V

Sets like Recursive Processes

Synopsis

This part is devoted to a coalgebraic presentation of Peter Aczel's theory of "non-well-founded sets" [Acz88]. A categorical duality is proved between the 'anti-foundation axiom' giving non-well-founded sets and the 'foundation axiom': it is shown that the former is equivalent to postulating that 'the universe $V = \mathcal{P}_S V$ is a final coalgebra', while the latter is equivalent to '$V = \mathcal{P}_S V$ is an initial algebra'. (The endofunctor $\mathcal{P}_S$ maps a class to the class of its (small) subsets.)

The semantic motivation for the use of anti-foundation is that it permits to prove the "Special Final Coalgebra Theorem" [Acz88] which states that, under mild assumptions, the greatest fixed point of an endofunctor on (possibly non-well-founded) sets is a final coalgebra.

The special final coalgebra theorem is stated in terms of the "Solution Lemma" [Acz88]. The final coalgebra presentation of anti-foundation adopted here renders this lemma (and its equivalence with anti-foundation) trivial. Correspondingly, the 'uniformity on maps' condition which an endofunctor has to satisfy in order for the special final coalgebra theorem to hold can be formulated in a more transparent way than in [Acz88].

Note. A preliminary version of this part has appeared as [RT93, §4].

Basic Set Theory

One way of understanding the abstract notion of *set* is as a collection x such that its elements have "no internal structure whatsoever" and x itself has "no internal structure except for equality and inequality of pairs of elements". (Cf [Law76, page 119].) Axiomatically, this corresponds to taking the membership relation '$\in$' as the only primitive notion of set theory and to postulating the following 'extensionality axiom', the first axiom of set theory.

Extensionality:

Two sets are equal iff they have the same elements.

Next, for every property P in a (first-order) language with membership and equality only, one would like the collection $\{x \mid \mathsf{P}(x)\}$ of sets which have the property P to be a set. However, Russel's paradoxical set $\{x \mid x \notin x\}$ shows that this 'strong comprehension axiom' cannot be stated in its full generality. One needs to consider properties *relative* to the elements of an already defined set. This leads to the 'comprehension axiom', the second axiom of set theory.

Comprehension:

For every property P and every set v, the collection

$$\{x \mid \mathsf{P}(x) \wedge x \in v\}$$

is a set.

As comprehension can be applied only to members of already defined sets, it is necessary to postulate the existence of some sets, either primitive or derived by applying some basic operators:

Empty Set:

There exists a set 0 with no elements.

Paring, Union, Power Set:

$\{x, y\}$, $\bigcup x$, $\mathcal{P}(x)$ are all sets, for x, y sets.

As usual, $\bigcup x$ and $\mathcal{P}(x)$ stand for the collection of all members of members of x and the collection of all subsets of x, respectively. In turn, the subset relation '$\subseteq$' can be derived from the membership relation:

$$x \subseteq y \iff \forall v\ (v \in x \Rightarrow v \in y)$$

By means of the union operator one can define an operator s acting as successor as follows: $s(x) = x \cup \{x\}$. The existence of an infinite set can be stated by postulating the existence of a set containing the natural numbers. That is:

Infinity:

> There exists a set containing 0 and closed under the successor operator s.

(The axioms above, as well as those given in the sequel, are written for convenience in natural language but note that they can also be expressed in the language of set theory – see, eg, [Lev79].)

Further useful notions can be derived from the above axioms, like, for instance, the notion of *ordered pair*:

$$<x, y> = \{x, \{x, y\}\}$$

A formal definition of function can then be given as a collection f of ordered pairs such that for every x there exists a unique y with $<x, y> \in f$. Two more axioms about functions are then usually added:

Replacement:

> The image of a set under a function is a set.

Choice:

> Every surjective function has a 'right inverse'.

A *right inverse* for a function $f : a \to b$ is a function $g : b \to a$ such that $f \circ g$ is the identity on b. The above axiom of choice is equivalent to postulate that for every set a there exists a *choice function*, that is, a function f such that, for every $x \in a$, $f(x) \in x$.

The above axioms (extensionality, comprehension, empty set, pairing, union, power set, infinity, replacement, choice) are the basic axioms of set theory; let us call the theory associated with (ie, the collection of all sentences derivable from) them **basic set theory** and the corresponding category of sets and functions **Set**. (Basic set theory is usually called ZFC^- in the literature – see, eg, [Lev79].)

Classes

Even though the collection $\{x \mid \mathsf{P}(x)\}$ of all sets x having a given property P might not be a set, it can still be of interest for set theory. Such 'specifiable' collections are called **classes**. Clearly, a set is a class, but the converse is not true, in which case one speaks of a **proper class**. (Also the terminology '*large set*', vs '*small set*', is used.) In the sequel, lower case letters are used for (small) sets and capital letters for classes.

The equality between classes is determined by their *small* elements. That is, two classes $X = \{x \mid \mathsf{P}(x)\}$ and $Y = \{x \mid \mathsf{P}'(x)\}$ are equal if and only if P and P' hold for the same (small) sets.

An example of a proper class is the **universe of sets**, namely the collection of all sets:

$$V \ = \ \{x \mid x = x\}.$$

(Since the property $x = x$ trivially holds for all sets, the class V is the collection of all sets indeed.) Notice that different properties may specify the same class. For instance, any property other than '$x = x$' which holds for all sets can be used to specify the universe.

Next, let **SET** be the category of classes and (class) functions corresponding to basic set theory. The claim is that the universe V can be seen as the carrier of both an algebra and a coalgebra structure of a suitable power-set endofunctor $\mathcal{P}_S$ on **SET**.

