpo}}$, like the function space which is of type $\underline{\mathrm{Cpo}}^{\mathrm{op}} \times \underline{\mathrm{Cpo}} \to \underline{\mathrm{Cpo}}$, as covariant functors $\overline{F^E}$ over $\underline{\mathrm{Cpo}}$.

If F is a $\underline{\mathrm{Cpo}}$-endofunctor, then the corresponding endofunctor F^E on $\underline{\mathrm{Cpo}}^E$ satisfies the hypotheses of the basic lemma and of its dual, hence it has both an initial algebra and a final coalgebra; moreover, the former is the initial algebra for the original endofunctor F on $\underline{\mathrm{Cpo}}$ as well. However, the final coalgebra of F^E is *not* the final F-coalgebra; for this, one need to move to the category $\underline{\mathbf{pCpo}}$ of cpos and *partial* continuous functions: every $\underline{\mathrm{Cpo}}$-endofunctor F on $\underline{\mathrm{pCpo}}$ has both an initial algebra and a final coalgebra, namely the initial algebra and the final coalgebra for the corresponding endofunctor F^E on $\underline{\mathrm{Cpo}}^E = \underline{\mathrm{pCpo}}^E$.

The 'coalgebraic completeness' of $\underline{\mathrm{pCpo}}$ makes of it a good candidate for the order-enriched version of the full abstraction theorem, but we find easier to work with a category of total morphisms which is isomorphic to $\underline{\mathrm{pCpo}}$, namely the category $\underline{\mathbf{Cppo}}_\perp$ of *pointed cpos* (ie cpos with a least element '$\perp$') and *strict* continuous *functions* (ie functions which also preserve the least element). (The above considerations on the construction of initial algebras and final coalgebras in $\underline{\mathrm{pCpo}}$ remain valid in the isomorphic category $\underline{\mathrm{Cppo}}_\perp$.)

When an endofunctor F has both an initial algebra $\psi : F\overline{F} \cong F$ and a final coalgebra $\varphi : F \cong F\widehat{F}$ there is a canonical morphism from the carrier $\overline{F}$ of the initial algebra to the carrier $\widehat{F}$ of the final coalgebra, namely the coinductive extension of the inverse $\psi^{-1} : \overline{F} \cong F\overline{F}$ of the initial algebra isomorphism. If this canonical morphism is itself an isomorphism, then the initial algebra and the final coalgebra are **canonically isomorphic**.

Categories in which all endofunctors have both an initial algebra and a final coalgebra and, moreover, they are canonically isomorphic are called **algebraically compact** in (Freyd, 1992). An example of such a category is $\underline{\mathrm{Cppo}}_\perp$, when regarded as a $\underline{\mathrm{Cpo}}$-category. This can be proved by means of the "*limit-colimit coincidence*" of categories of embedding-projection pairs (Smyth and Plotkin, 1982, Theorem 2). In particular, the one-element set $\{\perp\}$ (with trivial order) is a *null object* in $\underline{\mathrm{Cppo}}_\perp$ (and in the corresponding category of embedding-projections pairs), that is, it is both an initial and a final object; correspondingly, the diagram (14) whose colimit is an initial algebra can be turned into a diagram with the same objects whose limit is a final coalgebra by simply reversing the morphisms.

To summarize:

Theorem 6.1. (Algebraic Compactness) Every $\underline{\mathrm{Cpo}}$-endofunctor on $\underline{\mathrm{Cppo}}_{\perp}$ has both an initial algebra and a final coalgebra and, moreover, they are canonically isomorphic. □

Final coalgebras of $\underline{\mathrm{Cpo}}$-endofunctor can be characterized as the 'maximally final' ones. The definition of maximal finality requires the notion of a 'lax coalgebra morphism' from enriched category theory (Kelly, 1982) which, for the order-enrichment cuts down to the following notion. Let $F : \underline{\mathrm{Poset}} \to \underline{\mathrm{Poset}}$ be an endofunctor on $\underline{\mathrm{Poset}}$ and $\langle X, k\rangle$ and $\langle X', k'\rangle$ two F-coalgebras; a **lax coalgebra morphism** $f : \langle X, k\rangle \to \langle X', k'\rangle$ is a monotone function $f : X \to X'$ such that

$$Ff \circ k \geq k' \circ f$$

Diagrammatically:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "k"'] \arrow[dr, phantom, "\geq"] & X' \arrow[d, "k'"] \\
FX \arrow[r, "Ff"'] & FX'
\end{tikzcd}$$

Thus lax coalgebra morphisms generalize (strict) coalgebra morphisms.

