II

6 The Functorial Operational Semantics is Compositional

The semantics of a programming language is called **compositional** when the meaning of compound programs can be derived from the meaning of their subcomponents. A typical compositional semantics is obtained by defining the meaning of a program by induction, starting from a '**denotation**' $\llparenthesis\, \sigma \,\rrparenthesis$ for each n-ary program construct σ:

$$\llparenthesis\, \sigma(t_1, \ldots, t_n) \,\rrparenthesis^{\#} = \llparenthesis\, \sigma \,\rrparenthesis(\llparenthesis\, t_1 \,\rrparenthesis^{\#}, \ldots, \llparenthesis\, t_n \,\rrparenthesis^{\#})$$

This is called a **denotational semantics**.

A complete account of the meaning of a programming language requires both an operational and a denotational semantics. The former explains how a machine should execute programs, specifying their executable behaviours. The latter, because of its modularity, is better suited for reasoning about programs. The two meanings should be related in such a way that one should be able to infer from the denotational semantics the operational behaviour of the programs – up to a suitable abstraction. In other words, the denotational semantics of a language should be **adequate** wrt the operational semantics.

In general, much work is needed to prove the adequacy of a denotational semantics wrt an operational one. However, from operational semantics of transition systems defined by operational rules satisfying suitable *syntactic restrictions* (eg, the rules are in the *GSOS* format – see Section 11), it is possible to derive adequate denotational semantics systematically. (Cf notes below.)

Now, the novelty of the present functorial approach to operational semantics is that *every* functorial operational semantics coinduces a denotational semantics and, moreover, this denotational semantics is adequate wrt the operational one; as a corollary, every functorial operational semantics is compositional. Being formulated in terms of *abstract* notions of syntax and behaviour, this gives a general notion of 'well-behaved' operational semantics, based on purely mathematical properties. This encompasses and explains the 'syntactic' arguments otherwise used in the literature. (Cf Section 11.)

Assume, as usual, the (closed) programs of the language to be interpreted are the elements of the initial algebra of the endofunctor corresponding to some program constructs Σ. That is, let T be the syntactical monad of the language and $\Sigma T0 \cong T0$ the corresponding initial algebra of closed programs. Then, the problem of defining a denotational semantics can be reduced to the problem of finding a *suitable* Σ-algebra

$\langle D, \llparenthesis\text{-}\rrparenthesis \rangle$, whose carrier D is the **semantic domain** and whose structure

$$\llparenthesis\text{-}\rrparenthesis : \Sigma D \to D$$

is the set of denotations. The desired denotational interpretation of the programs is then the inductive extension of this algebra of denotations, that is, the unique algebra arrow $\llparenthesis\text{-}\rrparenthesis^{\#}$ from the initial algebra of programs to $\llparenthesis\text{-}\rrparenthesis : \Sigma D \to D$. Diagrammatically:

Initial Algebra Semantics

$$\begin{tikzcd}
\Sigma T0 \arrow[r, "\Sigma\llparenthesis\text{-}\rrparenthesis^{\#}"] \arrow[d, "\text{initial algebra}"', "\cong"] & \Sigma D \arrow[d, "\llparenthesis\text{-}\rrparenthesis\, =\, \text{denotations}"] \\
T0 \arrow[r, dashed, "\llparenthesis\text{-}\rrparenthesis^{\#}"'] & D
\end{tikzcd}$$

The restriction to closed programs is adopted only to simplify the presentation. In general, the interpretation of programs with variables $x \in X$, that is, for the elements of $TX \cong X + \Sigma TX$, is parametric in a 'valuation' function $\rho : X \to D$ mapping each variable to an element of the semantic domain D. Indeed, the inductive extension $\llparenthesis\text{-}\rrparenthesis^{\#}_{\rho} : TX \to D$ of the denotations $\llparenthesis\text{-}\rrparenthesis : \Sigma D \to D$ along the valuation $\rho : X \to D$ has the familiar clauses

$$\llparenthesis\, x \,\rrparenthesis^{\#}_{\rho} = \rho(x)$$

$$\llparenthesis\, \sigma(t_1, \ldots, t_n) \,\rrparenthesis^{\#}_{\rho} = \llparenthesis\, \sigma \,\rrparenthesis(\llparenthesis\, t_1 \,\rrparenthesis^{\#}_{\rho}, \ldots, \llparenthesis\, t_n \,\rrparenthesis^{\#}_{\rho})$$

The denotational model of a language is adequate wrt the operational one when it contains enough information to infer the *abstract behaviour* of the programs. Now, recall (from the previous section) that when the operational model of the (closed) programs can be expressed as a coalgebra structure $[\![\text{-}]\!] : T0 \to BT0$ of a behaviour B, then the abstract (global) behaviour of the programs is given by its coinductive extension, that is, by the corresponding final coalgebra semantics:

$$\begin{tikzcd}
T0 \arrow[r, dashed, "\text{final coalgebra semantics}", "{[\![\text{-}]\!]^{@}}"'] \arrow[d, "{[\![\text{-}]\!]}"'] & \widehat{B} \arrow[d, "\cong"', "\text{final coalgebra}"] \\
BT0 \arrow[r, "{B[\![\text{-}]\!]^{@}}"'] & B\widehat{B}
\end{tikzcd}$$

Then, in this setting, a denotational model is adequate wrt an operational one when its initial algebra semantics $\llparenthesis\text{-}\rrparenthesis^{\#} : T0 \to D$ is equal to the final coalgebra semantics $[\![\text{-}]\!]^{@} : T0 \to \widehat{B}$ corresponding to the operational model. Thus, in particular, the semantic domain D should be the carrier $\widehat{B}$ of the final coalgebra of the behaviour. Diagrammatically:

$$\begin{tikzcd}[column sep=huge, row sep=large]
\Sigma T0 \arrow[r, "\Sigma\llparenthesis\,\text{-}\,\rrparenthesis^{\#}"] \arrow[d, "\text{initial algebra}"'] \arrow[dr, phantom, "\text{initial algebra semantics}"] & \Sigma\widehat{B} \arrow[d, "\llparenthesis\,\text{-}\,\rrparenthesis"] \\
T0 \arrow[r, shift left, "\llparenthesis\,\text{-}\,\rrparenthesis^{\#}"] \arrow[r, shift right, "\llbracket\text{-}\rrbracket^{@}"'] \arrow[r, phantom, "\parallel"] \arrow[d, "\llbracket\text{-}\rrbracket"'] \arrow[dr, phantom, "\text{final coalgebra semantics}"] & \widehat{B} \arrow[d, "\text{final coalgebra}"] \\
BT0 \arrow[r, "B\llbracket\text{-}\rrbracket^{@}"'] & B\widehat{B}
\end{tikzcd}$$

That is, for all programs $t \in T0$,

$$\llparenthesis\, t\,\rrparenthesis^{\#} = \llbracket t \rrbracket^{@}$$

As a corollary, the equivalence relation corresponding to the final coalgebra semantics is a *congruence*, that is, if, for $i = 1, \ldots, n$,

$$\llbracket t_i \rrbracket^{@} = \llbracket t'_i \rrbracket^{@}$$

then

$$\llbracket \sigma(t_1, \ldots, t_n) \rrbracket^{@} = \llbracket \sigma(t'_1, \ldots, t'_n) \rrbracket^{@}$$

for every n-ary operator $\sigma \in \Sigma$.

The above equivalence relation

$$t \sim t' \iff \llbracket t \rrbracket^{@} = \llbracket t' \rrbracket^{@}$$

is the **observational equivalence** corresponding to the operational semantics of the language, as it is determined by the abstract global behaviour of the programs, which is their intended *observable behaviour*. Now, if observational equivalence of a language is a congruence, one can systematically derive a denotational model adequate wrt the operational semantics. In turn, to ensure that the observational equivalence is a congruence one can impose *suitable* syntactic restrictions on the format of the operational rules. (Eg, *GSOS* – see Section 11.) This gives a satisfactory method to derive adequate denotational models from operational semantics, but it strongly relies on the assumption that the operational semantics is given in terms of structural rules for transition systems.

The novelty of the present functorial approach to operational semantics is that it gives a general notion of 'well-behaved' operational semantics formulated in terms of *abstract* notions of syntax and behaviour: every functorial operational semantics coinduces a denotational model adequate wrt it. As shown in Section 11, this purely mathematical approach encompasses – and explains – the above 'syntactic' method.

The denotational model coinduced by Φ. Let us now look at the actual construction of the denotations corresponding to a functorial operational semantics Φ. Recall that the operational monad $\Phi = <\Phi, \eta, \mu>$ is a lifting of the syntactical monad $T = <T, \eta, \mu>$ freely generated by the program constructs Σ. It is convenient to use the isomorphism, illustrated in Section 2, between the categories of Σ-algebras and T-algebras, and define the desired denotational model as a T-algebra rather than as a Σ-algebra. That is, let us look for an arrow

$$\langle\!| \text{-} |\!\rangle : T\widehat{B} \to \widehat{B}$$

such that the following diagrams commute.

$$\begin{tikzcd} T^2\widehat{B} \arrow[r, "T\langle\!| \text{-} |\!\rangle"] \arrow[d, "\mu_{\widehat{B}}"'] & T\widehat{B} \arrow[d, "\langle\!| \text{-} |\!\rangle"] \\ T\widehat{B} \arrow[r, "\langle\!| \text{-} |\!\rangle"'] & \widehat{B} \end{tikzcd} \qquad \begin{tikzcd} \widehat{B} \arrow[r, "\eta_{\widehat{B}}"] \arrow[dr, equal] & T\widehat{B} \arrow[d, "\langle\!| \text{-} |\!\rangle"] \\ & \widehat{B} \end{tikzcd}$$

The idea is to exploit the fact that $\widehat{B} \cong B\widehat{B}$ is a final coalgebra and that the operational monad Φ maps a coalgebra structure $k : X \to BX$ to a coalgebra structure $\Phi k : TX \to BTX$. Thus, by applying Φ to the final coalgebra isomorphism $\varphi : \widehat{B} \cong B\widehat{B}$, one obtains a coalgebra structure on $T\widehat{B}$:

$$\Phi\varphi : T\widehat{B} \to BT\widehat{B}$$

Its coinductive extension $(\Phi\varphi)^{@} : T\widehat{B} \to \widehat{B}$

$$\begin{tikzcd} T\widehat{B} \arrow[r, dashed, "(\Phi\varphi)^{@}"] \arrow[d, "\Phi\varphi"'] & \widehat{B} \arrow[d, "\varphi"] \\ BT\widehat{B} \arrow[r, "B(\Phi\varphi)^{@}"'] & B\widehat{B} \end{tikzcd} \tag{1}$$

is then the natural candidate for the desired denotational model $\langle\!| \text{-} |\!\rangle : T\widehat{B} \to \widehat{B}$.

Let us prove, using finality, that this arrow is a T-algebra indeed. Consider first the multiplication law:

$$\begin{tikzcd} T^2\widehat{B} \arrow[r, "T(\Phi\varphi)^{@}"] \arrow[d, "\mu_{\widehat{B}}"'] & T\widehat{B} \arrow[d, "(\Phi\varphi)^{@}"] \\ T\widehat{B} \arrow[r, "(\Phi\varphi)^{@}"'] & \widehat{B} \end{tikzcd} \tag{2}$$

This is the upper side of the cube

$$\begin{tikzcd}
T^2\widehat{B} \arrow[rr, "T(\Phi\varphi)^{@}"] \arrow[dr, "\mu_{\widehat{B}}"] \arrow[dd, "\Phi^2\varphi"'] & & T\widehat{B} \arrow[dr, "(\Phi\varphi)^{@}"] & \\
& T\widehat{B} \arrow[rr, "(\Phi\varphi)^{@}"] \arrow[dd, "\Phi\varphi"'] & & \widehat{B} \arrow[dd, "\varphi"] \\
BT^2\widehat{B} \arrow[dr, "B\mu_{\widehat{B}}"'] & & & \\
& BT\widehat{B} \arrow[rr, "B(\Phi\varphi)^{@}"'] & & B\widehat{B}
\end{tikzcd}$$

whose vertical sides all commute:

> The front side and the other (not visible) side underlying the arrow $(\Phi\varphi)^{@} : T\widehat{B} \to \widehat{B}$ are two copies of the definition of $(\Phi\varphi)^{@}$, hence commute. The back (not visible) side is the square
>
> $$\begin{tikzcd}
T^2\widehat{B} \arrow[r, "T(\Phi\varphi)^{@}"] \arrow[d, "\Phi^2\varphi"'] & T\widehat{B} \arrow[d, "\Phi\varphi"] \\
BT^2\widehat{B} \arrow[r, "BT(\Phi\varphi)^{@}"'] & BT\widehat{B}
\end{tikzcd}$$
>
> which is nothing but the image under the functor Φ of, once more, the square defining $(\Phi\varphi)^{@} : T\widehat{B} \to \widehat{B}$, hence, by *functoriality*, it commutes. Finally, the last vertical side is a square which commutes by the fact that, by definition of lifting of a monad, multiplication $\mu : T^2 \Rightarrow T$ of the syntactical monad T lifts to the multiplication $\mu : \Phi^2 \Rightarrow \Phi$ of the operational monad Φ.

Therefore, both composites $(\Phi\varphi)^{@} \circ \mu_{\widehat{B}}$ and $(\Phi\varphi)^{@} \circ T(\Phi\varphi)^{@}$ fit as the (unique!) coinductive extension of the coalgebra structure $\Phi^2\varphi : T^2\widehat{B} \to BT^2\widehat{B}$, hence they must be the same.