Recall, from Section 10, that semi-lattices with sets as carriers and with arbitrary sets of joins give rise (by adjunction) to the power-set endofunctor $\mathcal{P} : \mathbf{Set} \to \mathbf{Set}$ and that, similarly, semi-lattices with finite joins give rise to the finite power-set functor $\mathcal{P}_{\text{fi}} : \mathbf{Set} \to \mathbf{Set}$. By considering semi-lattices with classes as carriers and joins of sets of classes one obtains then the following endofunctor on **SET**:

$$\mathcal{P}_S : \mathbf{SET} \to \mathbf{SET} \qquad\qquad X \mapsto \{x \mid x \text{ is a set } \wedge \ x \subseteq X\}$$

Notice that only (small) sub*sets* are taken into consideration. This makes possible that V be a fixed point of the power-set functor (which, by cardinality reasons, would not be the case if one would consider the collection of all sub*classes* of a given class):

> **The universe V is a fixed point** $V = \mathcal{P}_S V$.
>
> Indeed, V is the largest class. Thus, since $\mathcal{P}_S V$ is itself a class, $\mathcal{P}_S V \subseteq V$. For the converse it is sufficient to prove that every set x is a subset of V. That is, for every $y \in x$, y is also in V. This is immediate from the fact that y is a set.

Therefore, the identity on V can be seen both as a $\mathcal{P}_S$-algebra and as a $\mathcal{P}_S$-coalgebra structure for V.

Well-Founded Sets and Foundation

From the axioms of basic set theory alone it is not possible to draw a canonical picture of what the universe looks like, a picture independent of the specific interpretation one might give to the theory. This was felt as a problem already in the early developments of set theory. The solution was found in the '*foundation axiom*', which was then added to basic set theory. This axiom restricts the universe to the 'smallest' of all possible ones. Then the picture arises of a universe in which sets are hereditarily constructed from the empty set, by iterative applications of the power-set operator. Every set has a *rank*, namely the stage at which it appears in such a '*cumulative hierarchy*'.

In this section it is proved that the foundation axiom is equivalent to postulating that the universe $V = \mathcal{P}_S V$ is the initial algebra of the power-set endofunctor $\mathcal{P}_S$ on **SET**.

A set x is **well-founded** wrt the membership relation '$\in$' if either it is empty or has a least element wrt $\in$. In other words, there is no infinitely descending chain of elements starting from x. Correspondingly, let the class

$$W = \{x \mid x \text{ is well-founded wrt the relation} \in \}$$

be the **universe of well-founded sets**.

The 'foundations axiom' amounts to postulating that all sets in the universe V are well-founded, that is,

Foundation Axiom:

$$V = W$$

Now, notice that the class $\mathcal{P}_S W$ of (small) subsets of well-founded sets is the same as W, because the elements of a well-founded set are themselves well-founded. Thus

$$\mathcal{P}_S W = W$$

and the identity on W can be seen as a $\mathcal{P}_S$-algebra structure.

The universe of well-founded sets is the initial $\mathcal{P}_S$-algebra.

For every $\mathcal{P}_S$-algebra structure $h : \mathcal{P}_S X \to X$ there exists a unique function $h^{\#} : W \to X$ such that the following diagram commutes.

$$\begin{tikzcd} \mathcal{P}_S W \arrow[r, "\mathcal{P}_S(h^{\#})"] \arrow[d, equal] & \mathcal{P}_S X \arrow[d, "h"] \\ W \arrow[r, dashed, "h^{\#}"'] & X \end{tikzcd}$$

That is,

$$\begin{array}{rcl} h^{\#}(0) & = & h(0) \\ h^{\#}\{x_i\}_I & = & h\{h^{\#}(x_i)\}_I \end{array}$$

The proof is by straightforward induction on the (well-founded!) membership relation $\in$.

An immediate consequence of the initiality of W is the existence of a 'rank' function, mapping every well-founded set to a suitable 'ordinal'. An **ordinal** is a well-founded set which is *totally* ordered by the membership relation and which is 'transitive'. (A **transitive set** is a set x such that every element $y \in x$ is also a subset $y \subseteq x$.) Correspondingly, one can form the class $\mathbb{O}$ of all ordinals, which is a subclass of W.

If α and β are two ordinals such that $\beta \in \alpha$, one usually writes $\beta < \alpha$. The first ordinals are: 0, $s(0)$, $s^2(0)$, etc. The first limit ordinal is $\omega = \bigcup_{n \in N} s^n(0)$, which, by the infinity axiom, is indeed a set. In general, because every ordinal is totally ordered by $\in$, the union $\bigcup \{\alpha_i\}_I$ of a set $\{\alpha_i\}_I$ of ordinals is the least upper bound of the α_i's. As a consequence, the union operator is a $\mathcal{P}_S$-algebra structure on the class $\mathbb{O}$ of ordinals:

$$\bigcup : \mathcal{P}_S(\mathbb{O}) \to \mathbb{O} \qquad\qquad \{\alpha_i\}_I \mapsto \bigcup\{\alpha_i\}_I$$

The inductive extension $\mathsf{rank} = \bigcup^{\#} : W \to \mathbb{O}$ of this algebra structure on is the function assigning a 'rank' to every well-founded set. This can be thought of as the stage at which a well-founded set is constructed in an ideal construction starting from the empty set and then iteratively applying the power-set functor $\mathcal{P}_S$:

$$\begin{array}{rcl} \mathsf{rank}(0) & = & 0 \\ \mathsf{rank}\{x_i\}_I & = & \bigcup\{\mathsf{rank}(x_i)\}_I \end{array}$$

Another consequence of the initiality of W is that $W = \mathcal{P}_S W$ is the least (pre-) fixed point for $\mathcal{P}_S$:

$$W = \mathsf{lfp}[\mathcal{P}_S]$$

That is, for every class X such that $\mathcal{P}_S X \subseteq X$, one has that $W \subseteq X$. Indeed, regarding the inclusion of $\mathcal{P}_S X$ into X as a function $\kappa : \mathcal{P}_S X \hookrightarrow X$, one has that its inductive extension $\kappa^{\#} : W \to X$ is of the following form.

$$\begin{array}{rcl} \kappa^{\#}(0) & = & 0 \\ \kappa^{\#}\{x_i\}_I & = & \kappa\{\kappa^{\#}(x_i)\}_I \end{array}$$

Then, to see that $\kappa^{\#}$ is the inclusion of W into X, it suffices to notice that the power-set functor

$$\mathcal{P}_S \text{ **‘preserves inclusion functions’**}$$

that is, if $\iota : X \hookrightarrow Y$ is the inclusion of a subclass X of Y into Y, then the function $\mathcal{P}_S(\iota) : \mathcal{P}_S X \to \mathcal{P}_S Y$ is the inclusion of $\mathcal{P}_S X$ into $\mathcal{P}_S Y$.