A coalgebra of an endofunctor F on a $\underline{\mathrm{Poset}}$-category is **maximally final** if it is final and, for every F-coalgebra $\langle X, k\rangle$, the unique coinductive extension $k^{@}$ of k is maximal among all lax morphisms from $\langle X, k\rangle$ to the final F-coalgebra. That is, if $\varphi : \widehat{F} \cong F\widehat{F}$,

$$Ff \circ k \geq \varphi \circ f \ \Rightarrow \ k^{@} \geq f$$

Proposition 6.1. Final coalgebras of locally continuous endofunctors are maximally final.

Proof. See (Fiore, 1996b, Proposition 6.7). □

6.2. *Ordered Bisimulations*

The order-enriched generalization of the notion of coalgebraic bisimulation should ensure that (a) the inequality relation on a final coalgebra lifts to a coalgebraic ordered bisimulation and that (b) the inequality relation on a maximal final coalgebra should be maximally-final in the category of coalgebraic ordered bisimulations. This motivates the following definition from (Fiore, 1996b), improving a previous definition in (Rutten and Turi, 1993).

Definition 6.1. (Coalgebraic Ordered Bisimulation) Let $\langle X, k\rangle$ and $\langle Y, \ell\rangle$ be coalgebras of an endofunctor B on a $\underline{\mathrm{Poset}}$-category. A span $\langle R, r_1 : R \to X, r_2 : R \to Y\rangle$

between the carriers X and Y of $\langle X, k\rangle$ and $\langle Y, \ell\rangle$ *lifts to a coalgebraic ordered bisimulation* if there exists a coalgebra structure $\widetilde{R} : R \to BR$ making the first leg r_1 a lax coalgebra morphism and the second leg r_2 a (strict) coalgebra morphism:

$$\begin{tikzcd}
& R \arrow[dl, "r_1"'] \arrow[d, "\widetilde{R}"] \arrow[dr, "r_2"] & \\
X \arrow[d, "k"'] \arrow[r, phantom, "\not\leq"] & BR \arrow[dl, "Br_1"] \arrow[dr, "Br_2"'] & Y \arrow[d, "\ell"] \\
BX & & BY
\end{tikzcd}$$

$\square$

In an order-enriched category, the internal inequality $\mathrm{IQ}(X)$ of an object X can be defined by means of the following order-enriched version of the notion of pullbacks.

Definition 6.2. The **ordered pullback** of a pair $f : X \to Z$ and $g : Y \to Z$ of morphisms in a Poset-category having the same codomain is a span $\langle P, p : P \to X, q : P \to Y\rangle$ such that $f \circ p \leq g \circ q$ and for every span $\langle P', p' : P' \to X, q' : P' \to Y\rangle$ such that $f \circ p' \leq g \circ q'$ there exists a unique morphism $u : P' \to P$ such that $p' = p \circ u$ and $q' = q \circ u$. Diagrammatically:

$$\begin{tikzcd}
& P' \arrow[ddl, bend right, "p'"'] \arrow[d, dashed, "u"] \arrow[ddr, bend left, "q'"] & \\
& P \arrow[dl, "p"] \arrow[dr, "q"'] & \\
X \arrow[dr, "f"'] \arrow[rr, phantom, "\leq"] & & Y \arrow[dl, "g"] \\
& Z &
\end{tikzcd}$$

(Cf Diagram (6) in Section 4.) $\square$

(Ordered pullbacks are instances of the 'comma-objects' of (Kelly, 1989, §4).)

In Poset (and $\mathrm{Cppo}_\perp$), the ordered pullback of two morphisms $f : X \to Z$ and $g : Y \to Z$ is given by the set

$$\{<x, y> \in X \times Y \mid f(x) \leq g(y)\}$$

ordered componentwise ($<x, y> \leq <x', y'> \iff x \leq x',\ y \leq y'$). Thus, in particular, the **internal inequality** of a poset (or cppo) X

$$\mathrm{IQ}(X) = \{<x, x'> \in X \times X \mid x \leq x'\}$$

is the ordered kernel pair of the identity on X:

$$
\begin{tikzcd}
& \mathrm{IQ}(X) \arrow[dl, "i_1"'] \arrow[dr, "i_2"] & \\
X \arrow[rr, phantom, "\leq"] \arrow[dr, equal] & & X \arrow[dl, equal] \\
& X &
\end{tikzcd}
$$

Proposition 6.2. (Fiore, 1996b) For every coalgebra, the internal inequality lifts to a coalgebraic ordered bisimulation.

Proof. The proof is very much the same as that of Proposition 4.2; the coalgebra structure for $\mathrm{IQ}(X)$ is the one obtained by replacing $e_1 = e_2 : \mathrm{EQ}(X) \to X$ with $i_2 : \mathrm{IQ}(X) \to X$. Notice that the fact that i_1 needs only to be lax with respect to the resulting coalgebra allows for the proposition to hold. $\square$

Proposition 6.3. The internal inequality of the maximally-final coalgebra lifts to the maximally-final ordered bisimulation.