The proof of the other T-algebra law

$$\begin{tikzcd}
\widehat{B} \arrow[r, "\eta_{\widehat{B}}"] \arrow[dr, equal] & T\widehat{B} \arrow[d, "(\Phi\varphi)^{@}"] \\
& \widehat{B}
\end{tikzcd}$$

is similar and follows from the fact that $\eta : I \Rightarrow T$ lifts to the unit of the monad Φ:

$$\begin{tikzcd}
\widehat{B} \arrow[r, "\eta_{\widehat{B}}"] \arrow[d, "\varphi"'] & T\widehat{B} \arrow[d, "\Phi\varphi"] \\
B\widehat{B} \arrow[r, "B\eta_{\widehat{B}}"'] & BT\widehat{B}
\end{tikzcd}$$

(Notice this last commuting diagram tells us that, using the terminology of Section 3, the coalgebra $\langle T\widehat{B}, \Phi\varphi\rangle$ conservatively extends the final coalgebra $\langle \widehat{B}, \varphi\rangle$.)

Adequacy. Now, the claim is that the initial algebra semantics induced by the above denotational model

$$\llparenthesis\text{-}\rrparenthesis = (\Phi\varphi)^{@} : T\widehat{B} \to \widehat{B}$$

is the same as the final coalgebra semantics coinduced by the operational model

$$\llbracket\text{-}\rrbracket = \Phi 0 : T0 \to BT0$$

That is,

$$\begin{tikzcd}
T^2 0 \arrow[r, "T\llparenthesis\text{-}\rrparenthesis^{\#}"] \arrow[d, "\text{initial algebra}\quad \mu_0"'] \arrow[dr, phantom, "\text{initial algebra semantics}"] & T\widehat{B} \arrow[d, "\llparenthesis\text{-}\rrparenthesis = (\Phi\varphi)^{@}"] \\
T0 \arrow[r, shift left, "\llparenthesis\text{-}\rrparenthesis^{\#}"] \arrow[r, shift right, "\llbracket\text{-}\rrbracket^{@}"'] \arrow[r, phantom, "\shortparallel"] \arrow[d, "\Phi 0 = \llbracket\text{-}\rrbracket"'] \arrow[dr, phantom, "\text{final coalgebra semantics}"] & \widehat{B} \arrow[d, "\varphi\quad \text{final coalgebra}"] \\
BT0 \arrow[r, "B\llbracket\text{-}\rrbracket^{@}"'] & B\widehat{B}
\end{tikzcd}$$

(Formally, the initial algebra semantics $\llparenthesis\text{-}\rrparenthesis^{\#} : T0 \to \widehat{B}$ is the unique T-algebra arrow from the initial T-algebra $\langle T0, \mu_0\rangle$ to the denotational model $\langle \widehat{B}, \llparenthesis\text{-}\rrparenthesis\rangle$. By the isomorphism between T- and Σ-algebras, it is the same as the initial algebra semantics of the Σ-algebra corresponding to the T-algebra $\langle \widehat{B}, (\Phi\varphi)^{@}\rangle$.)

This follows from the fact that everything in sight in the diagram

$$\begin{tikzcd}
T^2 0 \arrow[r, "T^2 0^@"] \arrow[d, "\mu_0"', "\text{initial algebra}"'] & T^2\widehat{B} \arrow[r, "T(\Phi\varphi)^@"] \arrow[d, "\mu_{\widehat{B}}"] & T\widehat{B} \arrow[d, "(\Phi\varphi)^@ = \langle\!|\text{-}|\!\rangle"] \\
T0 \arrow[r, "T0^@"] \arrow[d, "\Phi 0 = [\![\text{-}]\!]"'] & T\widehat{B} \arrow[r, "(\Phi\varphi)^@"] \arrow[d, "\Phi\varphi"] & \widehat{B} \arrow[d, "\varphi", "\text{final coalgebra}"'] \\
BT0 \arrow[r, "BT0^@"'] & BT\widehat{B} \arrow[r, "B(\Phi\varphi)^@"'] & B\widehat{B}
\end{tikzcd}$$

commutes:

> The upper right square is the multiplication law (2) for the T-algebra structure $(\Phi\varphi)^@ : T\widehat{B} \to \widehat{B}$ and the lower right square is the defining square (1) of the arrow $(\Phi\varphi)^@ : T\widehat{B} \to \widehat{B}$. For the left squares first recall that 0 is the initial object in the base category $\mathbf{C}$. (Eg, in $\mathbf{C} = \mathbf{Set}$, 0 is the empty set.) and also recall the convention of writing $0 : 0 \to B0$ for the unique arrow from 0 to $B0$, which, by the way, is the structure of the initial B-coalgebra. The corresponding coinductive extension $0^@ : 0 \to \widehat{B}$ makes the diagram
>
> $$\begin{tikzcd}
0 \arrow[r, "0^@"] \arrow[d, "0"'] & \widehat{B} \arrow[d, "\varphi"] \\
B0 \arrow[r, "B0^@"'] & B\widehat{B}
\end{tikzcd}$$
>
> commute. (It is also the unique arrow from the initial object 0 to $\widehat{B}$.) Then the lower left square commutes because it is the image under the functor Φ of the above commuting square, and the upper left square commutes by the naturality of $\mu : T^2 \Rightarrow T$.

That is,

$$\langle\!|\text{-}|\!\rangle^{\#} = (\Phi\varphi)^@ \circ T0^@ = [\![\text{-}]\!]^@ : T0 \to \widehat{B}$$

Indeed, the composite arrow $(\Phi\varphi)^@ \circ T0^@ : T0 \to \widehat{B}$ is both a coalgebra arrow – hence the coinductive extension $[\![\text{-}]\!]^@$ – and an algebra arrow – hence the inductive extension $\langle\!|\text{-}|\!\rangle^{\#}$.

Equivalently,

$$((\Phi\varphi)^@)^{\#} = (\Phi\varphi)^@ \circ T0^@ = (\Phi 0)^@ : T0 \to \widehat{B}$$

Again, the restriction to closed programs is not essential. Given a set X of variables with a coalgebra structure $k : X \to BX$ on it, one has that the composite $(\Phi\varphi)^@ \circ Tk^@ : TX \to \widehat{B}$ is both the coinductive extension $(\Phi k)^@$ of the operational model $\Phi k : TX \to$

BTX and the inductive extension $\langle\!|\text{-}|\!\rangle_\rho^\#$ of the denotational model along the valuation function

$$\rho = (\Phi k)^@ \circ \eta_X : X \to \widehat{B}$$

That is,

$$\rho^\sharp = ((\Phi k)^@ \circ \eta_X)^\sharp = (\Phi\varphi)^@ \circ Tk^@ = (\Phi k)^@ : TX \to \widehat{B}$$

Example. Consider the functorial operational semantics corresponding to the rules $\mathcal{R}$ for the language

$$t ::= x \mid \mathsf{nil} \mid a \mid (t\ ;\ t)$$

The base category is **Set**. The syntactical monad $T =< T, \eta, \mu >$ is the one freely generated by the endofunctor

$$\Sigma X = 1 + \mathsf{Act} + X \times X$$

on **Set**. (Cf Section 2.) The behaviour is

$$BX = 1 + \mathsf{Act} \times X$$

whose coalgebras are the deterministic transition systems. (Cf Section 3.) Its final coalgebra $\langle \widehat{B}, \varphi\rangle$ has as carrier $\widehat{B}$ the set Act^∞ of finite and infinite streams of actions in Act and the isomorphism $\varphi : \mathsf{Act}^\infty \cong 1 + \mathsf{Act} \times \mathsf{Act}^\infty$ applied to a non-empty stream $p = a \cdot p'$ in Act^∞ returns a pair with first component a and second component the continuation p', while φ applied to the empty stream returns $*$. (Cf Section 5.) Equivalently, the final coalgebra $\langle \mathsf{Act}^\infty, \varphi\rangle$ can be seen as a deterministic transition system with finite and infinite streams as states and with transitions $p = a \cdot p' \xrightarrow{a} p'$.

Next, the set $T\widehat{B}$ is the set of terms over the constructs in Σ and with streams in Act^∞ as variables. Thus, for instance, the term $a\ ;\ (a \cdot b)$ is in this set. (Notice the distinction between the first a which is a constant of the language and the second a which is the first element of the stream $a \cdot b$, which is a variable.) Also, all closed terms of the language belong to the set $T\widehat{B}$ and the function $T0^@ : T0 \to T\widehat{B}$ is nothing but this inclusion.

The operational rules $\mathcal{R}$ for the language are the axioms

$$\mathsf{nil} \downarrow * \qquad \text{and} \qquad a \xrightarrow{a} \mathsf{nil}$$

and the three rules for sequential composition:

$$\frac{u \xrightarrow{a} u'}{u\ ;\ v \xrightarrow{a} u'\ ;\ v} \qquad \frac{u \downarrow * \quad v \xrightarrow{a} v'}{u\ ;\ v \xrightarrow{a} v'} \qquad \frac{u \downarrow * \quad v \downarrow *}{u\ ;\ v \downarrow *}$$

(Cf Section 3.) These rules induce an operational model denoted by

$$[\![\text{-}]\!]_{\mathcal{R}} : T0 \to BT0$$

such that $[\![\mathsf{nil}]\!]_{\mathcal{R}} = *$, $[\![a]\!]_{\mathcal{R}} = <a, \mathsf{nil}>$, and for all terms u, v,

$$[\![u\ ;\ v]\!]_{\mathcal{R}} = \begin{cases} <a, u'\ ;\ v> & \text{if } [\![u]\!]_{\mathcal{R}} = <a, u'> \\ <a, v'> & \text{if } [\![u]\!]_{\mathcal{R}} = * \text{ and } [\![v]\!]_{\mathcal{R}} = <a, v'> \\ * & \text{if } [\![u]\!]_{\mathcal{R}} = * = [\![v]\!]_{\mathcal{R}} \end{cases}$$

More generally, recall that every coalgebra structure $k : X \to BX$ can be seen as a set of axioms for the variables $x \in X$ by putting

$$x \xrightarrow{a} x'$$

if $k(x) = <a, x>$. Then, for every such k, the above rules $\mathcal{R}$ induce an operational model

$$[\![\text{-}]\!]^k_{\mathcal{R}} : TX \to BTX$$

which adds to the above the behaviours

$$[\![x]\!]^k_{\mathcal{R}} = k(x)$$

for every $x \in X$.

Consider now the inductive functorial operational semantics $\Phi = \widehat{\phi^{\mathcal{R}}}$ which the method in Section 4 assigns to the rules $\mathcal{R}$. It yields operational models

$$\Phi k = [\![\text{-}]\!]^k_{\lceil\mathcal{R}\rceil} : TX \to BTX$$

which differ from the above $[\![\text{-}]\!]^k_{\mathcal{R}}$ only in the treatment of the first case of sequential composition:

$$[\![u\ ;\ v]\!]^k_{\lceil\mathcal{R}\rceil} = <a, u'\ ;\ \gamma_{TX}[\![v]\!]^k_{\lceil\mathcal{R}\rceil}> \quad \text{if } [\![u]\!]^k_{\lceil\mathcal{R}\rceil} = <a, u'>$$

where, recall, the transformation $\gamma : B \Rightarrow T$ is the embedding of the behaviour into the syntax mapping $*$ to nil and $<a, x>$ to $a\ ;\ x$. It is a retraction for the operational semantics in the sense that, in particular,

$$[\![\gamma_{TX}[\![v]\!]^k_{\lceil\mathcal{R}\rceil}]\!]^k_{\lceil\mathcal{R}\rceil} = [\![v]\!]^k_{\lceil\mathcal{R}\rceil}$$

This equation allows one to use the compositionality of functorial operational semantics to prove that the coinductive extension of $[\![\text{-}]\!]^k_{\lceil\mathcal{R}\rceil}$ is equal to the coinductive extension of $[\![\text{-}]\!]^k_{\mathcal{R}}$, that is,

$$([\![\text{-}]\!]^k_{\lceil\mathcal{R}\rceil})^@ = ([\![\text{-}]\!]^k_{\mathcal{R}})^@ : TX \to \widehat{B}$$

which implies that the abstract global behaviours corresponding to the former are the same as those corresponding to the latter, so that the two models are 'observationally equivalent' as claimed in Section 4. The proof is as follows.

$\mathcal{R}$ is observationally equivalent to $\lceil\mathcal{R}\rceil$. Recall that the definition of coinductive extension gives, for $[\![u\,;\,v]\!]^k_{\lceil\mathcal{R}\rceil} = <a, u'\,;\,\gamma_{TX}[\![v]\!]^k_{\lceil\mathcal{R}\rceil}>$,

$$([\![u\,;\,v]\!]^k_{\lceil\mathcal{R}\rceil})^@ = <a, ([\![u'\,;\,\gamma_{TX}[\![v]\!]^k_{\lceil\mathcal{R}\rceil}]\!]^k_{\lceil\mathcal{R}\rceil})^@>$$

Then, it is enough to show that

$$([\![u'\,;\,\gamma_{TX}[\![v]\!]^k_{\lceil\mathcal{R}\rceil}]\!]^k_{\lceil\mathcal{R}\rceil})^@ = ([\![u'\,;\,v]\!]^k_{\lceil\mathcal{R}\rceil})^@$$

If $\langle\!|\,\text{-}\,|\!\rangle : T\widehat{B} \to \widehat{B}$ is the denotational model coinduced by the operational monad $\Phi = \widehat{\phi^{\mathcal{R}}}$, one has, by the above adequacy result,

$$\begin{array}{rcll} ([\![u'\,;\,\gamma_{TX}[\![v]\!]^k_{\lceil\mathcal{R}\rceil}]\!]^k_{\lceil\mathcal{R}\rceil})^@ & = & \langle\!|\,([\![u']\!]^k_{\lceil\mathcal{R}\rceil})^@\,;\,([\![\gamma_{TX}[\![v]\!]^k_{\lceil\mathcal{R}\rceil}]\!]^k_{\lceil\mathcal{R}\rceil})^@\,|\!\rangle & \text{(adequacy)} \\ & = & \langle\!|\,([\![u']\!]^k_{\lceil\mathcal{R}\rceil})^@\,;\,([\![v]\!]^k_{\lceil\mathcal{R}\rceil})^@\,|\!\rangle & \text{(retraction)} \\ & = & ([\![u'\,;\,v]\!]^k_{\lceil\mathcal{R}\rceil})^@ & \end{array}$$

This concludes the proof.