Usually, initial algebras are unique up to isomorphism, but in this setting one has a stronger result:

$$\mathcal{P}_S X = X \text{ is the initial } \mathcal{P}_S\text{-algebra} \iff X = W$$

That is, any other initial algebra which is a (strict) fixed point of $\mathcal{P}_S$ is not only isomorphic but equal to W. In order to prove this, ie the non-trivial implication from left to right, one can use very much the same argument as the one used above to prove that W is the least fixed point of $\mathcal{P}_S$.

Therefore, by replacing X by Y in the above equivalence, one has that the foundation axiom ‘$V = W$’ is equivalent to postulating that the universe V is the initial algebra of the power-set functor:

Foundation is Initiality:

$$V = W \iff \mathcal{P}_S V = V \text{ is the initial } \mathcal{P}_S\text{-algebra.}$$

Anti-Foundation and Finality

Not all sets occurring in the mathematical practice are well-founded. A typical example is given by recursive processes as occurring in the semantics of programming languages. (Cf Section 5.) In order to ensure the existence of *non-well-founded sets*, one can postulate the 'anti-foundation axiom'.

In this section, 'anti-foundation' is shown to be the dual of the initial algebra formulation of 'foundation':

Foundation: $\mathcal{P}_S V = V$ is an initial $\mathcal{P}_S$-algebra.

Anti-Foundation: $V = \mathcal{P}_S V$ is a final $\mathcal{P}_S$-coalgebra.

That is, anti-foundation postulates that the universe is the 'largest' possible one, while foundation postulates that it is the 'smallest'.

Let us consider the existence of the final coalgebra for the endofunctor

$$\mathcal{P}_S : \mathbf{SET} \to \mathbf{SET} \qquad\qquad X \mapsto \{x \mid x \text{ is a set } \wedge\ x \subseteq X\}$$

where, recall **SET** is the category of classes (ie large sets) which are definable within basic set theory. The proof that a final coalgebra for this functor exists can be carried out very much the same way as for the finite power-set functor

$$\mathcal{P}_{\mathrm{fi}} : \mathbf{Set} \to \mathbf{Set} \qquad\qquad x \mapsto \{y \mid y \text{ is finite } \wedge\ y \subseteq x\}$$

As shown in Section 13, the coalgebras of this finite power-set functor are the same as the directed finitely branching graphs and the final coalgebra is the set of rooted finitely branching trees (possibly of infinite depth) quotiented by $\mathcal{P}_{\mathrm{fi}}$-bisimulation.

Correspondingly, the coalgebras of the power-set functor $\mathcal{P}_S$ are the same as the directed 'locally small' graphs and the final coalgebra is the class of rooted 'locally small' trees (possibly of infinite depth) quotiented by $\mathcal{P}_S$-bisimulation. A (possibly large) graph is **locally small** if the collection of children of every node is a (small) set. Thus locally small graphs are in between *large graphs* (with a class of nodes each possibly having a class of children) and *small graphs* (with a set of nodes and a set of arcs).

Peter Aczel's original formulation of the anti-foundation axiom is in terms of small graphs and 'decorations'. A **decoration** for (the graph corresponding to) a

$\mathcal{P}_S$- coalgebra $\langle X, k\rangle$ is a coalgebra arrow from $\langle X, k\rangle$ to $V = \mathcal{P}_S V$

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "k"'] & V \arrow[d, equal] \\ \mathcal{P}_S X \arrow[r, "\mathcal{P}_S f"'] & \mathcal{P}_S V \end{tikzcd}$$

That is, a function f from X to the universe V such that, for every $x \in X$,

$$f(x) = \{f(x') \mid x' \in k(x)\}$$

In terms of graphs, this corresponds to a function mapping every node to a set in the following way.

$$f(x) = \{f(x') \mid x \longrightarrow x'\}$$

Therefore, by definition of final coalgebra, the coalgebra $V = \mathcal{P}_S V$ is final if and only if every (directed) locally small graph has a unique decoration. Now, the claim is that 'locally small' can be replaced by 'small' in the above equivalence. That is, every *locally small* graph has a unique decoration if (and only if) every *small* graph has a unique decoration. Indeed:

> (By contradiction.) Assume that every small graph has a unique decoration and that there are two distinct decorations f and g of (a coalgebra $\langle X, k\rangle$ corresponding to) a locally small graph. Then there is a node $x \in X$ such that
>
> $$f(x) \neq g(x)$$
>
> Now, the subgraph of $\langle X, k\rangle$ accessible from x is not only locally small but also (totally) small, that is, there are only set-many nodes accessible from x, because every node has only set-many children. But then f and g are both decorations for this small subgraph, which, by hypothesis, implies that
>
> $$f(x) = g(x)$$
>
> (The same argument can be used to prove that the class of small $\mathcal{P}_S$-coalgebras forms a *generating class* for the $\mathcal{P}_S$-coalgebras. (Cf Section 13.))

