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THE CATEGORY-THEORETIC SOLUTION OF RECURSIVE DOMAIN EQUATIONS*

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Abstract. Recursive specifications of domains plays a crucial role in denotational semantics as developed by Scott and Strachey and their followers. The purpose of the present paper is to set up a categorical framework in which the known techniques for solving these equations find a natural place. The idea is to follow the well-known analogy between partial orders and categories, generalizing from least fixed-points of continuous functions over *cpos* to initial ones of continuous functors over ω-categories. To apply these general ideas we introduce Wand's **O**-categories where the morphism-sets have a partial order structure and which include almost all the categories occurring in semantics. The idea is to find solutions in a derived category of embeddings and we give order-theoretic conditions which are easy to verify and which imply the needed categorical ones. The main tool is a very general form of the limit-colimit coincidence remarked by Scott. In the concluding section we outline how compatibility considerations are to be included in the framework. A future paper will show how Scott's universal domain method can be included too.

Key words. Domains, semantics, data-types, category, partial-order, fixed-point, computability

1. Introduction. Recursive specifications of domains play a crucial role in denotational semantics as developed by Scott and Strachey and their followers (Gordon [13], Milne and Strachey [26], Stoy [39], Tennent [40], [41]). For example, the equation

$$D \cong \mathrm{At} + (D \to D) \tag{1}$$

is just what is needed for the semantics of an untyped λ-calculus for computing over a domain, At, of atoms. Again, the simultaneous equations

$$T \cong \mathrm{At} \times F, \tag{2}$$

$$F \cong 1 + (T \times F) \tag{3}$$

specify a domain, T, of all finitarily branching trees and another, F, of forests of such trees. And recursively specified data types are also very useful [10], [19], [20].

The first tools for solving such equations were provided by Scott using his inverse limit constructions [33]. Later he showed how the inverse limits could be entirely avoided by using a universal domain and the ordinary least fixed point construction [34]. A systematic exposition of the inverse limit method was given by Reynolds [30], and the categorical aspects (already mentioned by Scott) were emphasized by Wand [43]. All of these treatments stuck to one category, such as, for example, **CL**, the category of countably based continuous lattices and continuous functions, although the details did not change much in other categories. Then Wand [44], gave an abstract treatment based on **O**-categories where the morphism sets are provided with a suitable order-theoretic structure. The relation between the category-theoretic treatment and the universal domain method has, until now, remained rather obscure.

The purpose of the present paper is to set up a categorical framework in which all known techniques for solving domain equations find a natural place. The idea as set out in § 2 is to follow the well-known analogy between partial orders and categories, and generalize from least fixed points to initial fixed points. These are constructed

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using the "basic lemma" which plays an organizational role: Most of the solution methods considered appear as ways of ensuring that the hypotheses of the lemma are fulfilled. Just as continuous functions over complete partial orders always have least fixed points, so continuous functors over ω-categories (as defined below) always have initial fixed points, which can be constructed by using the basic lemma; this seems to formalize some hints of Lawvere mentioned by Scott in [33]. The same idea appears in [1], [2].

All this is very general, and we introduce **O**-categories in § 3 in order to apply the basic lemma to the construction of the domains needed in denotational semantics. Here we are clearly greatly indebted to Wand [44], [45] who introduced **O**-categories, and indeed our work arose partly as an attempt to simplify and clarify his treatment. The idea is to apply the basic lemma not to a given **O**-category, **K**, but rather to a derived category, $\mathbf{K}^E$, of embeddings (equivalently, projections). We then look for easily verified conditions on **K** (whether categorical or order-theoretic) which imply the needed conditions on $\mathbf{K}^E$.

Our main tool is Theorem 2, which establishes a very general form of the limit-colimit coincidence remarked by Scott [33] and also gives an order-theoretic characterization of the relevant categorical limits. This improves Wand's work by removing the need for his troublesome "Condition A" (appearing in [44] rather than [45] which incorporates some of the ideas of the present paper); more positively we also introduce a useful notion of duality for **O**-categories.

With the aid of Theorem 2 (and the easy Theorem 1), one sees that simple conditions on an **O**-category, **K**, (mainly that it has all $\boldsymbol{\omega}^{\mathrm{op}}$-limits) ensure that $\mathbf{K}^E$ is an ω-category. Again with the aid of (Lemma 4 and) Theorem 3, one sees how to take any mixed contravariant-covariant functor over **K** (like the function-space one) which satisfies an order-continuity property (usually evident), and turn it into a covariant-continuous one over $\mathbf{K}^E$.