The denotational model $\langle\!|\,\text{-}\,|\!\rangle : T\widehat{B} \to \widehat{B}$ coinduced by the operational monad $\Phi = \widehat{\phi^{\mathcal{R}}}$ is the coinductive extension of the operational model $[\![\text{-}]\!]^{\varphi}_{\lceil\mathcal{R}\rceil} : T\widehat{B} \to BT\widehat{B}$ which, from the above result that $\mathcal{R}$ is observationally equivalent to $\lceil\mathcal{R}\rceil$, is the same as the coinductive extension of $[\![\text{-}]\!]^{\varphi}_{\mathcal{R}} : \widehat{B} \to BT\widehat{B}$, that is,

$$\langle\!|\,\text{-}\,|\!\rangle = ([\![\text{-}]\!]^{\varphi}_{\mathcal{R}})^@ : T\widehat{B} \to \widehat{B}$$

By definition of coinductive extension, this gives, for every term $t \in T\widehat{B}$,

$$\langle\!|\,t\,|\!\rangle \;=\; \begin{cases} * & \text{if } \; [\![t]\!]^{\varphi}_{\mathcal{R}} = * \\ <a, \langle\!|\,t'\,|\!\rangle> & \text{if } \; [\![t]\!]^{\varphi}_{\mathcal{R}} = <a, t'> \end{cases}$$

Thus, in particular, the nil constant is denoted by $*$,

$$\langle\!|\,\mathsf{nil}\,|\!\rangle = *$$

every action a is denoted by the pair $<a, *>$,

$$\langle\!|\,a\,|\!\rangle = <a, \langle\!|\,\mathsf{nil}\,|\!\rangle> = <a, *>$$

and the denotation of the sequential composition of two streams p and q is

$$\langle\!|\,p\,;\,q\,|\!\rangle \;=\; \begin{cases} <a, \langle\!|\,p'\,;\,q\,|\!\rangle> & \text{if } \; p = a \cdot p' \\ <a, q'> & \text{if } \; p = \epsilon \text{ and } q = a \cdot q' \\ * & \text{if } \; p = \epsilon = q \end{cases}$$

The adequacy of this denotational model wrt the operational semantics induced by the rules $\mathcal{R}$, that is, the commutativity of

$$\begin{tikzcd}
T^2 0 \arrow[r, "T\llbracket\text{-}\rrbracket^{@}_{\mathcal{R}}"] \arrow[d, "\mu_0"'] & T\widehat{B} \arrow[d, "\llparenthesis\text{-}\rrparenthesis = (\llbracket\text{-}\rrbracket^{\varphi}_{\mathcal{R}})^{@}"] \\
T0 \arrow[r, "\llbracket\text{-}\rrbracket^{@}_{\mathcal{R}}"'] & \widehat{B}
\end{tikzcd}$$

tells then that

$$\llparenthesis\, C[\, \llbracket t\rrbracket^{@}_{\mathcal{R}}\,]\, \rrparenthesis \;=\; \llbracket \mu_0(C[\, t\,])\rrbracket^{@}_{\mathcal{R}}$$

for every context $C[\,\text{-}\,]$ and term t. (The multiplication $\mu_0 : T^2 0 \to T0$ is needed in order to make of the context $C[\,\text{-}\,]$ and of the term t a term in $T0$.) In particular, for the context with two 'holes' $(\,\text{-}\ ;\ \text{-}\,)$ one has, omitting the multiplication μ_0, the equation

$$\llparenthesis\, \llbracket u\rrbracket^{@}_{\mathcal{R}}\ ;\ \llbracket v\rrbracket^{@}_{\mathcal{R}}\, \rrparenthesis \;=\; \llbracket u\ ;\ v\rrbracket^{@}_{\mathcal{R}}$$

used in the above proof of the equivalence between $\mathcal{R}$ and $\lceil\mathcal{R}\rceil$.

Another consequence of the above adequacy is that programs with the same abstract global behaviour can be interchanged in any context. That is, if $\llbracket u\rrbracket^{@}_{\mathcal{R}} = \llbracket v\rrbracket^{@}_{\mathcal{R}}$, then $\llbracket C[\, u\,]\rrbracket^{@}_{\mathcal{R}} = \llbracket C[\, v\,]\rrbracket^{@}_{\mathcal{R}}$, or, equivalently, in terms of the observational equivalence $\sim$ introduced earlier in this section,

$$u \sim v \quad \text{implies} \quad C[\, u\,] \sim C[\, v\,].$$

Finally, notice that a denotational model is adequate also if the final coalgebra semantics is not equal to but only 'included' in the initial algebra semantics; that is, one can be more liberal and define a denotational model $\langle D, \llparenthesis\text{-}\rrparenthesis\rangle$ to be adequate if it contains $\widehat{B}$ as a subalgebra and the inclusion sends the final coalgebra semantics to the initial algebra semantics:

$$\begin{tikzcd}
 & & D \\
T0 \arrow[urr, "\llparenthesis\text{-}\rrparenthesis^{\#}"] \arrow[r, "\llbracket\text{-}\rrbracket^{@}"'] & \widehat{B} \arrow[ur, tail] &
\end{tikzcd}$$

Notes. The relevance of initial algebras for semantics, type theory, and algebraic specification was recognized by the 'ADJ' group in the mid-seventies. (Some references on initial algebra semantics are [GTW78, MG85, Mos90, MT92].)

The idea of coupling initial algebra with final coalgebra semantics was first used in [RT94] to give a categorical account of the method described in [Rut92] for systematically deriving denotational models from structural operational semantics. (For precursors of this method see [Bad87, Rut90].) This method is based on results like those in [dS85, BIM88, GV92, Gro93] which show that the above notion of observational equivalence ('strong bisimulation') is a congruence if suitable restrictions are imposed on the syntactic format of the rules. (Cf Section 11.) This kind of results, although of great practical relevance, is very much dependent on the use of labelled transition systems and hard to export to other notions of operational model. Instead here the idea is that the structural rules correspond to the germ of an inductive functorial semantics, that is, they can be seen as an *action* of the syntax on the composite functor BT, for *abstract* notions of syntax T and behaviour B.

Like in the present approach, in [RT94] the denotational model is *coinduced* by the operational rules and the equivalence between initial algebra and final coalgebra semantics is proved by means of a four-squares diagram

$$\begin{tikzcd}
T^2 0 \arrow[r, "T^2 0^{@}"] \arrow[d, "\text{initial algebra}\ \mu_0"'] & T^2 \widehat{B} \arrow[r, "T\llparenthesis - \rrparenthesis"] \arrow[d, "\mu_{\widehat{B}}"] & T\widehat{B} \arrow[d, "\llparenthesis - \rrparenthesis"] \\
T0 \arrow[r, "T0^{@}"] \arrow[d, "\llbracket - \rrbracket_{\mathcal{R}}"'] & T\widehat{B} \arrow[r, dashed, "\llparenthesis - \rrparenthesis"] \arrow[d, "\llbracket - \rrbracket^{\varphi}_{\mathcal{R}}"] & \widehat{B} \arrow[d, "\varphi\ \ \text{final coalgebra}"] \\
BT0 \arrow[r, "BT0^{@}"'] & BT\widehat{B} \arrow[r, "B\llparenthesis - \rrparenthesis"'] & B\widehat{B}
\end{tikzcd}$$

The difference is that, in order to ensure the commutativity of the upper right square, it is *assumed* in [RT94] that the observational equivalence coinduced by the operational semantics is a congruence, which fact, instead, becomes here a trivial consequence of functoriality. In fact, the *functorial* description of 'well-behaved' operational rules is the essence of the present approach.

7 A Dual Lifting: Functorial Denotational Semantics

A functorial operational semantics is a *monad* lifting the syntactical monad (freely generated by the signature) to the coalgebras of the behaviour. As shown in the previous section, this operational monad coinduces a denotational model. In fact, this denotational model is just one particular action of a '*comonad*' coinduced by the operational monad. This 'denotational comonad' is a lifting (to the algebras of the syntax) of another comonad, namely the 'observational comonad' cofreely generated by the behaviour.

The property that every operational monad coinduces a denotational comonad is the **basic property** of the functorial approach to operational semantics. Its dual also holds, namely every denotational comonad induces an operational monad; this gives a useful method to derive an operational semantics from a denotational one.

The notion of comonad is dual to the one of monad: a **comonad** on a category $\mathbf{C}$ is a triple

$$D = < D, \varepsilon, \delta >$$

with D an endofunctor on $\mathbf{C}$

$$D : \mathbf{C} \to \mathbf{C}$$

and with the **counit** ε and the **comultiplication** δ natural transformations

$$\varepsilon : D \Rightarrow I \qquad \delta : D \Rightarrow D^2$$

which satisfy the following laws.

Comonad Laws

$$\begin{tikzcd}
D \arrow[r, Rightarrow, "\delta"] \arrow[d, Rightarrow, "\delta"'] & D^2 \arrow[d, Rightarrow, "\delta_D"] \\
D^2 \arrow[r, Rightarrow, "D\delta"'] & D^3
\end{tikzcd}
\qquad
\begin{tikzcd}
 & D \arrow[dl, equal] \arrow[d, Rightarrow, "\delta"] \arrow[dr, equal] & \\
ID & D^2 \arrow[l, Rightarrow, "\varepsilon_D"] \arrow[r, Rightarrow, "D\varepsilon"'] & DI
\end{tikzcd}$$

A first example of a comonad is given by the **observational comonad** $D = < D, \varepsilon, \delta >$ cofreely generated by the behaviour endofunctor $BX = 1 + \mathsf{Act} \times X$ on **Set**. For every set X, the value of D at X is the carrier DX of the final coalgebra

$$DX \cong X \times B(DX)$$

of the endofunctor $(X \times B) : \mathbf{Set} \to \mathbf{Set}$. (Cf definition of TX in Section 2.) In particular, the value of D at singleton 1 – the final object of $\mathbf{Set}$ – is the carrier $\widehat{B} = \mathsf{Act}^{\infty}$ of the final B-coalgebra, because $1 \times X = X$. Thus $D1$ is the set of *abstract* global behaviours corresponding to B, that is, the finite and infinite streams generated by Act. (See Section 5.)

Now, a stream $(a_1 a_2 \cdots)$ can be seen as a sequence of transitions

$$\bullet \xrightarrow{a_1} \bullet \xrightarrow{a_2} \bullet \cdots$$

in which the states have no name or, equivalently, have all the same name $* \in \{*\} = 1$. Therefore, $D1$ is the set of global behaviours with a single state.

In general, the set DX is the set of global behaviours observable with states $x \in X$, that is, the finite and infinite sequences of transitions

$$x \xrightarrow{a_1} x_1 \xrightarrow{a_2} x_2 \cdots$$

with states $x \in X$ and actions $a \in \mathsf{Act}$. Formally, one can check that

$$DX = X + \coprod_{1 \leq \alpha \leq \omega} (X \times \mathsf{Act})^{\alpha}$$

The final coalgebra isomorphism $DX \cong X \times BDX$ splits into two projections:

$$X \xleftarrow{\mathsf{fst}_X} DX \cong X \times BDX \xrightarrow{\mathsf{snd}_X} BDX$$

These are the operations which allow one to *observe* these global behaviours: the first projection extracts the root of a global behaviour, the second projection gives its continuation. For instance:

$$x \quad \xleftarrow{\mathsf{fst}_X} \quad x \xrightarrow{a_1} x_1 \xrightarrow{a_2} x_2 \cdots \quad \xmapsto{\mathsf{snd}_X} \quad \xrightarrow{a_1} x_1 \xrightarrow{a_2} x_2 \cdots$$

The first projection $\mathsf{fst}_X : X \times BDX$ is the natural candidate for the value of the counit $\varepsilon : D \Rightarrow I$ at X:

$$\varepsilon_X = \mathsf{fst}_X : DX \to X$$

while the second projection can be coinductively extended to yield the comultiplication $\delta : D \Rightarrow D^2$. Indeed, by finality, the coalgebra $DX \cong X \times BDX$ comes with a *coinduction principle* which can be used to extend the operator D to an endofunctor and to define its comultiplication:

> Every $(X \times B)$-coalgebra structure $Y \to X \times BY$ is a pair $< f, k >$, with $f : Y \to X$ and $k : Y \to BY$. The first function can be seen as a 'covaluation' function, while the second is a B-coalgebra structure. By duality with the definition of inductive extensions along valuation

functions, call the corresponding coinductive extension $f^\flat = < f, k >^@$: $Y \to DX$

$$\begin{tikzcd}
 & Y \arrow[r, "k"] \arrow[dl, "f"'] \arrow[d, dashed, "f^\flat"] & BY \arrow[d, "Bf^\flat"] \\
X & DX \arrow[l, "\varepsilon_X = \mathsf{fst}_X"] \arrow[r, "\mathsf{snd}_X"'] & BDX
\end{tikzcd}$$

the **coinductive extension of** k **along the** covaluation **function** f.