Proof. Let us adapt the proof of Proposition 4.3. Firstly, replace bisimulations by ordered bisimulations and take f and g in (7) to be *lax* coalgebra morphisms. Secondly, notice that replacing $\mathrm{EQ}(\widehat{B})$ by $\mathrm{IQ}(\widehat{B})$ in (8) one still has, by the universal property of ordered pullbacks, a unique mediating morphism $u : R \to \mathrm{IQ}(\widehat{B})$ from R to the internal inequality of the final coalgebra $\mathrm{IQ}(\widehat{B})$; this shows that $\mathrm{IQ}(\widehat{B})$ is final in the category of coalgebraic ordered bisimulations. It remains to prove maximality. For any other pair of lax morphisms f and g from $\langle X, k\rangle$ and $\langle Y, \ell\rangle$ to the final coalgebra one has a mediating morphism $v : R \to \mathrm{IQ}(\widehat{B})$ and, by Proposition 6.1, that $f \leq k^{@}$ and $g \leq \ell^{@}$. Then:

$$
\left.\begin{array}{rcl}
i_1 \circ v = f \circ r_1 & \leq & k^{@} \circ r_1 = i_1 \circ u \\
i_2 \circ v = g \circ r_2 & \leq & \ell^{@} \circ r_2 = i_2 \circ u
\end{array}\right\} \Rightarrow \; v \leq u
$$

because, as one can check, the two legs of the inequality are jointly order monic. $\square$

6.3. *Order-Enriched Final Coalgebra Semantics*

Following §4.2, if ordered pullbacks lift to ordered bisimulations of an endofunctor B on $\underline{\mathrm{Cppo}}_{\perp}$ then the *final ordered bisimulation* between two B-coalgebras $\langle X, k\rangle$ and $\langle Y, \ell\rangle$ exists and it is the ordered pullback of the coinductive extensions $k^{@}$ and $\ell^{@}$. Let us write '$\lesssim$' for this ordered bisimulation (omitting the superscript k, ℓ). Then, for every $x \in X$ and $y \in Y$,

$$
x \lesssim y \iff k^{@}(x) \leq \ell^{@}(y)
$$

(The implication from left to right follows from Proposition 6.3, while the converse follows from the assumption that ordered pullbacks lift to ordered B-bisimulations.) Semantically:

Theorem 6.2. (Ordered Full Abstraction) Let $[\![\text{-}]\!] : \mathit{Prog} \to B(\mathit{Prog})$ be a coalgebraic operational model of a language with programs $P \in \mathit{Prog}$ and behaviour B :

$\underline{\mathrm{Cppo}}_{\perp} \to \underline{\mathrm{Cppo}}_{\perp}$ such that ordered pullbacks lift to ordered B-bisimulations. Then for all $P, P' \in \mathit{Prog}$,

$$P \lesssim P' \iff [\![P]\!]^{@} \leq [\![P']\!]^{@}$$

□

6.4. *An Example: Deterministic Transition Systems with Divergence*

Let us give an example of fully abstract final coalgebra semantics in this order-enriched setting.

Consider the endofunctor

$$BX = 1_{\perp} + (Act_{\perp} \otimes X_{\perp}) \tag{15}$$

on $\underline{\mathrm{Cppo}}_{\perp}$, where: the endofunctor $X \mapsto X_{\perp}$ is the lifting functor adding a new least element to a cppo; the functor '$\otimes$' is the smash product of cppos (obtained from the cartesian product by identifying all pairs containing a bottom); the functor'+' is the categorical coproduct of $\underline{\mathrm{Cppo}}_{\perp}$.

The final coalgebra of the above endofunctor B is the set of finite, infinite, and *partial* words, the latter being finite words ending by $\perp$, like, eg, $aba\perp$. There is no relation between complete words, but any word w is greater than a partial word $w'\perp$ if w extends w' (ie $w = w'w''$ for some non-empty word w''); for example, $ab\perp < aba\perp < abaa$. In particular, if Act is a singleton, then the final coalgebra of the above behaviour is given by the 'lazy' or 'oblique natural numbers'. (Cf (Freyd, 1990).)