Section 4 presents several examples of useful categories which may be handled by the methods of §§ 2 and 3.

The method of universal domains, in relation to the ideas presented here, is treated in the sequel to this paper. An indication of our approach may be found in Plotkin and Smyth [28] (which may also be of help in getting a general overview of our results).

There is, however, one aspect of Scott's presentation of the universal domain approach [34] which must receive some mention here: the question of computability. The results presented in this paper would lose much of their point if we were forced to invoke a universal domain to handle computability. In the concluding § 5, we indicate briefly how this topic can in fact be handled at the level of generality of this paper; for a more detailed treatment we refer to Smyth [38].

We assume the reader possesses a basic knowledge of category theory; any gaps can be filled by consulting Arbib and Manes [4], Herrlich and Strecker [16], or MacLane [21].

2. Initial fixed points. In the categorical approach to recursive domain specifications we try to regard all equations such as (1) or (2) and (3) above as being of the form

$$X \cong F(X), \tag{4}$$

where X ranges over the objects of a category **K**, say, and $F: \mathbf{K} \to \mathbf{K}$ is an endofunctor of that category. For example, in the case of (1) we could take X to range over the objects of **K**, At to be a fixed object of **K**, and + and $\to$ to be covariant sum and function-space functors over $\mathbf{K}$; then $F:\mathbf{K}\to\mathbf{K}$ is defined by:

(5) $$F(X) \underset{\text{def}}{=} \mathrm{At}+(X\to X).$$

Let us spell the meaning of (5) out in detail. Recall that if $F_i:\mathbf{K}\to\mathbf{K}_i$ $(i=1,n)$ are functors then their *tupling* $F=\langle F_1,\cdots,F_n\rangle:\mathbf{K}\to\mathbf{K}_1\times\cdots\times\mathbf{K}_n$ is defined by putting for each object, X, of $\mathbf{K}$:

$$F(X)=\langle F_1(X),\cdots,F_n(X)\rangle$$

and for each morphism $f:X\to Y$ of $\mathbf{K}$:

$$F(f)=\langle F_1(f),\cdots,F_n(f)\rangle.$$

Then the functor F defined by (5) is just

$$F=+\circ\langle K_{\mathrm{At}},\to\circ\langle \mathrm{id}_{\mathbf{K}},\mathrm{id}_{\mathbf{K}}\rangle\rangle$$

where $K_{\mathrm{At}}:\mathbf{K}\to\mathbf{K}$ is the constantly-At functor and $\mathrm{id}_{\mathbf{K}}:\mathbf{K}\to\mathbf{K}$ is the identity functor.

Simultaneous equations are handled using product categories. For example, (2) and (3) can be regarded as having the form:

(6) $$X\cong F_0(X,Y),$$

(7) $$Y\cong F_1(X,Y),$$

where X, Y range over a category $\mathbf{K}$ (such as $\mathbf{CL}$) and F_0 and F_1 are bifunctors over $\mathbf{K}$ being defined by

$$F_0 \underset{\text{def}}{=} \times\circ\langle K_{\mathrm{At}},\pi_1\rangle,\qquad F_1 \underset{\text{def}}{=} +\circ\langle K_1,\times\circ\langle\pi_0,\pi_1\rangle\rangle$$

where At, 1 are objects of $\mathbf{K}$, and the $\pi_i:\mathbf{K}\times\mathbf{K}\to\mathbf{K}$ $(i=0,1)$ are the projection functors. Then (2) and (3) are put into the form (4) by using the product category $\mathbf{K}\times\mathbf{K}$ and taking F to be $\langle F_0,F_1\rangle$. Clearly this idea works for n simultaneous equations $X_i=F_i(X_1,\cdots,X_n)$ $(i=1,n)$ where X_i ranges over $\mathbf{K}_i$ $(i=1,n)$ and $F_i:\mathbf{K}_1\times\cdots\times\mathbf{K}_n\to\mathbf{K}_i$; we just take $\mathbf{K}$ to be $\mathbf{K}_1\times\cdots\times\mathbf{K}_n$ and F to be $\langle F_1,\cdots,F_n\rangle$.