Then, extend D to a functor by putting, for every function $f : X \to Y$,

$$Df = (f \circ \varepsilon_X)^\flat : DX \to DY$$

This function Df applied to a global behaviour $d_x = (x \xrightarrow{a_1} x_1 \xrightarrow{a_2} x_2 \cdots)$ substitutes every state in $d_x \in DX$ by its image under the 'renaming' f:

$$(Df)(d_x) = f(x) \xrightarrow{a_1} f(x_1) \xrightarrow{a_2} f(x_2) \cdots$$

Similarly, the value of the comultiplication $\delta : D \Rightarrow D^2$ at X is given by the coinductive extension of the second projection $\mathsf{snd}_X : DX \to BDX$ along the identity on DX:

$$\begin{tikzcd}
 & DX \arrow[r, "\mathsf{snd}_X"] \arrow[dl, phantom, "/\!/"] \arrow[d, dashed, "\delta_X = (\mathsf{id}_{DX})^\flat"] & BDX \arrow[d, "B\delta_X"] \\
DX & D^2X \arrow[l, "\varepsilon_{DX}"] \arrow[r, "\mathsf{snd}_{DX}"'] & BD^2X
\end{tikzcd}$$

The left triangle tells that ε is a left counit for δ. The proof that it is also a right counit and that δ is a comultiplication is dual to the proof in Section 2 for the unit η and the multiplication μ of the syntactical monad T.

Concretely, the comultiplication $\delta_X : DX \to D^2X$ maps a global behaviour $d_x = (x \xrightarrow{a_1} x_1 \xrightarrow{a_2} x_2 \cdots)$ to a global behaviour with the same transitions but with every state x_i replaced by its whole global behaviour d_{x_i}:

$$\delta_X(d_x) = (d_x \xrightarrow{a_1} d_{x_1} \xrightarrow{a_2} d_{x_2} \cdots)$$

In general, the coinductive extension of a coalgebra structure $k : Y \to BY$ along a function $f : Y \to X$ can be interpreted in terms of (deterministic) transition systems as follows. The B-coalgebra $\langle Y, k \rangle$ is a transition system with Y as set of states; the covaluation function $f : Y \to X$ maps every state $y \in Y$ to a state

$f(y) \in X$. Then, if the global behaviour of a state y in the transition system corresponding to $\langle Y, k\rangle$ is the (possibly infinite) sequence

$$y \xrightarrow{a_1} y_1 \xrightarrow{a_2} y_2 \cdots$$

the coinductive extension $f^\flat : Y \to DX$ maps y to this global behaviour, but replacing every state y_i by $f(y_i)$:

$$f^\flat(y_i) \;=\; f(y) \xrightarrow{a_1} f(y_1) \xrightarrow{a_2} f(y_2) \cdots$$

As an example, let the set Y of states be the set $\mathbb{Z}$ of integers and let the set Act of actions be trivial, that is, let Act be made of only one action a:

$$Y = \mathbb{Z} \qquad \text{and} \qquad \mathsf{Act} = \{a\}$$

Next, let the deterministic transition system corresponding to the coalgebra structure $k : \mathbb{Z} \to B(\mathbb{Z})$ be such that 0 is inert, a positive integer n performs a transition to its predecessor $n-1$, and a negative integer $-n$ performs a transition to its successor $-n+1$:

$$0\downarrow * \qquad n \xrightarrow{a} n-1 \qquad -n \xrightarrow{a} -n+1$$

Now, if X is the three-elements set $\{0, \clubsuit, \diamondsuit\}$ and $f : \mathbb{Z} \to \{0, \clubsuit, \diamondsuit\}$ is the function mapping 0 to 0, positive numbers to $\diamondsuit$, and negative numbers to $\clubsuit$, then the coinductive extension

$$f^\flat : \mathbb{Z} \to D\{0, \clubsuit, \diamondsuit\}$$

of the transition system along this covaluation function f maps every integer z to a sequence of a-transitions of length $|z|$ having 0 as last state and $\diamondsuit$ (resp., $\clubsuit$) as all other states if z is positive (resp., negative). Thus, for instance,

$$f^\flat(3) \;=\; \diamondsuit \xrightarrow{a} \diamondsuit \xrightarrow{a} \diamondsuit \xrightarrow{a} 0 \qquad f^\flat(-3) \;=\; \clubsuit \xrightarrow{a} \clubsuit \xrightarrow{a} \clubsuit \xrightarrow{a} 0$$

Notice that the same set $X = \{0, \clubsuit, \diamondsuit\}$ can be used to observe the global behaviours of the above transition system in quite a different way. Consider the function $g : \mathbb{Z} \to \{0, \clubsuit, \diamondsuit\}$ mapping odd numbers to $\clubsuit$ and even numbers to $\diamondsuit$. Then the coinductive extension $g^\flat : \mathbb{Z} \to \{0, \clubsuit, \diamondsuit\}$ of the transition system along g identifies n and $-n$. For instance:

$$g^\flat(3) \;=\; \clubsuit \xrightarrow{a} \diamondsuit \xrightarrow{a} \clubsuit \xrightarrow{a} 0 \;=\; g^\flat(-3)$$

The same identification can be obtained by setting $X = 1$ and thus forcing the covaluation function to map everything to the same state $\bullet \in \{\bullet\} = 1$. Then, the coinductive extension of k along this trivial function $Y \to 1$ is nothing but the simple coinductive extension

$$k^@ : Y \to \widehat{B} = D1 \cong 1{\times}BD1 \cong BD1 = B\widehat{B}$$

of k (see Section 5). In particular,

$$k^@(3) \;=\; \bullet \xrightarrow{a} \bullet \xrightarrow{a} \bullet \xrightarrow{a} \bullet \;=\; k^@(-3)$$

Consider now, for an arbitrary comonad $D =< D, \varepsilon, \delta >$ in a category $\mathbf{C}$, the **category $\mathbf{C}_D$ of D-coalgebras**. It is the category of coalgebras of the endofunctor D which 'respect' the counit ε and the comultiplication δ of the comonad D; that is, its objects are pairs $\langle X, k\rangle$, with X an object of $\mathbf{C}$ and $k : X \to DX$ an arrow of $\mathbf{C}$ satisfying the laws

$$
\begin{tikzcd}
X \arrow[r, "k"] \arrow[d, "k"'] & DX \arrow[d, "\delta_X"] \\
DX \arrow[r, "Dk"'] & D^2X
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
 & X \arrow[d, "k"] \arrow[dl, phantom, "/\!/"] \\
X & DX \arrow[l, "\varepsilon_X"]
\end{tikzcd}
$$

and its arrows $f : \langle X, k\rangle \to \langle Y, h\rangle$ are arrows $f : X \to Y$ of $\mathbf{C}$ such that $Df \circ k = h \circ f$.

> **B-coalgebras are D-coalgebras.** There is an isomorphism between the category of coalgebras of an endofunctor B and the coalgebras of its cofreely generated comonad D. This isomorphism maps every B-coalgebra $\langle X, k\rangle$ to the D-coalgebra with same carrier X and with structure the coinductive extension of k along the identity on X:
>
> $$
\begin{tikzcd}
 & X \arrow[r, "k"] \arrow[d, dashed, "(\mathsf{id}_X)^\flat"] \arrow[dl, phantom, "/\!/"] & BX \arrow[d, "B(\mathsf{id}_X)^\flat"] \\
X & DX \arrow[l, "\varepsilon_X"] \arrow[r, "\mathsf{snd}_X"'] & BDX
\end{tikzcd}
$$
>
> The inverse of this isomorphism is obtained by composing each D-coalgebra structure $k : X \to DX$ first with the second projection $\mathsf{snd}_X : X \times BDX \to BDX$ and then with $B\varepsilon_X : BDX \to BX$. That is:
>
> $$\langle X, k\rangle \mapsto \langle X, B\varepsilon_X \circ \mathsf{snd}_X \circ k\rangle$$
>
> The proof is simply the dual of the proof that *Σ-algebras are T-algebras* given in Section 2.

Notice that, under the above isomorphism of categories, the final B-coalgebra $\widehat{B} \cong B\widehat{B}$ corresponds to the cofree D-coalgebra over the final object, namely $\langle P1, \delta_1\rangle$, just like the initial Σ-algebra corresponds to $\langle T0, \mu_0\rangle$, the free T-algebra over the initial object.

The dualities between signature and syntactical monad on the one side and behaviour and observational comonad on the other side can be summarized as follows.

Signature $\Sigma : \mathbf{C} \to \mathbf{C}$	Behaviour $B : \mathbf{C} \to \mathbf{C}$
Algebras	Coalgebras
$X+\Sigma TX \cong TX$ = initial $(X+\Sigma)$-algebra	$DX \cong X\times BDX$ = final $(X\times B)$-coalgebra
Induction $(\text{-})^{\#}$	Coinduction $(\text{-})^{@}$
$\eta = \mathsf{inl} : I \Rightarrow T$	$\varepsilon = \mathsf{fst} : P \Rightarrow I$
$\mu = [\mathsf{id}, \mathsf{inr}]^{\#} = \mathsf{id}^{\sharp} : T^2 \Rightarrow T$	$\delta = <\mathsf{id}, \mathsf{snd}>^{@} = \mathsf{id}^{\flat} : P \Rightarrow P^2$
Syntactical Monad $T = <T, \eta, \mu>$	Observational Comonad $D = <D, \varepsilon, \delta>$
TX = Programs	DX = Global Behaviours
$\mathbf{C}^{\Sigma} \cong \mathbf{C}^{T}$	$\mathbf{C}_B \cong \mathbf{C}_D$
$\langle T0, \mu_0 \rangle$ = Initial Algebra	$\langle D1, \delta_1 \rangle$ = Final Coalgebra

Next, notice that the isomorphism between B- and D-coalgebras implies that every operational monad $\Phi = <\Phi, \eta, \mu>$ can be seen as a lifting of the syntactical monad $T = <T, \eta, \mu>$ to the coalgebras of the observational comonad D rather than to the coalgebras of the behaviour B (and vice versa). Thus, writing $U_D : \mathbf{C}_D \to \mathbf{C}$ for the forgetful functor mapping a D-coalgebra $\langle X, k \rangle$ to its carrier X, one has

Operational Monad

$$\begin{tikzcd} \mathbf{C}_D \arrow[r, "\Phi"] \arrow[d, "U_D"'] & \mathbf{C}_D \arrow[d, "U_D"] \\ \mathbf{C} \arrow[r, "T"'] & \mathbf{C} \end{tikzcd}$$

That is, for every D-coalgebra structure $k : X \to DX$, one has that $\Phi k : TX \to DTX$ is also a D-coalgebra structure and, moreover, the two squares in the diagram

$$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[d, "k"'] & TX \arrow[d, "\Phi k"] & T^2X \arrow[l, "\mu_X"'] \arrow[d, "\Phi^2 k"] \\ DX \arrow[r, "D\eta_X"'] & DTX & DT^2X \arrow[l, "D\mu_X"] \end{tikzcd}$$

commute.

In the above form, the definition of functorial operational semantics can be easily dualized to yield the definition of **functorial denotational semantics**, namely as a comonad Ψ lifting the observational comonad $D = < D, \varepsilon, \delta >$ to the T-algebras:

Denotational Comonad

$$\begin{tikzcd} \mathbf{C}^T \arrow[r, "\Psi"] \arrow[d, "U^T"'] & \mathbf{C}^T \arrow[d, "U^T"] \\ \mathbf{C} \arrow[r, "D"'] & \mathbf{C} \end{tikzcd}$$

That is, Ψ is a comonad with counit and comultiplication inherited from the observational comonad $D = < D, \varepsilon, \delta >$

$$\Psi = < \Psi, \varepsilon, \delta >$$

and with $\Psi : \mathbf{C}^T \to \mathbf{C}^T$ such that

$$U^T \Psi = DU^T : \mathbf{C}^T \to \mathbf{C}$$

Equivalently, Ψ is an **action** of the monad T on the composite functor $DU^T : \mathbf{C}^T \to \mathbf{C}$, ie a natural transformation

$$\Psi : TDU^T \Rightarrow DU^T$$

such that, for every T-algebra $h : TX \to X$, $\Psi h : TDX \to DX$ is also a T-algebra. Therefore, the fact that the counit and comultiplication of the observational comonad D lift to those of the denotational comonad Ψ is equivalent to the commutativity of the two squares in the following diagram.

$$\begin{tikzcd} TX \arrow[d, "h"'] & TDX \arrow[l, "T\varepsilon_X"'] \arrow[r, "T\delta_X"] \arrow[d, "\Psi h"'] & TD^2X \arrow[d, "\Psi^2 h"] \\ X & DX \arrow[l, "\varepsilon_X"] \arrow[r, "\delta_X"'] & D^2X \end{tikzcd}$$

(Cf Section 4.)

The basic property of the functorial approach to operational semantics can now be stated.

The denotational comonad $\Phi^@$ coinduced by an operational monad Φ.