Next, consider deterministic transition systems in which a state, beside performing an action or being inert, can diverge. These can be modelled as coalgebras of the above endofunctor as follows. Regard the set X of states of the transition system as a flat cppo (thus possibly adjoining a least element $\perp_X$). Notice that, for every cppo X, in $\underline{\mathrm{Cpo}}$ (or $\underline{\mathrm{Set}}$) one can prove the isomorphism

$$B(X) = 1_{\perp} + Act_{\perp} \otimes X_{\perp} \cong (1 + Act \times X)_{\perp}$$

where the product and coproduct inside the parentheses are in $\underline{\mathrm{Cpo}}$ rather than in $\underline{\mathrm{Cppo}}_{\perp}$. Let us denote the least element of BX as $\perp_{\mathrm{new}}$. One can then model deterministic transition systems with divergence as strict functions $k : X \to (1 + Act \times X)_{\perp}$ by putting $k(\perp_X) = \perp_{\mathrm{new}}$ and, for every other x in X,

$$k(x) = \begin{cases} \perp_{\mathrm{new}} & \text{if } x \text{ diverges} \\ * & \text{if } x \text{ is inert} \\ < a, x' > & \text{if } x \xrightarrow{a} x' \end{cases}$$

An ordered bisimulation R between the carriers X and Y of two such transition systems satisfies, for all xRy,

1 if x is inert (ie $k(x) = *$) then also y is inert and
2 if $x \xrightarrow{a} x'$ (ie $k(x) = < a, x' >$) then $y \xrightarrow{a} y'$ and $x'Ry'$
3 if x does not diverge (ie $k(x) \neq \perp_{\mathrm{new}}$) then
   (a) if y is inert then also x is inert and

(b) if $y \xrightarrow{a} y'$ then $x \xrightarrow{a} x'$ and $x' R y'$

4 if x diverges (ie $k(x) = \perp$) and $y \xrightarrow{a} y'$ then $x' R y'$ for some x' in X.

Conversely, every span R between the carriers X and Y of two (coalgebras modelling) deterministic transition systems with divergence which satisfies the properties 1-4 above is an ordered bisimulation for the behaviour endofunctor. These are known, in the non-deterministic setting, as *partial bisimulations* (Abramsky, 1991).

An example of an ordered B-bisimulation is the ordered pullback (in $\underline{\mathrm{Cppo}}_{\perp}$) of two coalgebra homomorphisms $f : \langle X, k\rangle \to \langle Z, h\rangle$ and $g : \langle Y, \ell\rangle \to \langle Z, h\rangle$. Indeed, a B-coalgebra structure lifting the ordered pullback R to an ordered B-bisimulation can be defined by putting, for every pair $<x, y>$ in R,

$$<x,y> \mapsto \begin{cases} * & \text{if } k(x) = \perp_{\text{new}},\ \ell(y) = * \\ <a, \perp_X, y'> & \text{if } k(x) = \perp_{\text{new}},\ \ell(y) = <a, y'> \\ \perp_{\text{new}} & \text{if } k(x) = \perp_{\text{new}} = \ell(y) \\ * & \text{if } k(x) = * = \ell(y) \\ <a, x', y'> & \text{if } k(x) = <a, x'>,\ \ell(y) = <a, y'> \end{cases}$$

Notice that these are all the case distinctions which can occur. For instance, $k(x) \neq \perp_{\text{new}} = \ell(y)$ is not possible because

$$(1 + Act \times f)_{\perp}(k(x)) = h(f(x)) \leq h(g(y)) = (1 + Act \times g)_{\perp}(\ell(y))$$

In particular, one can take f and g to be the coinductive extensions of k and ℓ, respectively. Therefore, by Theorem 6.2, the final coalgebra semantics of deterministic programs with divergence is fully abstract with respect to partial bisimulation.

7. Metric Spaces

One of the advantages of working in an order-enriched setting is that one can verify not only the equality between computations, but also when a computation is better than another in the sense that it contains more information. For instance, under the standard prefix ordering on words, a computation represented by the $a_1 \cdots a_n b$ is better than one represented by the word $a_1 \cdots a_n$. Still, in the order-enriched setting one is not able to compare computations which share a common prefix but then differ, like, eg, $a_1 \cdots a_n b$ and $a_1 \cdots a_n c$. In order to do this, one can introduce a natural notion of *distance* between computations.

The intuition is that the further two computations (programs) are (behave) the same, the smaller their distance should be. In the deterministic setting, with behaviours of type $BX = 1 + Act \times X$ (cf §2.2), one can say that the distance between two words is 2^{-n}, for n the length of the longest common prefix between the two words. This leads to the use of (ultra) metric spaces in semantics.