As a consequence, the postulate '$V = \mathcal{P}_S V$ is a final $\mathcal{P}_S$-coalgebra' is equivalent to Peter Aczel's original formulation of anti-foundation:

Anti-Foundation Axiom:

Every directed small graph has a unique decoration.

That is,

Anti-Foundation is Finality:

> Every directed small graph has a unique decoration if and only if $V = \mathcal{P}_S V$ is a final $\mathcal{P}_S$-coalgebra.

Notice that no axiom is needed in order to obtain a unique decoration for a *well-founded* graph: One can check that the class WG of well-founded directed small graphs is a (strict) fixed point for the power-set functor $\mathcal{P}_S$, and, moreover, that $\mathcal{P}_S(\mathsf{WG}) = \mathsf{WG}$ is an initial $\mathcal{P}_S$-algebra. Therefore WG is isomorphic to the universe of well-founded sets W and the image under this isomorphism of a well-founded graph is its unique decoration. (Cf "*Mostowski's collapsing lemma*" in [Acz88].)

When anti-foundation is postulated also non-well-founded graph have a unique decoration, but the converse is not true anymore. That is, there exist (non-well-founded) sets which 'decorate' different graphs. An example is the archetypal non-well-founded set, namely the *self-singleton* set

$$\Omega = \{\Omega\}$$

which is a member (and the only member) of itself. If anti-foundation is assumed, then both the root of the graph with one node and one arc

$$\begin{tikzcd} \bullet \arrow[loop below] \end{tikzcd}$$

and the root of the graph consisting in one infinite path

$$\bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \cdots$$

are necessarily mapped to Ω by the corresponding unique decorations.

Notes. Aczel's anti-foundation axiom is equivalent to Forti and Honsell's "X_1-axiom" [FH83].

Besides applications in the semantics of programming languages (eg, [Acz88, Muk91, RT93, Acz94, Bal94, HL95, Har96]), non-well-founded sets have been extensively used in *Situation Theory* (eg, [BE87]), where they are better known as *hypersets*. (Correspondingly, models of the universe of non-well-founded sets are also called *hyperuniverses*.)

Reasoning about non-well-founded sets: bisimulation. By the extensionality axiom, the equality between two sets is determined by the membership relation. One of the consequences of *foundation* is that, since then the membership relation is well-founded, one can use *induction* to reason about (the equality between) sets. Categorically, this induction principle follows from the fact that foundation postulates that the universe is an initial algebra. Dually, anti-foundation, by postulating that the universe is a final coalgebra, gives a *coinduction* principle for reasoning about (possibly non-well-founded) sets.

Now, as shown in Section 12, if an endofunctor *preserves weak pullbacks* then coinduction (wrt its final coalgebra) can be 'pulled back' to the corresponding coalgebraic notion of *bisimulation*. In particular, the power-set functor $\mathcal{P}_S$ does preserve weak pullbacks; the proof is essentially the same as the one given in Section 12 for the behaviour $BX = \check{\mathcal{P}}(1 + \mathsf{Act} \times X)$. Therefore, two sets are equal if and only they are $\mathcal{P}_S$-bisimilar. (Cf [Acz88] for this "*Strong extensionality*".)

By instantiating the general definition of coalgebraic bisimulation (Section 12) to the $\mathcal{P}_S$-coalgebras one has that a (possibly large) relation on the carrier X of a coalgebra $\langle X, k\rangle$ *lifts to a $\mathcal{P}_S$-bisimulation* when, for all $x_1, x_2 \in X$ such that $x_1 R x_2$,

- if $x_1 \to x_1'$ then $x_2 \to x_2'$ for some x_2' such that $x_1' R\, x_2'$
- and, conversely, if $x_2 \to x_2'$ then $x_1 \to x_1'$ for some x_1' such that $x_1' R\, x_2'$.

(Here the notation $x \to x'$ stands for 'there is an arc from x to x' in the graph corresponding to the coalgebra $\langle X, k\rangle$'.)

In particular, a relation R on the universe V lifts to a $\mathcal{P}_S$-bisimulation if, for every set x and y such that xRy, for every $x' \in x$ there exists a $y' \in y$ such that $x' R\ y'$ and, conversely, for every $y' \in y$ there exists an $x' \in x$ such that $x' R\ y'$. Therefore, by strong extensionality,

$$x = y \iff \text{there exists a relation } R \text{ such that:}$$

$$\begin{array}{ll} \bullet & x R\, y \\ \bullet & \forall x' \in x,\ \exists y' \in y,\ x' R\ y' \\ \bullet & \forall y' \in y,\ \exists x' \in x,\ x' R\ y' \end{array}$$

Systems of Set-Equations as Coalgebras

The self-singleton non-well-founded set $\Omega = \{\Omega\}$ can be seen as the unique solution of the 'set-equation'

$$x = \{x\}$$

In general, all non-well-founded sets arise from systems of set-equations with, on the left hand side, variables $x \in X$, and, on the right hand side, *well-founded* sets, possibly containing variables from X. This is the content of the "Solution Lemma".

In this section an elementary presentation of the solution lemma is given by means of the coalgebraic account of anti-foundation (and the initial algebra presentation of well-founded sets). This follows the coalgebraic treatment of recursive programs given in Section 5.

The definition of the universe of well founded sets W can be made parametric: for every (possibly large) set X, the **expanded universe of well-founded sets** WX is the class of all well-founded sets with variable $x \in X$. That is, every set in WX is either empty, or an element of X, or it has a least element wrt the membership relation $\in$. For $X = 0$ this yields the standard universe $W0$ of well-founded sets. Thus, in the sequel, W stands for an operator mapping a (large) set to the corresponding expanded universe of well-founded sets, rather than for the simple universe of well-founded sets.