Every operational monad $\Phi = <\Phi, \eta, \mu>$ lifting a syntactical monad $T = <T, \eta, \mu>$ to the coalgebras of an observational comonad $D = <D, \varepsilon, \delta>$ coinduces an endofunctor

$$\Phi^@ : \mathbf{C}^T \to \mathbf{C}^T$$

such that $\Phi^@ = <\Phi^@, \varepsilon, \delta>$ is a denotational comonad lifting D to the T-algebras:

$$\begin{tikzcd}
\mathbf{C}_D \arrow[r, "\Phi"] \arrow[d, "U_D"'] & \mathbf{C}_D \arrow[d, "U_D"] \\
\mathbf{C} \arrow[r, "T"'] & \mathbf{C}
\end{tikzcd}
\quad \longmapsto \quad
\begin{tikzcd}
\mathbf{C}^T \arrow[r, "\Phi^@"] \arrow[d, "U^T"'] & \mathbf{C}^T \arrow[d, "U^T"] \\
\mathbf{C} \arrow[r, "D"'] & \mathbf{C}
\end{tikzcd}$$

The endofunctor $\Phi^@$ on the T-algebras is defined by coinduction as follows. For simplicity, recalling the isomorphism $\mathbf{C}_D \cong \mathbf{C}_B$ between the categories of D- and B-coalgebras, consider the operational monad Φ to be on the B-coalgebras rather than on the D-coalgebras. Now, one needs, for every T-algebra structure $h : TX \to X$, a T-algebra structure $\Phi^@ h : TDX \to DX$. Therefore, first apply the given operational monad Φ to the B-coalgebra structure

$$\mathsf{snd}_X : DX \to BDX$$

obtaining the B-coalgebra structure

$$\Phi(\mathsf{snd}_X) : TDX \to BTDX$$

and then take the coinductive extension of this coalgebra structure $\Phi(\mathsf{snd}_X)$ along the composite arrow $h \circ T\varepsilon_X : TDX \to X$

$$\begin{tikzcd}
TX \arrow[d, "h"'] & TDX \arrow[l, "T\varepsilon_X"'] \arrow[r, "\Phi\,\mathsf{snd}_X"] \arrow[d, dashed, "\Phi^@ h = (h \circ T\varepsilon_X)^\flat"] & BTDX \arrow[d, "B\Phi^@ h"] \\
X & DX \arrow[l, "\varepsilon_X"] \arrow[r, "\mathsf{snd}_X"'] & BDX
\end{tikzcd}$$

That is,

$$\Phi^@ h = <h \circ T\varepsilon_X, \Phi(\mathsf{snd}_X)>^@ = (h \circ T\varepsilon_X)^\flat : TDX \to DX$$

The claim is threefold: (i) $\Phi^@ h : TDX \to DX$ is a T-algebra structure, (ii) the operation $\Phi^@$ is functorial, and (iii) the counit and comultiplication of the observational comonad $D = < D, \varepsilon, \delta >$ lift to counit and comultiplication for $\Phi^@$. The proofs are all by coinduction.

Let us start from (iii), that is, from the claim that the two squares in the diagram

$$\begin{tikzcd}
TX \arrow[d, "h"'] & TDX \arrow[l, "T\varepsilon_X"'] \arrow[r, "T\delta_X"] \arrow[d, "\Phi^{@}h"'] & TD^2X \arrow[d, "\Phi^{@^2}h"] \\
X & DX \arrow[l, "\varepsilon_X"] \arrow[r, "\delta_X"'] & D^2X
\end{tikzcd}$$

commute:

> The left square of the above diagram commutes by definition. As for the right square, it commutes because both composite arrows $\Phi^{@^2}h \circ T\delta_X$ and $\delta_X \circ \Phi^{@}h$ from TDX to X fit as the (unique!) coinductive extension
>
> $$\begin{tikzcd}
& TDX \arrow[r, "\Phi\,\mathsf{snd}_X"] \arrow[dd, dashed, "!"] & BTDX \arrow[dd, "B!"] \\
TDX \arrow[ur, phantom, sloped, "="] \arrow[d, "\Phi^{@}h"'] & & \\
DX & D^2X \arrow[l, "\varepsilon_{DX}"] \arrow[r, "\mathsf{snd}_{DX}"'] & BD^2X
\end{tikzcd}$$
>
> of the coalgebra structure $\Phi(\mathsf{snd}_X) : TDX \to BTDX$ along the arrow $\Phi^{@}h : TDX \to DX$. Indeed:
>
> $$\begin{tikzcd}
& TDX \arrow[r, "\Phi\,\mathsf{snd}_X"] \arrow[d, "T\delta_X"] & BTDX \arrow[d, "BT\delta_X"] \\
TDX \arrow[ur, phantom, sloped, "="] \arrow[d, "\Phi^{@}h"'] & TD^2X \arrow[l, "T\varepsilon_{DX}"] \arrow[r, "\Phi\,\mathsf{snd}_{DX}"'] \arrow[d, "\Phi^{@^2}h"] & BTD^2X \arrow[d, "B\Phi^{@^2}h"] \\
DX & D^2X \arrow[l, "\varepsilon_{DX}"] \arrow[r, "\mathsf{snd}_{DX}"'] & BD^2X
\end{tikzcd}
\qquad
\begin{tikzcd}
& TDX \arrow[r, "\Phi\,\mathsf{snd}_X"] \arrow[d, "\Phi^{@}h"] & BTDX \arrow[d, "B\Phi^{@}h"] \\
TDX \arrow[ur, phantom, sloped, "="] \arrow[d, "\Phi^{@}h"'] & DX \arrow[r, "\mathsf{snd}_X"'] \arrow[d, "\delta_X"] & BDX \arrow[d, "B\delta_X"] \\
DX \arrow[ur, phantom, sloped, "="] & D^2X \arrow[l, "\varepsilon_{DX}"] \arrow[r, "\mathsf{snd}_{DX}"'] & BD^2X
\end{tikzcd}$$

Next, consider the claim (i) that the arrow $\Phi^{@}h : TDX \to DX$ is a T-algebra structure, that is,

$$\Phi^{@}h \circ T\Phi^{@}h = \Phi^{@}h \circ \mu_{DX} \qquad \text{and} \qquad \Phi^{@}h \circ \eta_{DX} = \mathsf{id}_{DX}$$

The first equation holds because both $\Phi^{@}h \circ T\Phi^{@}h$ and $\Phi^{@}h \circ \mu_X$ fit as the coinductive extension of the coalgebra structure $\Phi^2(\mathsf{snd}_X) : T^2DX \to BT^2DX$ along the arrow $(h \circ Th) \circ T^2\varepsilon_X = (h \circ \mu_X) \circ T^2\varepsilon_X : T^2DX \to X$

$$\begin{tikzcd}
 & T^2X \arrow[dl, "Th"'] \arrow[dr, "\mu_X"] & & T^2DX \arrow[ll, "T^2\varepsilon_X"'] \arrow[r, "\Phi^2\mathsf{snd}_X"] \arrow[dd, dashed, "!"] & BT^2DX \arrow[dd, "B!"] \\
TX \arrow[dr, "h"'] & & TX \arrow[dl, "h"] & & \\
 & X & & DX \arrow[ll, "\varepsilon_X"] \arrow[r, "\mathsf{snd}_X"'] & BDX
\end{tikzcd}$$

$$\begin{tikzcd}
T^2DX \arrow[r, "\Phi^2\mathsf{snd}_X"] \arrow[d, "T\Phi^{@}h"'] & BT^2DX \arrow[d, "BT\Phi^{@}h"] \\
TDX \arrow[r, "\Phi\,\mathsf{snd}_X"'] & BTDX
\end{tikzcd}
\qquad
\begin{tikzcd}
T^2DX \arrow[r, "\Phi^2\mathsf{snd}_X"] \arrow[d, "\mu_{DX}"'] & BT^2DX \arrow[d, "B\mu_{DX}"] \\
TDX \arrow[r, "\Phi\,\mathsf{snd}_X"'] & BTDX
\end{tikzcd}$$

Similarly, the second equation, namely $\Phi^{@}h \circ \eta_{DX} = \mathsf{id}_{DX}$, holds because both $\Phi^{@}h \circ \eta_{DX}$ and the identity on DX fit as the coinductive extension of the coalgebra structure $\Phi(\mathsf{snd}_X) : TDX \to BTDX$ along $\varepsilon_X = \mathsf{id}_X \circ \varepsilon_X = (h \circ \eta_X) \circ \varepsilon_X : DX \to X$

$$\begin{tikzcd}
 & X \arrow[dl, "\eta_X"'] \arrow[dd, phantom, "\|"] & DX \arrow[l, "\varepsilon_X"'] \arrow[r, "\Phi\,\mathsf{snd}_X"] \arrow[dd, dashed, "!"] & BTDX \arrow[dd, "B!"] \\
TX \arrow[dr, "h"'] & & & \\
 & X & DX \arrow[l, "\varepsilon_X"] \arrow[r, "\mathsf{snd}_X"'] & BD^2X
\end{tikzcd}$$

Finally, the claim (ii) that, for every T-algebra arrow $f : \langle X, h\rangle \to \langle Y, k\rangle$, the operation

$$\begin{tikzcd}
TX \arrow[r, "Tf"] \arrow[d, "h"'] & TY \arrow[d, "k"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}
\quad\longmapsto\quad
\begin{tikzcd}
TDX \arrow[r, "TDf"] \arrow[d, "\Phi^{@}h"'] & TDY \arrow[d, "\Phi^{@}k"] \\
DX \arrow[r, "\Phi^{@}f = Df"'] & DY
\end{tikzcd}$$

is functorial amounts to

$$\Phi^{@}g \circ \Phi^{@}f = \Phi^{@}(g \circ f) \qquad \text{and} \qquad \Phi^{@}\mathsf{id}_X = \mathsf{id}_{DX}$$

for every T-algebra arrow $g : \langle Y, k\rangle \to \langle Z, l\rangle$. Its proof is similar to the one of (i) and left to the reader.

Notice that the above construction applies to any lifting of a (not necessarily freely generated) monad to the coalgebras of a cofreely generated comonad on any category.

As an example, consider the denotational comonad coinduced by the operational monad $\Phi = \widehat{\phi^{\mathcal{R}}}$ corresponding to the rule $\mathcal{R}$ for the sample language

$$t ::= x \mid \mathsf{nil} \mid a \mid (t \mathbin{;} t)$$

Thus, using the notation of Section 4, $\Phi\,\mathsf{snd}_X = [\![\text{-}]\!]^{\mathsf{snd}_X}_{\lceil\mathcal{R}\rceil} : TDX \to BTDX$. As a shorthand, write

$$\Phi\,\mathsf{snd}_X = [\![\text{-}]\!]_X : TDX \to BTDX$$

Next, recall the set DX is the set of global behaviours

$$d_x \;=\; x \xrightarrow{a_1} x_1 \xrightarrow{a_2} x_2 \cdots$$

with $x_i \in X$; the counit $\varepsilon_X : DX \to X$ is the operation returning the root x of a global behaviour d_x and the second projection $\mathsf{snd}_X : DX \to BDX$ returns its continuation.

Then, the value of the corresponding coinduced denotational comonad $\Phi^@$ at a T-algebra structure $h : TX \to X$ is

$$\begin{tikzcd}
TX \arrow[d, "h"'] & TDX \arrow[l, "T\varepsilon_X"'] \arrow[r, "{[\![\text{-}]\!]_X}"] \arrow[d, dashed, "\Phi^@ h = <h \circ T\varepsilon_X, [\![\text{-}]\!]_X>^@"'] & BTDX \arrow[d, "B\Phi^@ X"] \\
X & DX \arrow[l, "\varepsilon_X"] \arrow[r, "\mathsf{snd}_X"'] & BDX
\end{tikzcd}$$

which gives, for all global behaviours $d_x, d_y \in DX$,

$$(\Phi^@ h)(d_x \mathbin{;} d_y) \;=\; \langle h(x \mathbin{;} y), (\Phi^@ h)(\mathsf{snd}_X(d_x) \mathbin{;} d_y)\rangle$$

if $\mathsf{snd}_X(d_x)$ is different from $*$. Thus, for instance, the term

$$d_x \mathbin{;} d_y \;=\; (x \xrightarrow{a_1} x_1 \xrightarrow{a_2} x_2) \mathbin{;} (y \xrightarrow{b_1} y_1 \cdots)$$

is mapped to the global behaviour

$$h(x \mathbin{;} y) \xrightarrow{a_1} h(x_1 \mathbin{;} y) \xrightarrow{a_2} h(x_2 \mathbin{;} y) \xrightarrow{b_1} y_1 \cdots$$

That is, the meaning of the sequential composition of two global behaviours d_x and d_y is obtained by first concatenating d_y to d_x and then replacing the states x_i of d_x by $h(x_i \mathbin{;} y)$, where y is the root of d_y, while apart from y which is removed, all states of d_y are left the same.

In particular, consider X equal to the singleton 1, the final object in **Set**. There exists only one function from $T1$ to 1, namely the trivial function $1 : T1 \to 1$ mapping every term of $T1$ to the state $\bullet \in \{\bullet\} = 1$. Next, the set $D1 \cong 1 \times BD1 \cong BD1$ is the carrier of the final coalgebra $\widehat{B} \cong B\widehat{B}$ and, moreover, the structure $\mathsf{snd}_1 : D1 \to BD1$ is isomorphic to the final coalgebra isomorphism $\varphi : \widehat{B} \cong B\widehat{B}$. That is,

$$\langle D1, \mathsf{snd}_1 \rangle \cong \langle \widehat{B}, \varphi \rangle$$

Then, the T-algebra structure $\Phi^{@}1 : TD1 \to D1$ is isomorphic to the canonical denotational model

$$\begin{tikzcd} T\widehat{B} \arrow[r, dashed, "(\Phi\varphi)^{@}"] \arrow[d, "\Phi\varphi"'] & \widehat{B} \arrow[d, "\varphi"] \\ BT\widehat{B} \arrow[r, "B(\Phi\varphi)^{@}"'] & B\widehat{B} \end{tikzcd}$$

given in the previous section.

Finally, consider the dual of the above construction, namely

The operational monad $\Psi^{\#}$ induced by a denotational comonad Ψ.