A *metric space* is a set X together with a *distance* function $d_X : X \times X \to [0, \infty]$ mapping each two elements of X to a non-negative real number. (It is convenient to include ∞ in the range of this distance function, even though one can always down-scale distances to the interval $[0, 1]$.) Further, a distance has to satisfy the following three axioms:

1 $d_X(x,y) = 0 \iff x = y$
2 $d_X(x,z) \leq d_X(x,y) + d_X(y,z)$
3 $d_X(x,y) = d_X(y,x)$

The second axiom is called the *triangular inequality.* When a stronger form of triangular inequality holds, namely $d_X(x,z) \leq \max\{d_X(x,y), d_X(y,z)\}$, then $\langle X, d_X \rangle$ is an *ultra-metric space.* In semantics one usually works with this more specific structures, but it is convenient to state the general theorems in the full generality of metric spaces.

In metric spaces, the mathematical tool most relevant for semantics available is Banach's Theorem. This theorem ensures the existence of a *unique* fixed point for a large class of endofunctions on *complete metric spaces*, that is, on metric spaces closed under limits of 'Cauchy sequences'. The endofunctions at stake are the **contractive** ones, that is, functions $f : X \to X$ such that, for some $0 \leq \epsilon < 1$,

$$d(f(x), f(x')) \leq \epsilon \cdot d(x, x')$$

for all $x, x' \in X$. (Here the subscript X is omitted from d_X; the same is done in the sequel whenever possible.)

Theorem 7.1. (Banach) Every contractive endofunction f on a complete metric space has a unique fixed point $\mathit{fix}(f) = f(\mathit{fix}(f))$. □

Banach's Theorem carries over to the higher level of categories and functors as follows. Consider the category **<u>Metric</u>** with objects metric spaces and with morphisms the **non-expansive functions** between them, that is, the functions $f : X \to Y$ such that

$$d_Y(f(x), f(x')) \leq d_X(x, x')$$

for all $x, x' \in X$.

Observation 7.1. The set $\underline{\text{Metric}}(X,Y)$ of non-expansive functions between two metric spaces is itself a metric space, with distance

$$d(f,g) = \sup_{x \in X}\{d_Y(f(x), g(x))\} \tag{16}$$

for all non-expansive functions $f, g : X \to Y$; in other words, the hom-sets of <u>Metric</u> are objects of <u>Metric</u> itself. Similarly, one has that in the subcategory **<u>Cms</u>** of *complete metric spaces* and non-expansive functions the hom-sets are themselves complete metric spaces, with (16) as distance.

Therefore, one can define an endofunctor F on <u>Metric</u> (or on the subcategory <u>Cms</u>) to be **locally contractive** if, for some $0 \leq \epsilon < 1$,

$$d(F(f), F(g)) \leq \epsilon \cdot d(f,g)$$

for all metric spaces X, Y, and for all pairs of parallel non-expansive functions $f, g : X \to Y$ between them.

Theorem 7.2. Every locally contractive endofunctor F on the category of complete metric spaces and non-expansive functions has a unique fixed point $\mathit{Fix}(F) \cong F(\mathit{Fix}(F))$.

Proof. The proof is based on (America and Rutten, 1989), where a category $\underline{\text{Cms}}^E$ of complete metric spaces is considered with as morphisms (a metric version of Smyth and

Plotkin's) 'embedding-projection' pairs. (Cf §6.1.) Then a notion of 'contractivity' for endofunctors on that category is defined, and it is shown that contractive functors have a unique fixed point. The present theorem follows from the observation (due to Gordon Plotkin) that every locally contractive functor on $\underline{\text{Cms}}$ induces a contractive functor on $\underline{\text{Cms}}^E$. (For details see (Rutten and Turi, 1993).) $\square$

As shown by the following proposition, the above theorem gives a foundation for both induction and coinduction.

Proposition 7.1. Every fixed point $\mathit{Fix}(F) \cong F(\mathit{Fix}(F))$ of a locally contractive functor $F : \underline{\text{Cms}} \to \underline{\text{Cms}}$ is both an initial algebra and a final coalgebra for F.

Proof. Let $\varphi : \mathit{Fix}(F) \cong F(\mathit{Fix}(F))$ be the F-coalgebra structure given by the fixed point isomorphism and let $\langle X, k\rangle$ be an F-coalgebra. Define $\Phi : \underline{\text{Cms}}(X, \mathit{Fix}(F)) \to \underline{\text{Cms}}(X, \mathit{Fix}(F))$ by, for all f,

$$\Phi(f) = \varphi^{-1} \circ F(f) \circ k$$

Suppose F is locally contractive with factor ϵ. Then Φ is a contraction with factor ϵ: for all $f_1, f_2 \in \underline{\text{Cms}}(X, \mathit{Fix}(F))$,