Let us now decide what a solution to (4) might be and which particular ones we want. In the case where $\mathbf{K}$ is a partial order, F is then just a monotonic function, solutions are just fixed points of F (that is, elements A of K such that $A=F(A)$), and we can look for least solutions. Further, we can define *prefixed points* as elements A such that $F(A)\sqsubseteq A$, and it turns out that the least prefixed point, if it exists, is always the least fixed point as well. In the categorical case we need to know the isomorphism as well as the object:

DEFINITION 1. Let $\mathbf{K}$ be a category and $F:\mathbf{K}\to\mathbf{K}$ be an endofunctor. Then a *fixed point* of F is a pair (A,α) where A is an object of K and $\alpha:FA\cong A$ is an isomorphism of K; a *prefixed point* is a pair (A,α) where A is an object of $\mathbf{K}$ and $\alpha:FA\to A$ is a morphism of $\mathbf{K}$.

We also call prefixed points of F, *F-algebras* (same as *F-dynamic* of Arbib and Manes [4]). The F-algebras are the objects of a category:

DEFINITION 2. Let (A,α) and (A',α') be F-algebras. A *morphism* $f:(A,\alpha)\to(A',\alpha')$ (*F-homomorphism*) is just a morphism $f:A\to A'$ in $\mathbf{K}$ such that the following commutes:

$$\begin{tikzcd} FA \arrow[r, "\alpha"] \arrow[d, "Ff"'] & A \arrow[d, "f"] \\ FA' \arrow[r, "\alpha'"] & A' \end{tikzcd}$$

It is easily verified that this gives a category: the identity and composition are both inherited from $\mathbf{K}$. Following on the above remarks on partial orders, we look for initial F-algebras rather than just initial fixed points of F and this is justified by the following lemma (which also appears in Arbib [5], and in Barr [8], where it is credited to Lambek).

LEMMA 1. *The initial F-algebra, if it exists, is also the initial fixed point.*

Proof. Let (A, α) be the initial F-algebra. We only have to prove that α is an isomorphism. Now as (A, α) is an F-algebra so is $(FA, F\alpha)$ and so there is an F-homomorphism $f:(A, \alpha) \to (FA, F\alpha)$; one also easily sees that $\alpha:(FA, F\alpha) \to (A, \alpha)$ is an F-homomorphism and so $\alpha \circ f: (A, \alpha) \to (A, \alpha)$ is also one and it must be id_A, the identity on A as (A, α) is initial. Then as $f:(A, \alpha) \to (FA, F\alpha)$ we also have $f \circ \alpha = (F\alpha) \circ (Ff) = F(\alpha \circ f) = F(\mathrm{id}_A) = \mathrm{id}_{FA}$, which shows that α is an isomorphism with two-sided inverse f. $\square$

Note that we have to do more than specify an object A such that $A \cong F(A)$ when looking for the initial fixed point. First we have to specify an isomorphism $\alpha: FA \cong A$, and secondly we must establish the initiality property. Both are vital in applications. When giving the semantics of programming languages using recursively specified domains the isomorphism is needed just to be able to make the definitions. Initiality is closely connnected to structural induction principles and both can be used for making proofs about elements of the specified domains. When the equations are used to specify data-type definitions within a language following the approach in Lehmann and Smyth [19], [20], the isomorphism carries the basic operations, and initiality is again essential for proofs. (The paper [20] also contains more information on simultaneous equations and on equations with parameters; in many ways, it is a companion to the present paper.)

When K is a partial order, the least fixed point can, as is well known, be constructed as $\bigsqcup_K F^n(\perp)$, the l.u.b. of the increasing sequence $\langle F^n(\perp)\rangle_{n\in\omega}$ where $\perp$ is the least element of K. This works if the least element exists, the l.u.b. exists and F preserves the l.u.b.—that is, $F(\bigsqcup_K F^n(\perp)) = \bigsqcup_K F(F^n(\perp))$. Our basic lemma merely generalizes these remarks to the case of a category.