Every denotational comonad $\Psi = <\Psi, \varepsilon, \delta>$ lifting an observational comonad $D = < D, \varepsilon, \delta>$ to the algebras of a syntactical monad $T = <T, \eta, \mu>$ induces an endofunctor

$$\Psi^{\#} : \mathbf{C}_D \to \mathbf{C}_D$$

such that $\Psi^{\#} = <\Psi^{\#}, \eta, \mu>$ is an operational monad lifting T to the D-coalgebras:

$$\begin{tikzcd} \mathbf{C}^T \arrow[r, "\Psi"] \arrow[d, "U^T"'] & \mathbf{C}^T \arrow[d, "U^T"] \\ \mathbf{C} \arrow[r, "D"'] & \mathbf{C} \end{tikzcd} \quad \longmapsto \quad \begin{tikzcd} \mathbf{C}_D \arrow[r, "\Psi^{\#}"] \arrow[d, "U_D"'] & \mathbf{C}_D \arrow[d, "U_D"] \\ \mathbf{C} \arrow[r, "T"'] & \mathbf{C} \end{tikzcd}$$

The endofunctor $\Psi^{\#}$ on the D-coalgebras is defined by induction as follows.

$$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[d, "k"'] & TX \arrow[d, dashed, "\Psi^{\#}k = (D\eta_X \circ k)^{\sharp}"] & \Sigma TX \arrow[l, "\mathsf{inr}_X"'] \arrow[d, "\Sigma\Psi^{\#}h"] \\ DX \arrow[r, "D\eta_X"'] & DTX & \Sigma DTX \arrow[l, "\Psi\,\mathsf{inr}_X"] \end{tikzcd}$$

That is, for every coalgebra structure $k : X \to DX$,

$$\Psi^{\#}k = [D\eta_X \circ k, \Psi(\mathsf{inr}_X)]^{\#} = (D\eta_X \circ k)^{\sharp} : TX \to DTX$$

(Again, for simplicity, the denotational monad Ψ is assumed to be on the Σ-algebras rather than on the isomorphic category of T-algebras.)

Notes. Comonads in semantics appear in Brookes and Geva's work [BG92], which bears resemblance with Moggi's work on computational monads [Mog91]. The *computational comonads* defined in [BG92] are comonads $D{=}{<}D, \varepsilon, \delta{>}$ with an extra operation $\gamma : I \Rightarrow D$ such that

$$\varepsilon \circ \gamma = \mathsf{id} \qquad \text{and} \qquad \delta \circ \gamma = \gamma_T \circ \gamma$$

The type D is the type of computations and the operation γ embeds data into computations. For instance, the observational comonad D cofreely generated by the endofunctor $X \mapsto 1 + X$ is a computational comonad as well: the set DX is the set X^∞ of finite and infinite sequences of $x \in X$ and the operation $\gamma : I \Rightarrow D$ 'saturates' every $x \in X$ by mapping it to the infinite sequence x^ω.

Brookes and Geva's work focuses on the ('co-Kleisli') subcategory of cofree coalgebras of a computational comonad rather than on the full category of coalgebras as in the present work. It would be interesting to understand whether there is a closer relationship between the two notions "computational comonad" and "observational comonad".

As pointed out to this author by Axel Poigné, liftings of functors to algebras of monads were studied in [Joh75]. In particular, Lemma 1 of [Joh75] shows that such liftings are in one-to-one correspondence with distributive laws (cf Section 4); in particular, every lifting of an *endofunctor* (thus without comonad operations!) D to the T-algebras is equivalent to a distributive law of the monad T over the endofunctor D.

The systematic method introduced in this section for deriving operational models from denotational ones is simply the dual to the already known method for deriving denotational models from operational ones. The existence of such a method had been forecasted in Section 5.3 of [RT94] (thanks to the mixed algebraic/coalgebraic approach used there which already allowed for a dualization), yet it had never been described before. (In general, one of the advantages of bringing to light the categorical structure underlying a given phenomenon is that then the mighty *duality principle* can be applied.) A concrete example of an operational monad $\Psi^\#$ induced by a denotational comonad Ψ is given in Section 10, where it is used to prove that 'basic process algebra' is functorial.

8 Operational is Denotational

The coinductive construction $\Phi \mapsto \Phi^{@}$ is a bijection between operational monads and denotational comonads whose inverse is the inductive construction $\Psi \mapsto \Psi^{\#}$. The proof of this fact is given in terms of adjunctions.

Let us rephrase the inductive construction at the end of the previous section of an operational monad $\Psi^{\#}$ from a denotational comonad Ψ in terms of adjunctions.

Recall, for every D-coalgebra structure $k : X \to DX$, the structure $\Psi^{\#}k : TX \to DTX$ is defined as the inductive extension of k along the composite $D\eta_X \circ k : X \to DTX$.

$$\begin{tikzcd}
X \arrow[r, "\eta_X"] \arrow[d, "k"'] & TX \arrow[d, dashed, "\Psi^{\#}k = (D\eta_X \circ k)^{\sharp}"] & \Sigma TX \arrow[l, "\mathsf{inr}_X"'] \arrow[d, "\Sigma\Psi^{\#}h"] \\
DX \arrow[r, "D\eta_X"'] & DTX & \Sigma DTX \arrow[l, "\Psi\,\mathsf{inr}_X"]
\end{tikzcd}$$

But this is the same as saying that $\Psi^{\#}k$ is obtained by taking the left adjunct of the function

$$D\eta_X \circ k : X \to DTX = U^{\Sigma}\langle DTX, \Psi\,\mathsf{inr}_X\rangle$$

wrt the adjunction $F^{\Sigma} \dashv U^{\Sigma}$, where $U^{\Sigma} : \mathbf{Set}^{\Sigma} \to \mathbf{Set}$ is the forgetful functor mapping Σ-algebras to their carriers and $F^{\Sigma} : \mathbf{Set} \to \mathbf{Set}^{\Sigma}$ is its left adjoint mapping a set X to the free Σ-algebra $\langle TX, \mathsf{inr}_X\rangle$ over X. (Cf Section 2.)

$$\begin{tikzcd}
X \arrow[r, "\eta_X"] \arrow[d, "k"'] & TX \arrow[d, "\Psi^{\#}k"'] \\
DX \arrow[r, "D\eta_X"'] & DTX
\end{tikzcd}
\qquad
\begin{tikzcd}
TX \arrow[d, dashed, "\Psi^{\#}k = (D\eta_X \circ k)^{\sharp}"] & \Sigma TX \arrow[l, "\mathsf{inr}_X"'] \arrow[d, "\Sigma\Psi^{\#}k"] \\
DTX & \Sigma DTX \arrow[l, "\Psi\,\mathsf{inr}_X"]
\end{tikzcd}$$

This is for an operational monad Ψ on the Σ-algebras. If, instead, the monad Ψ is on the isomorphic categories of T-algebras, one can use the similar adjunction $F^{T} \dashv U^{T}$ regarding DTX as carrying the T-algebra structure $\Psi\mu_X : TDTX \to DTX$ and thus obtaining $\Phi^{@}k$ as the left adjunct of

$$D\eta_X \circ k : X \to DTX = U^{T}\langle DTX, \Psi\mu_X\rangle$$

wrt this latter adjunction. That is:

$$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[d, "k"'] & TX \arrow[d, "\Psi^{\#}k"'] \\ DX \arrow[r, "D\eta_X"'] & DTX \end{tikzcd} \qquad \begin{tikzcd} TX \arrow[d, dashed, "\Psi^{\#}k", "= (D\eta_X \circ k)^{\sharp}"'] & T^2X \arrow[l, "\mu_X"'] \arrow[d, "T\Psi^{\#}k"] \\ DTX & TDTX \arrow[l, "\Psi\mu_X"] \end{tikzcd}$$

Next, recall that, while the syntactical monad T is freely generated by the signature Σ, the observational comonad is cofreely generated by the behaviour B. Then, by duality, the forgetful functor $U_B : \mathbf{Set}_B \to \mathbf{Set}$ mapping coalgebras to their carriers has a *right adjoint*, namely the functor

$$G_B : \mathbf{Set} \to \mathbf{Set}_B \qquad\qquad X \mapsto \langle DX, \mathsf{snd}_X \rangle$$

mapping a set X to the cofree coalgebra over it. (This holds for arbitrary endofunctors $B : \mathbf{C} \to \mathbf{C}$, provided that the endofunctor $(X \times B) : \mathbf{C} \to \mathbf{C}$ has a final coalgebra for every object X in $\mathbf{C}$.) Similarly, the forgetful functor $U_D : \mathbf{C}_D \to \mathbf{C}$ mapping the coalgebras of a comonad $D = < D, \varepsilon, \delta >$ to their carriers has a right adjoint

$$G_D : \mathbf{C} \to \mathbf{C}_D \qquad\qquad X \mapsto \langle DX, \delta_X \rangle$$

and the counit $\varepsilon : U_D G_D = D \Rightarrow I$ of this adjunction $U_D \dashv G_D$ is simply the counit of the comonad D. Therefore, the coinductive construction of the denotational monad $\Phi^{@}$ from an operational monad Φ on the B-coalgebras

$$\begin{tikzcd} TX \arrow[d, "h"'] & TDX \arrow[l, "T\varepsilon_X"'] \arrow[r, "\Phi\,\mathsf{snd}_X"] \arrow[d, dashed, "\Phi^{@}h", "= (h \circ T\varepsilon_X)^{\flat}"'] & BTDX \arrow[d, "B\Phi^{@}h"] \\ X & DX \arrow[l, "\varepsilon_X"] \arrow[r, "\mathsf{snd}_X"'] & BDX \end{tikzcd}$$

can be rephrased in terms of operational monads Φ on the D-coalgebras as the right adjunct wrt the adjunction $U_D \dashv G_D$ of the arrow

$$h \circ T\varepsilon_X : U_D \langle TDX, \Psi\delta_X \rangle = TDX \to X$$

That is:

$$\begin{tikzcd} TX \arrow[d, "h"'] & TDX \arrow[l, "T\varepsilon_X"'] \arrow[d, "\Phi^{@}h"'] \\ X & DX \arrow[l, "\varepsilon_X"] \end{tikzcd} \qquad \begin{tikzcd} TDX \arrow[r, "\Phi\delta_X"] \arrow[d, dashed, "\Phi^{@}h", "= (h \circ T\varepsilon_X)^{\flat}"'] & DTDX \arrow[d, "D\Phi^{@}h"] \\ DX \arrow[r, "\delta_X"'] & D^2X \end{tikzcd}$$

In order to calculate the value of this right adjunct $\Phi^{@}h = (h \circ T\varepsilon_X)^\flat$, one can use the standard formula

$$f^\flat = Gf \circ \eta_X$$

valid for every adjunction $F \dashv G$ (with unit $\eta : I \Rightarrow GF$), which, pictorially, amounts to the following bijection.

$$\frac{FX \xrightarrow{f} Y}{X \xrightarrow[\eta_X]{} GFX \xrightarrow[Gf]{} GY}$$

In particular, the unit itself is the right adjunct of the identity. For the adjunction $U_D \dashv G_D$ this gives that the unit at a coalgebra $\langle X, k\rangle$ is the structure $k : X \to DX$ of the coalgebra itself, since, by the D-coalgebra laws,

$$\begin{tikzcd} & X \arrow[d, "k"] \arrow[dl, equal] \\ X & DX \arrow[l, "\varepsilon_X"] \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "k"] \arrow[d, "k"'] & DX \arrow[d, "\delta_X"] \\ DX \arrow[r, "Dk"'] & D^2X \end{tikzcd}$$

Therefore

$$\frac{TDX \xrightarrow{h \circ T\varepsilon_X} X}{TDX \xrightarrow[\Phi\delta_X]{} DTDX \xrightarrow[D(h \circ T\varepsilon_X)]{} DX}$$

and thus

$$\Phi^{@}h = (h \circ T\varepsilon)^\flat = Dh \circ DT\varepsilon_X \circ \Phi\delta_X$$

Finally, notice that, by using the adjunction, the comonad D needs not to be cofreely generated by an endofunctor, the coinduction principle being replaced by the more general adjunction principle. Dually, also the induction principle can be replaced by the adjunction principle, which holds for every monad T.