$$\begin{array}{rcl}
d\,(\Phi(f_1), \Phi(f_2)) & = & \sup\limits_{x\in X}\{d_{\mathrm{Fix}(F)}(\Phi(f_1)(x), \Phi(f_2)(x))\} \\
& = & \sup\limits_{x\in X}\{d_{\mathrm{Fix}(F)}((\varphi^{-1}\circ F(f_1)\circ k)(x), (\varphi^{-1}\circ F(f_2)\circ k)(x))\} \\
& \leq & \sup\limits_{y\in F(X)}\{d_{\mathrm{Fix}(F)}((\varphi^{-1}\circ F(f_1))(y), (\varphi^{-1}\circ F(f_2))(y))\} \\
& \leq & \sup\limits_{y\in F(X)}\{d_{\mathrm{Fix}(F)}(F(f_1)(y), F(f_2)(y))\}\ (\varphi^{-1} \text{ is non-expansive}) \\
& = & d(F(f_1), F(f_2)) \\
& \leq & \epsilon \cdot d(f_1, f_2)\ (F \text{ is locally contractive}).
\end{array}$$

By Banach's theorem Φ has a unique fixed point $\mathit{fix}(\Phi) : X \to \mathit{Fix}(F)$. Moreover:

$$\varphi \circ \mathit{fix}(\Phi) = \varphi \circ \Phi(\mathit{fix}(\Phi)) = \varphi \circ \varphi^{-1} \circ F(\mathit{fix}(\Phi)) \circ k = F(\mathit{fix}(\Phi)) \circ k,$$

which shows that $\mathit{fix}(\Phi)$ is a coalgebra morphism from $\langle X, k\rangle$ to $\langle \mathit{Fix}(F), \varphi\rangle$. Since any such coalgebra morphism is also a fixed point of Φ, $\mathit{fix}(\Phi)$ is also the unique coalgebra morphism, which shows that $\langle \mathit{Fix}(F), \varphi\rangle$ is a final F-coalgebra. A dual argument shows that $\langle \mathit{Fix}(F), \varphi^{-1}\rangle$ is an initial F-algebra. $\square$

The proof of the above proposition does not require that the metric space X be complete, because for the hom-set $\underline{\text{Metric}}(X, Y)$ to be complete it is sufficient that Y be complete. Therefore:

Corollary 7.1. A final coalgebra (initial algebra) of a locally contractive endofunctor F on $\underline{\text{Cms}}$ which extends to an endofunctor F' on $\underline{\text{Metric}}$ is also a final F'-coalgebra (initial F'-algebra) in $\underline{\text{Metric}}$. $\square$

As a consequence, one can consider metric operational models which are not complete and still have a final coalgebra semantics. Similarly, bisimulations in the metric setting need not be complete objects.

7.1. *Metric-Enriched Categories*

The categories Metric and Cms are both self-enriched categories in the sense of §6.1. The corresponding notion of a Metric-functor however is not local contractivity but the milder one of 'locally non-expansiveness', which comprehends all functors used in metric semantics. However, every locally non-expansive endofunctor can be turned into a locally contractive one by composing it with a trivial 'shrinking' endofunctor.

The categorical product of two metric spaces $\langle X, d_X \rangle$ and $\langle Y, d_Y \rangle$ is the cartesian product of their carriers equipped with the distance

$$d\,(<x, x'>, \ <y, y'>) = \max\{d_X(x, x'), \ d_Y(y, y')\}$$

However, the composition of non-expansive functions is not non-expansive with respect to the categorical product of hom-sets. Therefore, a different notion of distance is needed, namely

$$d\,(<x, x'>, \ <y, y'>) = d_X(x, x') + d_Y(y, y')$$

Equipped with this distance, the cartesian product $X \times Y$ of the carriers is called the *tensor product* $\langle X, d_X \rangle \otimes \langle Y, d_Y \rangle$ of two metric spaces. (This tensor product is associative and commutative, and it has the singleton metric space 1 as a unit, making of Cms a '(symmetric) monoidal category' in the sense of (Mac Lane, 1971, §VII.1).) One can check that the composition of non-expansive functions, viewed, for all metric spaces X, Y, Z, as a function

$$\circ : \underline{\mathrm{Metric}}(Y, Z) \otimes \underline{\mathrm{Metric}}(X, Y) \to \underline{\mathrm{Metric}}(X, Z)$$

is non-expansive. The same holds for its restriction to Cms. Therefore, because of Observation 7.1, the categories Metric and Cms are Metric-enriched categories and Cms is Cms-enriched. A **Metric-endofunctor** F on a Metric-category is then an endofunctor such that

$$d\,(F(f), F(g)) \leq d\,(f, g) \tag{17}$$

for all metric spaces X, Y, and for all pairs of parallel non-expansive functions $f, g : X \to Y$ between them. In other words, the Metric-endofunctors are the '**locally non-expansive**' ones.