First we give some notation and terminology which are not quite standard. By an $\boldsymbol{\omega}$*-chain* in a category $\mathbf{K}$ we understand a diagram of the form $\Delta = D_0 \to^{f_0} D_1 \to^{f_1} \cdots$ (that is, a functor from $\boldsymbol{\omega}$ to $\mathbf{K}$); for $m \leqq n$, we write $f_{mn}: D_m \to D_n$ for the morphism $f_{n-1} \circ \cdots \circ f_m$. Dually an $\boldsymbol{\omega}^{\mathrm{op}}$*-chain* in a category $\mathbf{K}$ is a diagram of the form $\Delta = D_0 \leftarrow^{f_0} D_1 \leftarrow^{f_1} \cdots$ (that is, a functor from $\boldsymbol{\omega}^{\mathrm{op}}$ to $\mathbf{K}$); for $m \geqq n$ we have the evident $f_{mn}: D_m \to D_n$. By the *mediating* morphism between a limiting (colimiting) μ over a diagram Δ and any other cone ν over Δ, we must understand the unique morphism given by universality from (to) the vertex of ν to (from) the vertex of μ.

Notation. The initial object of a category $\mathbf{K}$ is written as $\perp_{\mathbf{K}}$ and the unique arrow from it to an object A is written as $\perp_A$. If $\Delta = D_0 \to^{f_0} D_1 \to^{f_1} \cdots$ is an $\boldsymbol{\omega}$-chain in K and $\mu: \Delta \to A$ is a cone over Δ, then Δ^- is the $\boldsymbol{\omega}$-chain $D_1 \to^{f_1} D_2 \to^{f_2} \cdots$ and $\mu^-: \Delta^- \to A$ is the cone $\langle \mu_{n+1}\rangle_{n\in\omega}$; if, further, $F:\mathbf{K} \to \mathbf{L}$ is a functor, then $F\Delta$ is the $\boldsymbol{\omega}$-chain $FD_0 \to^{Ff_0} FD_1 \to^{Ff_1}$ and $F\mu: F\Delta \to FA$ is the cone $\langle F\mu_n\rangle_{n\in\omega}$.

Now given a functor $F: \mathbf{K} \to \mathbf{K}$ we can define the $\boldsymbol{\omega}$-chain $\Delta = \langle F^n(\perp_{\mathbf{K}}), F^n(\perp_{F\perp})\rangle$ (if $\perp_{\mathbf{K}}$ exists) and try to justify the calculation analogous to that for partial orders:

$$F \varinjlim \Delta \cong \varinjlim F\Delta = \varinjlim \Delta^- = \varinjlim \Delta.$$

The basic lemma gives conditions for this to work and characterizes $\lim_{\to} \Delta$ (with an appropriate morphism) as the initial F-algebra.

LEMMA 2 (basic lemma). *Let* $\mathbf{K}$ *be a category with initial object* $\perp_{\mathbf{K}}$ *and let* $F: \mathbf{K} \to \mathbf{K}$ *be a functor. Define the* $\boldsymbol{\omega}$*-chain* Δ *to be* $\langle F^n(\perp_{\mathbf{K}}), F^n(\perp_{F\perp})\rangle$. *Suppose that both* $\mu: \Delta \to A$ *and* $F\mu: F\Delta \to FA$ *are colimiting cones. Then the initial F-algebra exists and is* (A, α) *where* $\alpha: FA \to A$ *is the mediating morphism from* $F\mu$ *to* μ^-.

Proof. Let $\alpha': FA' \to A'$ be any F-algebra. We show there is a unique F-homomorphism $f: (A, \alpha) \to (A', \alpha')$. First suppose f is such a homomorphism. Define a cone $\nu: \Delta \to A'$ by putting $\nu_0 = \perp_{A'}: \perp \to A'$ and $\nu_{n+1} = \alpha' \circ F(\nu_n)$. To see ν is a cone we prove by induction on n that the following diagram commutes.

$$\begin{tikzcd}
F^n(\perp) \arrow[r, "F^n(\perp_{F\perp})"] \arrow[dr, "\nu_n"'] & F^{n+1}(\perp) \arrow[d, "\nu_{n+1}"] \\
& A'
\end{tikzcd}$$