To summarize:

Monad $\mathsf{T} = < T, \eta, \mu >$

$$F^T X = \langle TX, \mu_X \rangle$$

$$F^T \dashv U^T$$

$$\begin{tikzcd}
\mathbf{C}_D \arrow[r, "\Phi"] \arrow[d, "U_D"'] & \mathbf{C}_D \arrow[d, "U_D"] \\
\mathbf{C} \arrow[r, "T"'] & \mathbf{C}
\end{tikzcd}$$

$$\begin{aligned}
\Psi^{\#} &= (D\eta \circ _)^{\sharp} \\
&= \Psi\mu_X \circ TD\eta_X \circ T(_)
\end{aligned}$$

$$\begin{tikzcd}
\mathbf{C}_D \arrow[r, "\Psi^{\#}"] \arrow[d, "U_D"'] & \mathbf{C}_D \arrow[d, "U_D"] \\
\mathbf{C} \arrow[r, "T"'] & \mathbf{C}
\end{tikzcd}$$

Comonad $\mathsf{D} = < D, \varepsilon, \delta >$

$$G_D = \langle DX, \delta_X \rangle$$

$$U_D \dashv G_D$$

$$\begin{tikzcd}
\mathbf{C}^T \arrow[r, "\Phi^{@}"] \arrow[d, "U^T"'] & \mathbf{C}^T \arrow[d, "U^T"] \\
\mathbf{C} \arrow[r, "D"'] & \mathbf{C}
\end{tikzcd}$$

$$\begin{aligned}
\Phi^{@} &= (_ \circ T\varepsilon)^{\flat} \\
&= D(_) \circ DT\varepsilon_X \circ \Phi\delta_X
\end{aligned}$$

$$\begin{tikzcd}
\mathbf{C}^T \arrow[r, "\Psi"] \arrow[d, "U^T"'] & \mathbf{C}^T \arrow[d, "U^T"] \\
\mathbf{C} \arrow[r, "D"'] & \mathbf{C}
\end{tikzcd}$$

Operational is Denotational. The mapping $\Phi \mapsto \Phi^{@}$ is a bijection between operational monads and denotational comonads with $\Psi \mapsto \Psi^{\#}$ as inverse:

$$\begin{tikzcd}
\Phi \arrow[dd, mapsto] & \mathbf{C}_D \arrow[r, "\Phi"] \arrow[d, "U_D"'] & \mathbf{C}_D \arrow[d, "U_D"] & \Psi^{\#} \\
& \mathbf{C} \arrow[r, "T"'] & \mathbf{C} & \\
\Phi^{@} & \mathbf{C}^T \arrow[r, "\Psi"] \arrow[d, "U^T"'] & \mathbf{C}^T \arrow[d, "U^T"] & \Psi \arrow[uu, mapsto] \\
& \mathbf{C} \arrow[r, "D"'] & \mathbf{C} &
\end{tikzcd}$$

In order to prove that, for every D-coalgebra structure $k : X \to DX$, one has $\Phi k = (\Phi^{@})^{\#} k$, let us first rewrite $(\Phi^{@})^{\#} k$ in terms of Φ: because $\Phi^{@} h = Dh \circ DT\varepsilon_X \circ \Phi\delta_X$ for every T-algebra structure h and hence

$$\Phi^{@}\mu_X = D\mu_X \circ DT\varepsilon_{TX} \circ \Phi\delta_{TX}$$

and because

$$\Psi^{\#} k = \Psi\mu_X \circ TD\eta_X \circ Tk$$

one has

$$\begin{aligned} (\Phi^{@})^{\#} k &= \Phi^{@}\mu_X \circ TD\eta_X \circ Tk \\ &= D\mu_X \circ DT\varepsilon_{TX} \circ \Phi\delta_{TX} \circ TD\eta_X \circ Tk \end{aligned}$$

But then everything in sight in the following diagram commutes.

$$\begin{tikzcd}
TX \arrow[rrr, "\Phi k"] & & & DTX \\
TX \arrow[u, equal] \arrow[r, "T\eta_X"] \arrow[d, "Tk"'] \arrow[rrru, bend right=90, looseness=1.5, "(\Phi^{@})^{\#} k"'] & TDX \arrow[ul, "\mu_X"'] \arrow[r, "\Phi^2 k"] \arrow[d, "T\Phi k"'] & DT^2X \arrow[ur, "D\mu_X"] \arrow[d, "DT\Phi k"'] \arrow[dr, equal] & \\
TDX \arrow[r, "TD\eta_X"'] & TDTX \arrow[r, "\Phi\delta_{TX}"'] & DTDTX \arrow[r, "DT\varepsilon_{TX}"'] & DT^2X \arrow[uu, "D\mu_X"']
\end{tikzcd}$$

> The only non-trivial fact is the commutativity of the sub-diagram in the middle, but this follows from the fact that it is the image under the functor Φ of one of the two D-coalgebra laws for the structure $\Phi k : TX \to DTX$. That is,

$$\begin{tikzcd}
TX \arrow[r, "\Phi k"] \arrow[d, "\Phi k"'] & DTX \arrow[d, "\delta_{TX}"] & & T^2X \arrow[r, "T\Phi k"] \arrow[d, "\Phi^2 k"'] & TDTX \arrow[d, "\Phi\delta_{TX}"] \\
DTX \arrow[r, "D\Phi k"'] & D^2TX & \overset{\Phi}{\longmapsto} & DT^2X \arrow[r, "DT\Phi k"'] & DTDTX
\end{tikzcd}$$

This proves that $\Phi k = (\Phi^{@})^{\#} k$ and, by duality, $\Psi h = (\Psi^{\#})^{@} h$.

Notes. The original proof of "operational is denotational" was more complex: the above simplified proof is due to Bart Jacobs.

9 A Category of Models

The algebras (ie the denotational models) of an operational monad Φ are the same as the coalgebras (ie the operational models) of its coinduced denotational comonad $\Phi^{@}$. Therefore, one can define a general category of Φ-*models* (ie Φ-algebras or, equivalently, $\Phi^{@}$-coalgebras) where both operational and denotational aspects are displayed: this is the proper setting for understanding the *adequacy* results of functorial semantics. In particular, the unique arrow from the initial to the final Φ-model is both the initial algebra and the final coalgebra semantics corresponding to Φ.

By instantiating the general definition of algebras of a monad to a monad $\Phi =< \Phi, \eta, \mu >$ on the D-coalgebras one has that a Φ-algebra has as carrier a D-coalgebra $\langle X, k\rangle$ and as structure a D-coalgebra arrow $h : \Phi\langle X, k\rangle \to \langle X, k\rangle$ such that

$$
\begin{tikzcd}
\Phi^2\langle X,k\rangle \arrow[r, "\Phi h"] \arrow[d, "\mu_{\langle X,k\rangle}"'] & \Phi\langle X,k\rangle \arrow[d, "h"] \\
\Phi\langle X,k\rangle \arrow[r, "h"'] & \langle X,k\rangle
\end{tikzcd}
\qquad
\begin{tikzcd}
\langle X,k\rangle \arrow[r, "\eta_{\langle X,k\rangle}"] \arrow[dr, equal] & \Phi\langle X,k\rangle \arrow[d, "h"] \\
 & \langle X,k\rangle
\end{tikzcd}
$$

If, like in functorial operational semantics, the monad Φ is a lifting of a monad $T =< T, \eta, \mu >$ to the D-coalgebras, then the structure $h : \Phi\langle X, k\rangle \to \langle X, k\rangle$ is of the form

$$
\begin{tikzcd}
TX \arrow[r, "h"] \arrow[d, "\Phi k"'] & X \arrow[d, "k"] \\
DTX \arrow[r, "Dh"'] & DX
\end{tikzcd}
$$

Moreover, $h : \Phi\langle X, k\rangle \to \langle X, k\rangle$ is a Φ-algebra structure if and only if the underlying $h : TX \to X$ is a T-algebra structure. Indeed, for instance, the first Φ-algebra law for h amounts to the commutativity of the following cube

$$\begin{tikzcd}
T^2X \arrow[rr, "Th"] \arrow[dr, "\mu_X"] \arrow[dd, "\Phi^2 k"'] & & TX \arrow[dr, "h"] & \\
 & TX \arrow[rr, "h"] \arrow[dd, "\Phi k"'] & & X \arrow[dd, "k"] \\
DT^2X \arrow[dr, "D\mu_X"'] & & & \\
 & DTX \arrow[rr, "Dh"'] & & DX
\end{tikzcd}$$

> The front side and the other (not visible) side underlying h are two copies of the definition of h, hence commute. The back (not visible) side is the image of the front side under the functor Φ, hence it commutes. The remaining vertical side commutes because the multiplication μ of T lifts to the multiplication of Φ. The bottom (not visible) side is the image under the functor D of the top side, hence to prove the commutativity of the whole cube it suffices to prove that the top side commutes. But this is nothing but the first T-algebra law for h.

Therefore a **Φ-algebra** is a triple $\langle X, k, h\rangle$ with $k : X \to DX$ a D-coalgebra and $h : TX \to X$ a T-algebra structure such that

$$\begin{tikzcd}
TX \arrow[r, "\Phi k"] \arrow[d, "h"'] & DTX \arrow[d, "Dh"] \\
X \arrow[r, "k"'] & DX
\end{tikzcd}$$

commutes.

Similarly, a Φ-algebra arrow $f : \langle X, k, h\rangle \to \langle Y, m, l\rangle$ is an arrow $f : X \to Y$ such that everything in sight in the diagram

$$\begin{tikzcd}
 & TY \arrow[rr, "\Phi m"] \arrow[dd, "l"'] & & DTY \arrow[dd, "Dl"] \\
TX \arrow[ur, "Tf"] \arrow[rr, "\Phi k"] \arrow[dd, "h"'] & & DTX \arrow[ur, "DTf"'] \arrow[dd, "Dh"] & \\
 & Y \arrow[rr, "m"'] & & DY \\
X \arrow[ur, "f"] \arrow[rr, "k"'] & & DX \arrow[ur, "Df"] &
\end{tikzcd}$$

commutes, but for this it suffices that

$$\begin{tikzcd} & TY \arrow[d, "l"] & \\ TX \arrow[ur, "Tf"] \arrow[d, "h"'] & Y \arrow[r, "m"] & DY \\ X \arrow[ur, "f"] \arrow[r, "k"'] & DX \arrow[ur, "Df"'] & \end{tikzcd}$$

commutes, that is, f is both a D-coalgebra arrow $f : \langle X, k \rangle \to \langle Y, m \rangle$ and a T-algebra arrow $f : \langle X, h \rangle \to \langle Y, l \rangle$.

Dually, given a lifting Ψ of a comonad D on a category of T-algebras, a **Ψ-coalgebra** is a triple $\langle X, h, k \rangle$ with $h : TX \to X$ a T-algebra and $k : X \to DX$ a D-coalgebra structure such that

$$\begin{tikzcd} TX \arrow[r, "Tk"] \arrow[d, "h"'] & TDX \arrow[d, "\Psi h"] \\ X \arrow[r, "k"'] & DX \end{tikzcd}$$

commutes. The arrows $f : \langle X, h, k \rangle \to \langle Y, l, m \rangle$ of the corresponding category $\mathbf{C}^{T}{}_{\Psi}$ are again arrows $f : X \to Y$ which preserve both the T-algebra and the D-coalgebra structure.

The claim now is that a triple $\langle X, k, h \rangle$ is a Φ-algebra if and only if $\langle X, h, k \rangle$ is a $\Phi^{@}$-coalgebra, that is,

Φ-algebras are $\Phi^{@}$-coalgebras

$$\begin{tikzcd} TX \arrow[r, "\Phi k"] \arrow[d, "h"'] & DTX \arrow[d, "Dh"] \\ X \arrow[r, "k"'] & DX \end{tikzcd} \iff \begin{tikzcd} TX \arrow[r, "Tk"] \arrow[d, "h"'] & TDX \arrow[d, "\Phi^{@} h"] \\ X \arrow[r, "k"'] & DX \end{tikzcd}$$

Equivalently, the claim is that the diagram

$$\begin{tikzcd} TX \arrow[r, "\Phi k"] \arrow[d, "Tk"'] & DTX \arrow[d, "Dh"] \\ TDX \arrow[r, "\Phi^{@} h"'] & DX \end{tikzcd}$$

commutes. But then fill this last diagram as follows and notice that all sub-diagrams commute.

$$\begin{tikzcd} TX \arrow[rr, "\Phi k"] \arrow[dd, "Tk"'] & & DTX \arrow[dl, "DTk"'] \arrow[d, equal] \\ & DTDX \arrow[r, "DT\varepsilon_X"'] & DTX \arrow[d, "Dh"] \\ TDX \arrow[ur, "\Phi\delta_X"'] \arrow[rr, "\Phi^{@} h"'] & & DX \end{tikzcd}$$

> The only non-trivial sub-diagram is the one corresponding to the upper left corner but this is the image under the functor Φ of one of the two D-coalgebra laws for the structure $k : X \to DX$. That is,
>
> $$\begin{tikzcd} X \arrow[r, "k"] \arrow[d, "k"'] & DX \arrow[d, "\delta_X"] \\ DX \arrow[r, "Dk"'] & D^2X \end{tikzcd} \quad \overset{\Phi}{\longmapsto} \quad \begin{tikzcd} TX \arrow[r, "Tk"] \arrow[d, "\Phi k"'] & TDX \arrow[d, "\Phi\delta_X"] \\ DTX \arrow[r, "DTk"'] & DTDX \end{tikzcd}$$

Thus, up to the permutation $\langle X, k, h\rangle \mapsto \langle X, h, k\rangle$, for any monad Φ lifting a monad T to the coalgebras of a comonad D, the two categories of Φ-algebras and $\Phi^{@}$-coalgebras are the same:

$$\mathbf{C}_D{}^{\Phi} = \mathbf{C}^T{}_{\Phi^{@}}$$

Dually,

$$\mathbf{C}^T{}_{\Psi} = \mathbf{C}_D{}^{\Psi^{\#}}$$

that is, **Ψ-coalgebras are $\Psi^{\#}$-algebras**.

Notice that, since every monad is defined by its algebras and, dually, every comonad is defined by its coalgebras, this gives an alternative proof that the mapping $\Phi \mapsto \Phi^{@}$ is a bijection with $\Psi \mapsto \Psi^{\#}$ as inverse.

Φ-Models. If Φ is an operational monad, then the category $\mathbf{C}_D{}^{\Phi} = \mathbf{C}^T{}_{\Phi^@}$ can be seen as the category of **models of** Φ:

$$\Phi\text{-}\mathbf{Mod} = \mathbf{C}_D{}^{\Phi} = \mathbf{C}^T{}_{\Phi^@}$$

This category has both an initial and a final object which are 'lifted' from the initial T-algebra and the final D-coalgebra, respectively.