Definition 7.1. (Down-Scalers and Shrunk Functors) The *down-scaler endofunctor* $Id_\epsilon : \underline{\mathrm{Metric}} \to \underline{\mathrm{Metric}}$, with scaling factor $0 \leq \varepsilon < 1$, is the functor which 'shrinks' the distance d of a metric space by mapping it to $\epsilon \cdot d$, while on carriers and on morphisms it behaves as the identity:

$$Id_\epsilon \langle X, d \rangle = \langle X, \epsilon \cdot d \rangle$$

A *shrunk endofunctor* $F : \underline{\mathrm{Metric}} \to \underline{\mathrm{Metric}}$ is an endofunctor which can be decomposed either as $F' \circ Id_\epsilon$ or as $Id_\epsilon \circ F'$, for some endofunctor F' and some $0 \leq \epsilon < 1$. □

Clearly, every shrunk locally non-expansive endofunctor on Metric (and on Cms) is locally contractive, therefore, by Theorem 7.2 and Proposition 7.1, the following holds.

Theorem 7.3. Every shrunk Metric-endofunctor on Cms has both an initial algebra and a final coalgebra and, moreover, they are canonically isomorphic. □

(Cf Theorem 6.1.)

For example, consider the behaviour endofunctor $BX = 1 + Act \times X$ on Set. It extends to a Metric-endofunctor on Cms by imposing the *discrete bounded ultrametric* on Act (ie $d(a, a') = 0$ if $a = a'$ and $d(a, a') = 1$ if $a \neq a'$). Notice that products and coproducts of spaces with discrete metric are like in Set. By precomposing this behaviour with the (typical) down-scaler $Id_{\frac{1}{2}}$ one obtains the shrunk Cms-endofunctor $B\langle X, d\rangle = 1 + Act \times \langle X, \frac{1}{2}d\rangle$. One can check that the carrier of its final coalgebra and initial algebra given by the above theorem is the set $Act^\infty = Act^* \cup Act^\omega$ of finite and infinite words over the alphabet Act. Its distance is the one described at the beginning of this section on metric spaces, namely two words have distance 2^{-n} if n is the length of the longest common prefix between them.

7.2. *Reconciling Partial Orders with Metric Spaces*

Structures which combine the qualitative (asymmetric) information given by partial orders with the quantitative information given by (symmetric) metric spaces are the **quasimetric spaces**, that is, metric spaces in which the (third) axiom stating the symmetry of the distance functions is omitted. Thus a quasimetric space is a set X equipped with a function $d : X \times X \to [0, \infty]$ such that, for all x, y, and z in X,

1 $d(x, y) = 0$ and $d(y, x) = 0 \iff x = y$

2 $d(x, z) \leq d(x, y) + d(y, z)$

Clearly, the category **QMetric** of quasimetric spaces and non-expansive functions contains Metric as a full subcategory. Moreover, thanks to the asymmetry of the distance, the category Poset of partial orders and monotone functions fully and faithfully embeds in QMetric. This embedding maps a partial order $\leq$ on a set X to the following distance $d_\leq$ on X.

$$d_\leq(x, y) = \begin{cases} 0 & \text{if } x \leq y \\ \infty & \text{otherwise} \end{cases}$$

Monotone functions between partial orders are non-expansive functions with respect to their corresponding quasi-metrics.

A crucial contribution to the theory of quasimetric spaces has been the introduction in (Smyth, 1988) of notions of limit and completeness for quasimetric spaces generalizing both the notions of least upper bound of an ω-chain and of (ordinary) metric limit of a Cauchy sequence. The corresponding category **Cqms** of *complete* quasimetric spaces has as morphisms the non-expansive functions which are *continuous* in the sense that they preserve Cauchy sequences and the corresponding limits. Notice that for metric spaces, non-expansive functions are always continuous, while for quasimetric spaces the two notions are incomparable. (See (Rutten, 1996, Remark 3.6).) Also contractive functions are not necessarily continuous and Banach's Theorem holds for complete quasimetric spaces only under the additional assumption that the endofunction at stake is continuous.

Like for Cms, Banach's Theorem carries over to the level of Cqms (Rutten, 1996, Theorem 7.3). This allows for the following generalization of Theorem 7.3. First notice that the category Cqms is, like Cms, self-enriched (with respect to the same tensor product as for ordinary metric spaces), thus a **Cqms-endofunctor** on $\underline{\text{Cqms}}$ is an endofunctor which is both locally non-expansive and locally continuous. Then:

Theorem 7.4. Every shrunk $\underline{\text{Cqms}}$-endofunctor on $\underline{\text{Cqms}}$ has both an initial algebra and a final coalgebra and, moreover, they are canonically isomorphic.