This is clear for $n = 0$. For $n+1$ we have: $\nu_{n+2} \circ F^{n+1}(\perp_{F\perp}) = \alpha' \circ F(\nu_{n+1}) \circ F^{n+1}(\perp_{F\perp}) = \alpha' \circ F(\nu_{n+1} \circ F^n(\perp_{F\perp})) = \alpha' \circ F(\nu_n)$ (by induction assumption) $= \nu_{n+1}$. Now the uniqueness of f will follow when we show it is the mediating morphism from μ to ν; here we use induction on n to show $\nu_n = f \circ \mu_n$. This is clear for $n = 0$. For $n+1$ we have: $f \circ \mu_{n+1} = f \circ \alpha \circ F(\mu_n)$ (by the definition of α) $= \alpha' \circ F(f) \circ F(\mu_n)$ (f is a homomorphism) $= \alpha' \circ F(f \circ \mu_n) = \alpha' \circ \nu_n$ (by induction assumption) $= \nu_{n+1}$.

Secondly, to show that f exists, let it be the mediating morphism from μ to ν (so that $\nu_n = f \circ \mu_n$ for all integers, n) where ν is defined as above. We will show that $f \circ \alpha$ and $\alpha' \circ Ff$ are both mediating arrows from $F\mu$ to ν^-, thus demonstrating that they are equal and that f is an F-homomorphism as required.

In the first case, $(f \circ \alpha) \circ F\mu_n = f \circ \mu_{n+1}$ (by definition of α) $= \nu_{n+1}$ (by definition of f). In the second case, $(\alpha' \circ Ff) \circ F\mu_n = \alpha' \circ F(f \circ \mu_n) = \alpha' \circ F\nu_n$ (by definition of f) $= \nu_{n+1}$ (by definition of ν). This concludes the proof. □

In the case of partial orders, our method of constructing least fixed-points always works if K is an ω-complete partial order and $F: K \to K$ is ω-continuous. Here an *ω-complete pointed partial order* (cpo) is a partial order which has l.u.b.s of all increasing ω-sequences and which has a least element; it is termed an "ω-complete partial order" or even just a "complete partial order" elsewhere. Also a function $F: K \to L$ of partial orders is *ω-continuous* if and only if it is monotonic and preserves l.u.b.s of increasing ω-sequences, that is, if whenever $\langle x_n\rangle_{n\in\omega}$ is an increasing ω-sequence such that $\bigsqcup_K x_n$ exists, then $F(\bigsqcup_K x_n) = \bigsqcup_L F(x_n)$. We make analogous definitions for categories:

DEFINITION 3. A category, K, is an *ω-complete pointed category* (shortened below to *ω-category*) if and only if it has an initial element, and every ω-chain has a colimit.

DEFINITION 4. Let $F: \mathbf{K} \to \mathbf{L}$ be a functor. It is *ω-continuous* if and only if it preserves $\boldsymbol{\omega}$-colimits; that is, whenever Δ is an $\boldsymbol{\omega}$-chain and $\mu: \Delta \to A$ is a colimiting cone, then $F\mu : F\Delta \to FA$ is also a colimiting cone. (The reader is warned that this definition is dual to the notion of continuity of functors usual in category theory (MacLane [21]); this is done in order to maintain the analogy with partial orders.)

Clearly, when $\mathbf{K}$ is an ω-category and $F: \mathbf{K} \to \mathbf{K}$ is ω-continuous, the conditions of the basic lemma are satisfied. In § 3 we will give the conditions for this to be the case. In the sequel to this paper (see also Plotkin and Smyth [28]), we will show that, in the presence of a universal object, the conditions of the basic lemma may be satisfied without requiring that $\mathbf{K}$ be an ω-category and that F be ω-continuous. Usually, we can completely avoid direct verification of the conditions of the basic lemma, or of whether something is an ω-category or is ω-continuous. Of course sometimes, as in the case of **Set**, it is already known that we have an ω-category and that such functors as $+$ and $\times$ are ω-continuous (see Lehmann and Smyth [20]). One case in which there is, so far, no alternative to direct verification is with Lehmann's category, **Dom** of small ω-categories and ω-continuous functors [18] (in this case **Dom**-categories might provide a good general setting).

Note that it is only necessary to check that the basic categories are ω-categories and the basic functors are ω-continuous; one easily proves that any denumerable product of ω-categories is an ω-category, that all constant and projection functors are ω-continuous, and that composition and tupling preserve ω-continuity. Thus to solve (1) one only needs to check that $+$ and $\to$ are ω-continuous; for (2) and (3) one looks at $+$ and $\times$.