The claim is that the *initial Φ-model* is the Φ-algebra

$$T^2 0 \xrightarrow{\mu_0} T0 \xrightarrow{\Phi 0} DT0$$

where, recall, the set $T0$ is the set of closed programs, the structure μ_0 is the initial T-algebra structure, and the structure

$$\Phi 0 = [\![\text{-}]\!] : T0 \to DT0$$

is the initial operational model corresponding to Φ. Dually, the *final Φ-model* is the $\Phi^@$-coalgebra

$$TD1 \xrightarrow{\delta_1} D1 \xrightarrow{\Phi^@ 1} D^2 1$$

where, recall, the set $D1$ is the set of abstract global behaviours, the structure δ_1 is the final D-coalgebra structure, and the structure

$$\Phi^@ 1 = \llparenthesis\text{-}\rrparenthesis : TD1 \to D1$$

is the denotational model coinduced by Φ on the final coalgebra.

If the above holds, then one has, by the very definition of Φ-algebra and $\Phi^@$-coalgebra arrows, that the unique arrow from the initial to the final Φ-model is both the initial algebra and the final coalgebra semantics corresponding to Φ

$$\begin{tikzcd}
 & & TD1 \arrow[d, "\Phi^@ 1 = \llparenthesis\text{-}\rrparenthesis"] & \\
T^2 0 \arrow[urr, "T!"] \arrow[d, "\mu_0"'] & & D1 \arrow[r, "\delta_1"] & D^2 1 \\
T0 \arrow[r, "\Phi 0 = [\![\text{-}]\!]"'] \arrow[urr, dashed, "\llparenthesis\text{-}\rrparenthesis^{\#} = {!} = [\![\text{-}]\!]^{@}"] & DT0 \arrow[urr, "D!"'] & &
\end{tikzcd}$$

That is,

$$[\![\text{-}]\!]^{@} = \llparenthesis\text{-}\rrparenthesis^{\#} : T0 \to D1$$

The fact that the triple $\langle T0, \mu_0, \Phi 0\rangle$ is the initial Φ-model can be proved directly, but it is more informative to obtain it by means of an adjunction as follows. First notice that the Φ-model $\langle T0, \mu_0, \Phi 0\rangle$ can be obtained by applying the functor

$$\widetilde{F^T} : \mathbf{C}_D \to \Phi\text{-}\mathbf{Mod} \qquad\qquad \langle X, k\rangle \mapsto \langle TX, \mu_X, \Phi k\rangle$$

to the initial D-coalgebra:

$$(0 \xrightarrow{0} D0) \overset{\widetilde{F^T}}{\longmapsto} (T^2 0 \xrightarrow{\mu_0} T0 \xrightarrow{\Phi 0} DT0)$$

Next, if a functor has a right adjoint, then it 'preserves colimits' (see, eg, §V.5 of [Mac71]), thus, in particular, if the functor $\widetilde{F^T}$ has a right adjoint then it maps the initial D-coalgebra to the initial Φ-model. Now, the claim is that this right adjoint exists and it is the functor

$$\widetilde{U^T} : \Phi\text{-}\mathbf{Mod} \to \mathbf{C}_D \qquad\qquad \langle X, h, k\rangle \mapsto \langle X, k\rangle$$

which forgets the T-algebra structure in a Φ-model. Moreover, this adjunction

$$\widetilde{F^T} \dashv \widetilde{U^T}$$

is a 'lifting' of the adjunction $F^T \dashv U^T$ corresponding to the algebras of the monad T (see Section 2).

Let us prove this claim in its dual form, namely that the adjunction $U_D \dashv G_D$, corresponding to the coalgebras of the comonad D (see previous section), lifts to an adjunction

$$\widetilde{U_D} \dashv \widetilde{G_D}$$

between the forgetful functor

$$\widetilde{U_D} : \Phi\text{-}\mathbf{Mod} \to \mathbf{C}^T \qquad\qquad \langle X, k, h\rangle \mapsto \langle X, h\rangle$$

and the functor

$$\widetilde{G_D} : \mathbf{C}^T \to \Phi\text{-}\mathbf{Mod} \qquad\qquad \langle X, h\rangle \mapsto \langle DX, \delta_X, \Phi^{@} h\rangle$$

The adjunction $\widetilde{U_D} \dashv \widetilde{G_D}$ splitting the comonad $\Phi^@$. Given a monad Φ lifting a monad T to the coalgebras of a comonad D, the composition $\widetilde{U_D}\widetilde{G_D} : \mathbf{C}^T \to \mathbf{C}^T$ of the above functors $\widetilde{G_D} : \mathbf{C}^T \to {\mathbf{C}_D}^\Phi = \Phi\text{-}\mathbf{Mod}$ and $\widetilde{U_D} : \Phi\text{-}\mathbf{Mod} = {\mathbf{C}_D}^\Phi \to \mathbf{C}^T$ is equal to the endofunctor $\Phi^@ : \mathbf{C}^T \to \mathbf{C}^T$. Indeed,

$$(TX \xrightarrow{h} X) \overset{\widetilde{G_D}}{\longmapsto} (TDX \xrightarrow{\Phi^@ h} DX \xrightarrow{\delta_X} D^2X) \overset{\widetilde{U_D}}{\longmapsto} (TDX \xrightarrow{\Phi^@ h} DX)$$

for every T-algebra structure $h : TX \to X$. The claim is that $\widetilde{G_D}$ is right adjoint to $\widetilde{U_D}$ and the whole comonad $\Phi^@ = < \Phi^@, \varepsilon, \delta >$ arises from the adjunction $\widetilde{U_D} \dashv \widetilde{G_D}$. Moreover, the adjunction $\widetilde{U_D} \dashv \widetilde{G_D}$ 'lifts' the adjunction $U_D \dashv G_D$ (which splits the comonad D):

$$\begin{tikzcd}
\mathbf{C}^T \arrow[r, shift right, "\widetilde{G_D}"'] \arrow[d, "U^T"'] \arrow[r, phantom, "\perp"] & {\mathbf{C}_D}^\Phi \arrow[l, shift right, "\widetilde{U_D}"'] \arrow[d, "U^\Phi"] \\
\mathbf{C} \arrow[r, shift right, "G_D"'] \arrow[r, phantom, "\perp"] & \mathbf{C}_D \arrow[l, shift right, "U_D"']
\end{tikzcd}$$

That is,

$$\begin{array}{rcl}
U^T\widetilde{U_D} & = & U_D U^\Phi : {\mathbf{C}_D}^\Phi \to \mathbf{C} \\
G_D U^T & = & U^\Phi \widetilde{G_D} : \mathbf{C}^T \to \mathbf{C}_D \\
\varepsilon_{U^T} & = & U^\Phi \widetilde{\varepsilon}_{U^\Phi} : DU^T \Rightarrow U^T
\end{array}$$

The first and second equation are immediate, while the third is to be checked: by definition of T-algebra arrows, it tells that the counit of the upper adjunction is the same as the counit $\varepsilon : U_D G_D = D \Rightarrow I$ of the lower one. That is, the claim is that, for every T-algebra arrow $f : \widetilde{U_D}\langle Y, k, l\rangle = \langle Y, l\rangle \to \langle X, h\rangle$, the right adjunct $U^T f^\flat = f^\flat : \langle Y, k\rangle \to \langle DX, \delta_X\rangle$ of $U^T f = f : U_D\langle Y, k\rangle = Y \to X$ wrt the adjunction $U_D \dashv G_D$ is the unique Φ-algebra arrow from $\langle Y, k, l\rangle$ to $\widetilde{G_D}\langle X, h\rangle = \langle DX, \delta_X, \Phi^@ h\rangle$ factorizing f through ε_X. Diagrammatically:

$$\begin{tikzcd}
 & & TY \arrow[lldd, bend right, "Tf"'] \arrow[ldd, "Tf^\flat"'] \arrow[d, "l"] \\
 & & Y \arrow[lldd, bend right, "f"'] \arrow[ldd, "f^\flat"] \\
TX \arrow[d, "h"'] & TDX \arrow[l, "T\varepsilon_X"'] \arrow[d, "\Phi^@ h"'] & \\
X & DX \arrow[l, "\varepsilon_X"] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & TY \arrow[rr, "\Phi k"] \arrow[d, "l"] \arrow[ldd, "Tf^\flat"'] & & DTY \arrow[d, "Dl"] \arrow[ldd, "DTf^\flat"'] \\
 & Y \arrow[rr, "k"'] \arrow[ldd, dashed, "f^\flat"] & & DY \arrow[ldd, "Df^\flat"] \\
TDX \arrow[rr, "\Phi\delta_X"'] \arrow[d, "\Phi^@ h"'] & & DTDX \arrow[d, "D\Phi^@ h"'] & \\
DX \arrow[rr, "\delta_X"'] & & D^2X &
\end{tikzcd}$$

All sub-diagrams commute either by definition or because they are obtained by applying a functor to a commuting diagram, except for

$$\begin{tikzcd}
TY \arrow[r, "Tf^\flat"] \arrow[d, "l"'] & TDX \arrow[d, "\Phi^@ h"] \\
Y \arrow[r, "f^\flat"'] & DX
\end{tikzcd}$$

(and its image under D). But the commutativity of the latter follows from the fact that both composite arrows $f^\flat \circ l$ and $\Phi^@ h \circ Tf^\flat$ fit as the (unique!) arrow $(h \circ Tf)^\flat : \langle TY, \Phi k\rangle \to \langle DX, \delta_X\rangle$. (If the comonad D is cofreely generated, then this arrow is the unique coinductive extension of Φk along the composite $h \circ Tf : TY \to X$.)

This shows that $\widetilde{G_D}$ is right adjoint to $\widetilde{U_D}$ and that ε is the counit of the adjunction. The unit of the adjunction is obtained by taking the right adjunct of the identity and, by the D-coalgebra laws, its value at a Φ-algebra $\langle X, k, h\rangle$ is the coalgebraic component of the Φ-algebra, namely $k : X \to DX$.

$$\begin{tikzcd}
 & & TX \arrow[ddl, "Tk"'] \arrow[d, "h"] \\
 & & X \arrow[ddl, "k"] \\
TX \arrow[d, "h"'] & TDX \arrow[l, "T\varepsilon_X"'] \arrow[d, "\Phi^@ h"'] & \\
X & DX \arrow[l, "\varepsilon_X"] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & TX \arrow[rr, "\Phi k"] \arrow[d, "h"] \arrow[ddl, "Tk"'] & & DTX \arrow[d, "Dh"] \arrow[ddl, "DTf^\flat"'] \\
 & X \arrow[rr, "k"'] \arrow[ddl, "k"] & & DX \arrow[ddl, "Dk"] \\
TDX \arrow[rr, "\Phi\delta_X"'] \arrow[d, "\Phi^@ h"'] & & DTDX \arrow[d, "D\Phi^@ h"'] & \\
DX \arrow[rr, "\delta_X"'] & & D^2X &
\end{tikzcd}$$

Finally, notice that also the comultiplication of the comonad $\Phi^@ =< \Phi^@, \varepsilon, \delta >$ arises from this adjunction by first taking the unit at $\widetilde{G_D}\langle X, h\rangle$ and then applying the functor $\widetilde{U_D}$ to it. In general, every adjunction $F \dashv G$ defines a comonad $< FG, \varepsilon, F\eta_G >$, where ε and η are the counit and the unit of the adjunction respectively. (Cf Section 2 for the dual ‘every adjunction defines a monad’.)

To summarize, there are two adjunctions for the category of Φ-models, namely

$$\begin{tikzcd} \mathbf{C}^T \arrow[r, shift right, "\widetilde{G_D}"'] \arrow[r, phantom, "\perp"] & {\mathbf{C}_D}^{\Phi} = \Phi\text{-}\mathbf{Mod} = {\mathbf{C}^T}_{\Phi^@} \arrow[l, shift right, "\widetilde{U_D}"'] \arrow[r, shift right, "\widetilde{U^T}"'] \arrow[r, phantom, "\perp"] & \mathbf{C}_D \arrow[l, shift right, "\widetilde{F^T}"'] \end{tikzcd}$$

and the unique arrow from the initial Φ-model $\widetilde{F^T}(0) = \langle T0, \mu_0, \Phi 0\rangle$ to the final Φ-model $\widetilde{G_D}(1) = \langle D1, \delta_1, \Phi^@ 1\rangle$ is both the *initial algebra semantics* induced by the denotational model $\Phi^@ 1 = \llparenthesis - \rrparenthesis$ and the *final coalgebra semantics* coinduced by the operational model $\Phi 0 = [\![-]\!]$. Diagrammatically:

$$\begin{tikzcd}
 & & TD1 \arrow[dd, "\Phi^@ 1 = \llparenthesis - \rrparenthesis"] & & & T1 \arrow[dd, dashed, "1"] \\
T^2 0 \arrow[urr, "T!"] \arrow[dd, "\mu_0"'] & & & & {} & {} \arrow[l, mapsto, "\widetilde{F^T}"'] \\
 & & D1 \arrow[r, "\delta_1"] & D^2 1 & & 1 \\
T0 \arrow[r, "\Phi 0 = [\![-]\!]"'] \arrow[urr, dashed, "! = [\![-]\!]^@ = \llparenthesis - \rrparenthesis^\#"] & DT0 \arrow[urr, "D!"'] & & & & \\
 & {} & & & & \\
 & {} \arrow[u, mapsto, "\widetilde{G_D}"] & & & & \\
0 \arrow[r, dashed, "0"'] & D0 & & & &
\end{tikzcd}$$

This is a more compact and symmetric formulation of the adequacy result given in Section 6.

Notes. The idea that adequacy results 'live' in categories of "algebras over coalgebras" is due to Gordon Plotkin and it has been fundamental for the development of the present functorial approach to operational semantics.

Liftings of adjunctions are treated in [Joh75]. In particular, the adjunction splitting the comonad $\Phi^@$ can be obtained by applying Theorem 4 of [Joh75] (see also, eg, [HJ95a] for a 2-categorical account of this theorem).