The fact that $W0$ is the least (strict) fixed point of the power-set functor $\mathcal{P}_S$ and that $\mathcal{P}_S W0 = W0$ is an initial $\mathcal{P}_S$-algebra generalizes as follows: the class WX is the least (strict) fixed point of the endofunctor $X + \mathcal{P}_S(\text{-})$ on **SET** and

$$X + \mathcal{P}_S WX = WX$$

is an initial algebra for this endofunctor. As usual, this initiality can be used to extend the operator W to a functor (cf Section 1):

$$\begin{tikzcd}
X \arrow[r, "\eta_X = \mathsf{inl}_X"] \arrow[d, "f"'] & WX = X + \mathcal{P}_S WX \arrow[d, dashed, "Wf = {[\eta_Y \circ f, \mathsf{inr}_Y]}^{\#}"] & \mathcal{P}_S WX \arrow[l, "\mathsf{inr}_X"'] \arrow[d, "\mathcal{P}_S Wf"] \\
Y \arrow[r, "\eta_Y = \mathsf{inl}_Y"'] & WY = Y + \mathcal{P}_S WY & \mathcal{P}_S WY \arrow[l, "\mathsf{inr}_Y"]
\end{tikzcd}$$

That is, for every function $f : X \to Y$, the function $Wf : WX \to WY$ is the inductive extension of the algebra structure $\mathsf{inr}_Y : \mathcal{P}_S WY \to WY$ along the composite

$\eta_Y \circ f : X \to WY$, where the left injection $\eta_Y = \mathsf{inl}_Y : Y \to WY$ is the usual insertion-of-variables function. In other words,

W is *freely generated* by $\mathcal{P}_S$.

Now, the idea is that a system of 'set-equations' like, eg,

$$\begin{array}{rcl} x & = & \{x, \{y\}\} \\ y & = & \{y, 0\} \end{array}$$

can be seen as a function k mapping the variables $x, y, \ldots \in X$ of the system to elements of $\mathcal{P}_S WX$, ie sets of well-founded sets possibly with variables in X. For instance, the above system corresponds to a function $k : \{x, y\} = X \to \mathcal{P}_S WX$ mapping x to $\{x, \{y\}\}$ and y to $\{y, 0\}$. Therefore, in general, a **system of set-equations in** X is a coalgebra $\langle X, k\rangle$ of the composite endofunctor $\mathcal{P}_S W$ on **SET**.

In order to solve a system of set-equations $\langle X, k\rangle$ one can (postulate anti-foundation and) use the finality of the universe $V = \mathcal{P}_S V$. For this, one first needs to extend the $\mathcal{P}_S W$-coalgebra structure $k : X \to \mathcal{P}_S WX$ to a $\mathcal{P}_S$-coalgebra structure as follows. Since $WX = X + \mathcal{P}_S WX$ is a coproduct, one can form the copair of k and the identity id on $\mathcal{P}_S WX$

$$\begin{tikzcd} X \arrow[r, tail] \arrow[dr, "k"'] & WX \arrow[d, dashed, "{[k, \mathsf{id}]}"] & \mathcal{P}_S WX \arrow[l, tail] \arrow[dl, equal] \\ & \mathcal{P}_S WX & \end{tikzcd}$$

This is a $\mathcal{P}_S$-coalgebra structure behaving as k on $x \in X$ and as the identity on $v \in \mathcal{P}_S WX$. Its coinductive extension $\overline{k} = [k, \mathsf{id}]^{@} : WX \to V$ wrt the final $\mathcal{P}_S$-coalgebra $V = \mathcal{P}_S V$ is then the (unique) **solution of the system** $k : X \to \mathcal{P}_S WX$ **of set-equations**:

$$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[dr, "k"'] & WX \arrow[r, dashed, "{\overline{k} = [k, \mathsf{id}]^{@}}"] \arrow[d, "{[k, \mathsf{id}]}"] & V \arrow[d, equal] \\ & \mathcal{P}_S WX \arrow[r, "\mathcal{P}_S \overline{k}"'] & \mathcal{P}_S V \end{tikzcd}$$

Omitting, as usual, the injections, and letting v and v' range over objects of type $\mathcal{P}_S W$, one has that

$$\overline{k}(x) = \{\overline{k}(v) \mid v \in hx\}$$

and

$$\overline{k}(v) = \{\overline{k}(v') \mid v' \in v\}$$

For example, the unique solution of equation $k(x) = \{x\}$ is the self-singleton (non-well-founded) set

$$\overline{k}(x) = \{\overline{k}(x)\}$$

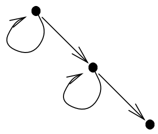
that is, $\overline{k}(x) = \Omega$. Similarly, the solution of the above system

$$k(x) = \{x, \{y\}\} \quad k(y) = \{y, 0\}$$

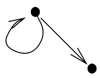
is

$$\begin{aligned} \overline{k}(x) &= \{\overline{k}(x), \{\overline{k}(y)\}\} \\ \overline{k}(y) &= \{\overline{k}(y), 0\} \end{aligned}$$

In terms of graphs, the sets $\overline{k}(x)$ and $\overline{k}(y)$ correspond to



and



respectively.