Proof. See (Rutten, 1996, Theorem 7.3). □

It is an open question whether by, on the one hand, removing the shrunkness condition and by, on the other hand, working in the category of *pointed* complete quasi metric spaces and *strict* continuous and non-expansive functions, one would still have the same result, thus generalizing Theorem 6.1.

Generalized Metric Spaces. Smyth's notion of Cauchy sequence for quasimetric spaces has been generalized both in (Flagg and Kopperman, 1997) and in K.R. Wagner's thesis (Wagner, 1994). Wagner's notion of limit is made parametric in a *quantale* Ω: for Ω equal to the two-elements lattice $\bot \leq \top$, it specializes to the standard notion of an ω-chain in a partial order; for Ω equal to $[0, \infty]$ it specializes to Smyth's Cauchy sequence. Wagner use the parameter Ω to generalize the standard constructions of initial algebras (alias final coalgebras) for $\underline{\text{Cppo}_\bot}$ and $\underline{\text{Cms}}$.

Wagner's work is based on a seminal article by Lawvere (1973), where metric spaces, partial orders, and categories are all shown to be instances of the notion, described in §6.1, of a $\underline{\text{V}}$-category, for different (monoidal) categories $\underline{\text{V}}$.

Metric spaces are obtained by putting $\underline{\text{V}} = [0, \infty]$. The morphisms of $[0, \infty]$ are given by its reverse order, that is, $r \to r'$ if and only if $r \geq r'$; the tensor product is the sum of reals. Then, a $[0, \infty]$-category is a **generalized metric space**, that is, a set X equipped with a function $d : X \times X \to [0, \infty]$ such that, for all x, y, and z in X,

1' $d(x, x) = 0$

2 $d(x, z) \leq d(x, y) + d(y, z)$

Notice that two elements with distance 0 need not to be equal. Clearly, every metric space is also a generalized metric space. Interestingly, if the 'max' operator rather than the sum between reals is taken as tensor of $[0, \infty]$, one has that a $[0, \infty]$-category is a generalized *ultrametric* space, with the max operator replacing the sum in the triangular inequality axiom.

Partial orders are obtained by putting $\underline{\text{V}} = \underline{2}$, the category with two elements $\bot$ and $\top$ and only one non-identity morphism, namely $\bot \leq \top$. With the logical 'and' as the tensor product of $\underline{2}$, a $\underline{2}$-category is then nothing but a pre-order.

Next, the notion of a $\underline{\text{V}}$-functor specializes to that of a monotone function for $\underline{\text{V}} = \underline{2}$ and to that of a non-expansive function for $\underline{\text{V}} = [0, \infty]$. Thus, at a higher level, a $\underline{\text{Poset}}$-functor can be seen as a locally $\underline{2}$-functor and, similarly, a $\underline{\text{Metric}}$-functor can be seen as a locally $[0, \infty]$-functor.

The conceptual advantage of regarding (generalized) metric spaces as $[0, \infty]$-categories goes well beyond providing a unifying framework. For instance, in $\underline{\text{V}}$-category one has the notion of a *bimodule* and, as argued in (Lawvere, 1973), one can regard bimodules as being '$\underline{\text{V}}$-valued relations'. In particular, a bimodule R between two metric spaces X

and Y (regarded as $[0,\infty]$-categories) is a function

$$R : X \times Y \to [0,\infty]$$

satisfying suitable conditions (Lawvere, 1973, page 159). The value of R at a pair (x,y) is the "truth value of the R-relatedness of x to y". With such a notion of $[0,\infty]$-valued relations, one can then define a new notion of **metric bisimulation** as follows. Consider, for simplicity the deterministic transition systems given by the coalgebras in $\underline{\text{Metric}}$ of the behaviour

$$B = 1 + (Act \times Id_{\frac{1}{2}}) : \underline{\text{Metric}} \to \underline{\text{Metric}}$$

Then a $[0,\infty]$-valued relation R on the states of one such deterministic transition system $k : X \to BX$ could be called a 'metric bisimulation' if it satisfies:

$$xRy = \begin{cases} 0 & \text{if } k(x) = * = k(y) \\ \frac{1}{2} \cdot \epsilon & \text{if } k(x) = <a, x'>,\ k(y) = <a, y'>,\ x'Ry' = \epsilon \\ 1 & \text{otherwise} \end{cases}$$

The idea is that $xRy = 2^{-n}$ if for n transition steps x and y behave the same and then not anymore. In particular, $xRy = 0$ if x and y are bisimilar in the ordinary sense.

A coalgebraic generalization of the above notion of a metric bisimulation is outside the scope of the present work.

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