> **The Solution Lemma is equivalent to Anti-Foundation.** The above property that every system of set-equations has a unique solution, is called the **solution lemma** in [Acz88]. (See also [BE87, Chapter 3].) It is obtained assuming the anti-foundation axiom. Conversely, postulating the solution lemma, one can prove that $V = \mathcal{P}_S V$ is the final $\mathcal{P}_S$-coalgebra. Indeed, for every $\mathcal{P}_S$-coalgebra $\langle X, k \rangle$, one obtains
>
> $$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[d, "k"'] & WX \arrow[r, dashed, "\overline{\mathcal{P}_S(\eta_X) \circ k}"] \arrow[d, "{[\mathcal{P}_S(\eta_X) \circ k, \mathsf{id}]}"] & V \arrow[d, equal] \\ \mathcal{P}_S X \arrow[r, "\mathcal{P}_S(\eta_X)"'] & \mathcal{P}_S W X \arrow[r, "\mathcal{P}_S(\overline{\mathcal{P}_S(\eta_X) \circ k})"'] & \mathcal{P}_S V \end{tikzcd}$$
>
> The desired coinductive extension of the coalgebra structure $k : X \to \mathcal{P}_S X$ is given by the composite coalgebra arrow
>
> $$k^{@} = \overline{\mathcal{P}_S(\eta_X) \circ k} \circ \eta_X : X \to V$$

Notice that, assuming anti-foundation, the upper rectangle in the following diagram commutes, because all other sub-diagrams commute.

$$\begin{tikzcd}
& & \mathcal{P}_S WX \arrow[r, "\mathcal{P}_S\overline{k}"] \arrow[dl, "\mathsf{inr}_X"'] \arrow[ddl, equal] & \mathcal{P}_S V \arrow[dl, equal] \\
X \arrow[r, "\eta_X"] \arrow[dr, "k"'] & WX \arrow[r, "\overline{k}"] \arrow[d, "{[k,\mathsf{id}]}"'] & V \arrow[d, equal] & \\
& \mathcal{P}_S WX \arrow[r, "\mathcal{P}_S\overline{k}"'] & \mathcal{P}_S V &
\end{tikzcd}$$

Therefore, the solution $\overline{k} : WX \to V$ of a system of set equations $\langle X, k\rangle$ is not only a $\mathcal{P}_S$-coalgebra arrow but also a $\mathcal{P}_S$-algebra arrow from $\langle WX, \mathsf{inr}_X\rangle$ to $\mathcal{P}_S V = V$. The algebra $\langle WX, \mathsf{inr}_X\rangle$ is a *free $\mathcal{P}_S$-algebra* over X.

The Substitution Lemma from Freeness. In the present approach, the proof of the solution lemma is trivial. The original proof, instead, makes use of a **substitution lemma** [Acz88]. This lemma asserts that, for every function $f : X \to V$, there exists a unique extension $f^\sharp : WX \to V$ of f to $WX = X + \mathcal{P}_S WX$ such that, omitting the injections,

$$f^\sharp(x) = f(x)$$

and

$$f^\sharp(v) = \{f^\sharp(v') \mid v' \in v\}$$

Now, also this becomes trivial here, because of the initial algebra presentation of the expanded universe of well-founded sets WX. Indeed, the desired function $f^\sharp : WX \to V$ is the inductive extension of the $\mathcal{P}_S$-algebra structure $\mathcal{P}_S V = V$ along $f : X \to V$. That is:

$$\begin{tikzcd}
X \arrow[r, "\eta_X"] \arrow[dr, "f"'] & WX \arrow[d, dashed, "f^\sharp"] & \mathcal{P}_S WX \arrow[l, "\mathsf{inr}_X"'] \arrow[d, "\mathcal{P}_S f^\sharp"] \\
& V \arrow[r, phantom, "="] & \mathcal{P}_S V
\end{tikzcd}$$

Notice that, in contrast with [Acz88], anti-foundation is *not* used here.

Notes. In general, every free $\mathcal{P}_S$-algebra over a (possibly large) set X can be used to model the universe of Zermelo-Fraenkel set theory expanded with elements of X as *atoms*. This fact can be seen as an instance of a more general result in [JM95] (Theorem II.5.5) stated in terms of free "*Zermelo-Fraenkel algebras*" and intuitionistic set theory.

From Greatest Fixed Points to Final Coalgebras

The greatest (strict) fixed point $V = \mathcal{P}_S V$ of the power-set functor $\mathcal{P}_S$ can be seen as the final coalgebra of the restriction of the functor $\mathcal{P}_S$ to the subcategory $\mathbf{SET}_\subset$ of inclusion functions. Anti-foundation postulates that this final coalgebra lifts to a final coalgebra in **SET**. If an endofunctor is 'uniform on maps', then, assuming anti-foundation, its final coalgebra in the subcategory $\mathbf{SET}_\subset$ also lifts to a final coalgebra in **SET**. This is the content of the "Special Final Coalgebra Theorem".

In this section, a new formalization of the notion of uniformity on maps in terms of natural transformations is given. The proof of the theorem is then rephrased in terms of this definition.

Let F be an endofunctor on **SET**. A post-fixed point $X \subseteq FX$ for F can be seen as an inclusion function $X \hookrightarrow FX$, hence as an F-coalgebra structure on X. If the endofunctor F preserves inclusion functions, ie F applied to $X \hookrightarrow Y$ is an inclusion $FX \hookrightarrow FY$, then one can restrict F to the subcategory $\mathbf{SET}_\subset$ of classes and inclusion functions. The post-fixed points of F are then its coalgebras in this subcategory. In particular, the final F-coalgebra in $\mathbf{SET}_\subset$, if it exists, is the greatest (post-)fixed point

$$\mathsf{gfp}[F] = F(\mathsf{gfp}[F])$$

of F. The claim is that if F is 'uniform on maps' then, assuming anti-foundation, $\mathsf{gfp}[F] = F(\mathsf{gfp}[F])$ is also a final coalgebra.

Intuitively, an endofunctor on **SET** is uniform on maps if it is completely determined by its action on objects (ie classes). Most of endofunctors are thus uniform on maps. For instance, consider the endofunctor $X \mapsto A \times X$ mapping a class X to its product with a fixed class A. Given a function $f : X \to Y$, the value of $A \times f$ at an element $<a, x>$ of $A \times X$ is the pair $<a, f(x)> \in A \times Y$ which is obtained by applying f to the $x \in X$ in $A \times X$. This suggests that the class X should be regarded as a class of variables and that, in general, the action of a functor F uniform on maps on a function f should simply be the substitution of the variables x occurring in FX by $f(x)$.

Formally, this can be expressed by means of the expanded universe of well-founded sets $WX = X + \mathcal{P}_S WX$. What one needs is a natural transformation

$$\rho : F \Rightarrow \mathcal{P}_S W$$

which, for every X, 'embeds' FX into $\mathcal{P}_S WX$ – the class of sets of (well-founded) sets having $x \in X$ as variables.

Naturality amounts to having, for every function $f : X \to Y$, the following diagram commute.

$$\begin{tikzcd} FX \arrow[r, "Ff"] \arrow[d, "\rho_X"'] & FY \arrow[d, "\rho_Y"] \\ \mathcal{P}_S WX \arrow[r, "\mathcal{P}_S Wf"'] & \mathcal{P}_S WY \end{tikzcd}$$

It should be an 'embedding' in the sense that, for every X and for every $v \in FX$, by 'forgetting' the distinction between variables and sets in $\rho_X(v) \in \mathcal{P}_S WX$ one should get back the original set v. This operation of forgetting the distinction between variables and sets in objects of type $\mathcal{P}_S W$ can be made formal as follows.

Consider the inductive extension $\varepsilon_V : WV \to V$ of the $\mathcal{P}_S$-algebra structure $\mathcal{P}_S V = V$ along the identity on V:

$$\begin{tikzcd} V \arrow[r, "\eta_V"] \arrow[dr, equal] & WV \arrow[d, dashed, "\varepsilon_V"] & \mathcal{P}_S WV \arrow[l, "\mathsf{inr}_V"'] \arrow[d, "\mathcal{P}_S(\varepsilon_V)"] \\ & V \arrow[r, equal] & \mathcal{P}_S V \end{tikzcd}$$

Omitting, as usual, the injections, one has that, for every $v \in WV$, $\varepsilon_V(v) = v$ if v is a variable and $\varepsilon_V(v) = \{\varepsilon_V(v_i)\}_I$ if $v = \{v_i\}_I$. Then, an endofunctor $F : \mathbf{SET} \to \mathbf{SET}$ is **uniform on maps** if there exists a natural transformation

$$\rho : F \Rightarrow \mathcal{P}_S W$$

such that

$$\begin{tikzcd} FV \arrow[r, hook] \arrow[d, "\rho_V"'] & V \arrow[d, equal] \\ \mathcal{P}_S WV \arrow[r, "\mathcal{P}_S(\varepsilon_V)"'] & \mathcal{P}_S V \end{tikzcd}$$

commutes.

Before setting out to prove the special final coalgebra theorem, notice that, since W is freely generated by $\mathcal{P}_S$, the forgetful functor mapping $\mathcal{P}_S$-algebras to their carriers is right adjoint to the functor mapping a class X to the (free) $\mathcal{P}_S$-algebra with carrier WX and structure

$$\mathsf{inr}_X : \mathcal{P}_S WX \to WX$$

(Cf Section 2.) The other injection $\eta_X = \mathsf{inl}_X : X \to WX$ is the unit of the adjunction at X, while the value of the counit at an algebra $\langle Y, h\rangle$ is given by the inductive extension of the right injection $\mathsf{inr}_Y : \mathcal{P}_S WY \to WY$ along the identity on Y.

$$
\begin{tikzcd}
Y \arrow[r, "\eta_Y"] \arrow[dr, phantom, "\setminus\!\!\setminus"] & WY \arrow[d, dashed, "\varepsilon_{\langle Y,h\rangle}"] & \mathcal{P}_S WY \arrow[l, "\mathsf{inr}_Y"'] \arrow[d, "\mathcal{P}_S(\varepsilon_{\langle Y,h\rangle})"] \\
 & Y & \mathcal{P}_S Y \arrow[l, "h"]
\end{tikzcd}
$$

Thus, in particular, the above function $\varepsilon_V : WV \to V$ is the value of the counit at the algebra $\mathcal{P}_S V = V$. (Formally, $\varepsilon_V = U\varepsilon_{(\mathcal{P}_S V = V)} = \varepsilon_{(\mathcal{P}_S V = V)}$, where U is the forgetful functor mapping algebras to their carriers.) By adjunction, there is a bijection (natural in X and $\langle Y, h\rangle$) between functions $f : X \to Y$ and $\mathcal{P}_S$-algebra arrows $g : \langle WX, \mathsf{inr}_X\rangle \to \langle Y, h\rangle$. This bijection maps f to

$$f^\sharp = \varepsilon_{\langle Y,h\rangle} \circ Wf$$

and g to

$$g^\flat = Ug \circ \eta_X = g \circ \eta_X$$

> **The Special Final Coalgebra Theorem.** Let F be a endofunctor on **SET** which cuts down to an endofunctor on the subcategory $\mathbf{SET}_\subset$ of inclusion functions.
>
> If F is uniform on maps, then, assuming anti-foundation, its final coalgebra
>
> $$\mathsf{gfp}[F] = F(\mathsf{gfp}[F])$$
>
> in $\mathbf{SET}_\subset$ lifts to a final F-coalgebra in **SET**.

Proof: Consider an F-coalgebra structure

$$k : X \to FX$$

By uniformity on maps, there exists a function $\rho_X : FX \to \mathcal{P}_S WX$, hence k can be made into a system of set-equations in X by composing it with ρ_X. Take its solution $\overline{\rho_X \circ k} : WX \to V$ and define a function f from X to V as the right adjunct of this solution wrt the above adjunction; that is,

$$f = (\overline{\rho_X \circ k})^\flat = \overline{\rho_X \circ k} \circ \eta_X : X \to V$$

Diagrammatically:

$$\begin{tikzcd}
X \arrow[r, "\eta_X"'] \arrow[d, "k"'] \arrow[rr, bend left, "f = (\overline{\rho_X \circ k})^\flat"] & WX \arrow[r, dashed, "\overline{\rho_X \circ k}"'] \arrow[d, "{[\rho_X \circ k, \mathsf{id}]}"'] & V \arrow[d, equal] \\
FX \arrow[r, "\rho_X"'] & \mathcal{P}_S WX \arrow[r, "\mathcal{P}_S(\overline{\rho_X \circ k})"'] & \mathcal{P}_S V
\end{tikzcd}$$

The claim is that, under the above hypotheses, f is an F-coalgebra arrow from $\langle X, k\rangle$ to $\mathsf{gfp}[F] = F(\mathsf{gfp}[F])$, that is, the diagram

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "k"'] & \mathsf{gfp}[F] \arrow[d, equal] \\
FX \arrow[r, "Ff"'] & F(\mathsf{gfp}[F])
\end{tikzcd}$$

commutes. More precisely: Let Y be the image under f of X. The function $f : X \to V$ can be factorized, like every function in **SET**, as

$$X \overset{f}{\twoheadrightarrow} Y \hookrightarrow V$$

The claim is then as follows.

> The class Y is a post-fixed point for F, ie $Y \subseteq FY$, and f is a coalgebra arrow from $\langle X, k\rangle$ to $Y \hookrightarrow FY$, ie
>
> $$\begin{tikzcd}
X \arrow[r, two heads, "f"] \arrow[d, "k"'] & Y \arrow[d, hook] \\
FX \arrow[r, "Ff"'] & FY
\end{tikzcd}$$
>
> commutes.

If the above holds, since F cuts down to an endofunctor on the subcategory $\mathbf{SET}_{\subset}$ of inclusions, the composition of f the inclusion $Y \hookrightarrow \mathsf{gfp}[F]$ of Y into the greatest fixed point of F is an F-coalgebra arrow:

$$\begin{tikzcd}
X \arrow[r, two heads, "f"] \arrow[d, "k"'] & Y \arrow[r, hook] \arrow[d, hook] & \mathsf{gfp}[F] \arrow[d, equal] \\
FX \arrow[r, "Ff"'] & FY \arrow[r, hook] & F(\mathsf{gfp}[F])
\end{tikzcd}$$

In order to prove the above claim, notice that everything in sight in the following diagram commutes.

$$\begin{tikzcd}
X \arrow[rr, "\eta_X"'] \arrow[rrrr, bend left, "f"] \arrow[dd, "k"'] & & WX \arrow[rr, "f^\sharp"] \arrow[dr, "Wf"'] \arrow[dd, "{[\rho_X \circ k, \mathsf{id}]}"'] & & V \arrow[dd, equal] \\
& & & WV \arrow[ur, "\varepsilon_V"'] & \\
FX \arrow[rr, "\rho_X"] \arrow[dr, "Ff"'] & & \mathcal{P}_S WX \arrow[rr, "\mathcal{P}_S(f^\sharp)"] \arrow[dr, "\mathcal{P}_S Wf"] & & \mathcal{P}_S V \\
& FV \arrow[rr, "\rho_V"'] \arrow[uuurrr, hook, bend right] & & \mathcal{P}_S WV \arrow[ur, "\mathcal{P}_S \varepsilon_V"'] &
\end{tikzcd}$$

In particular, the outer diagram does commute, hence:

$$\begin{tikzcd}
X \arrow[r, two heads, "f"] \arrow[dd, "k"'] & Y \arrow[dr, hook] & \\
& & V \\
FX \arrow[r, "Ff"'] & FY \arrow[ur, hook] &
\end{tikzcd}$$

Therefore, for all $x \in X$,

$$f(x) = (Ff \circ k)(x)$$

which implies that the image Y of X under f is included in the image of FX under Ff, hence

$$Y \subseteq FY$$

and f is a coalgebra arrow from $\langle X, k\rangle$ to $Y \hookrightarrow FY$.

Therefore, for every F-coalgebra $\langle X, k\rangle$, there exists a coalgebra arrow to $\mathsf{gfp}[F] = F(\mathsf{gfp}[F])$. Moreover, this arrows is unique. Indeed, the above arguments also show that every coalgebra arrow from $\langle X, k\rangle$ to $\mathsf{gfp}[F] = F(\mathsf{gfp}[F])$ fits as the right adjunct $(\overline{\rho_X \circ k})^\flat$ of the unique solution of a system of set-equations, hence it is unique. This concludes the proof.

Notes. An alternative (but more restrictive) form of the special final coalgebra theorem in the standard category of ordinary sets is presented in [Pau95].

The special final coalgebra theorem is the 'dual' of the standard fact that least (strict) fixed points of most endofunctors on **SET** are initial algebras. (Cf [Acz88, Theorem 7.6].) It gives an elementary way of finding final coalgebras, at the price of assuming anti-foundation. For instance, under foundation, the endofunctor $BX = \mathsf{Act} \times X$ has the empty set 0 as the unique fixed point, while, under anti-foundation, the empty set is the least fixed point and the set Act^ω of infinite words over the alphabet Act is the greatest fixed point of B: the special final coalgebra theorem tells then that $\mathsf{Act}^\omega = \mathsf{Act} \times \mathsf{Act}^\omega$ is a final B-coalgebra.

Notice that one can prove the (non-strict!) fixed point $\mathsf{Act}^\omega \cong \mathsf{Act} \times \mathsf{Act}^\omega$ is a final B-coalgebra in **Set**, *independently* of the use of anti-foundation. In general, as shown in [AM89], endofunctors to which the special final coalgebra theorem applies always have a final coalgebra in the category of ordinary (possibly large) sets. Thus, unless one is really interested in strict fixed points $\widehat{B} = B\widehat{B}$ rather than fixed points up to isomorphism $\widehat{B} \cong B\widehat{B}$, the interest can be shifted from non-well-founded sets and greatest fixed points to ordinary sets and final coalgebras.