The original work on models of the pure λ-calculus (Scott [34], Wadsworth [42]) did not solve (4) as $\lim_{\to} \langle F^n(\perp), F^n(\perp_{F\perp})\rangle$, but rather as $\lim_{\to} \langle F^n(D), F^n(\pi)\rangle$ for an object D and a morphism $\pi : D \to F(D)$. It turns out that this solution is, essentially, the initial fixed point of a functor F_π, derived from F, over the comma category $(D{\downarrow}\mathbf{K})$ of "objects over D" (see MacLane [21]). The analogous idea in partial orders is that of a least fixed point greater than some fixed element, d.

The comma category $(D{\downarrow}\mathbf{K})$ has as objects pairs (A, d) where A is an object of $\mathbf{K}$ and $d: D \to A$; the morphisms $f: (A, d) \to (A', d')$ are the morphisms $f: A \to A'$ of $\mathbf{K}$ such that the following diagram commutes.

$$\begin{tikzcd} D \arrow[r, "d"] \arrow[dr, "d'"'] & A \arrow[d, "f"] \\ & A' \end{tikzcd}$$

Now the endofunctor $F_\pi : (D{\downarrow}\mathbf{K}) \to (D{\downarrow}\mathbf{K})$ can be obtained by putting $F(A, d) = (FA, (Fd) \circ \pi)$ for objects and $F_\pi(f) = F(f)$ for morphisms. Then one can see that an F_π-algebra is a pair $((A, d), \alpha)$ where A is an object of $\mathbf{K}$, and $d: D \to A$ and $\alpha: FA \to A$ and the following diagram commutes.

$$\begin{tikzcd} D \arrow[r, "\pi"] \arrow[drr, "d"'] & F(D) \arrow[r, "Fd"] & F(A) \arrow[d, "\alpha"] \\ & & A \end{tikzcd}$$

A homomorphism $f: ((A, d), \alpha) \to ((A', d'), \alpha')$ is a morphism $f: A \to A'$ such that the following two diagrams commute.

$$\begin{tikzcd} F(A) \arrow[r, "\alpha"] \arrow[d, "F(f)"'] & A \arrow[d, "f"] \\ F(A') \arrow[r, "\alpha'"] & A' \end{tikzcd} \qquad \begin{tikzcd} D \arrow[r, "d"] \arrow[dr, "d'"'] & A \arrow[d, "f"] \\ & A' \end{tikzcd}$$

Let us assume, for simplicity, that $\mathbf{K}$ is an ω-category and F is ω-continuous. Then $(D\downarrow\mathbf{K})$ is also an ω-category. Its initial object is (D, id_D). For colimits suppose $\Delta=\langle(A_n, d_n), f_n\rangle$ is an $\boldsymbol{\omega}$-chain in $D\downarrow\mathbf{K}$. Then it is straightforward to check that $\mu:\langle A_n, f_n\rangle\to A$ is a (colimiting) cone in $\mathbf{K}$ if and only if $\mu:\Delta\to(A, \mu_0\circ d_0)$ is in $(D\downarrow\mathbf{K})$ (and they have the same mediating morphisms). This makes it easy to show that F is ω-continuous.

Now, applying the basic lemma to $(D\downarrow\mathbf{K})$ and F_π, we have to find a colimiting cone $\mu:\langle F_\pi^n(\perp), F_\pi^n(\perp_{F_\pi(\perp)})\rangle\to(A, d)$. One sees, by induction on n, that $F_\pi^n(\perp_{(D\downarrow\mathbf{K})})$ is $\langle F^n(D), d_n\rangle$ where $d_n=F^{n-1}(\pi)\circ\cdots\circ\pi:D\to F^n(D)$ and that $F_\pi^n(\perp_{F_\pi(\perp)})=F^n(\pi)$. So from the above remarks one can take μ to be a colimiting cone, $\mu:\langle F^n(D), F^n(\pi)\rangle\to A$, also defining A, and put $d=\mu_0\circ d_0$. Then, by the basic lemma, the initial F_π-algebra is $((A, d), \alpha)$ where α is the mediating morphism from $F\mu$ to μ^- (which can be taken in $\mathbf{K}$). Thus we see that $A=\lim_\to\langle F^n(D), F^n(\pi)\rangle$, together with its colimiting cone, determines the initial F_π-algebra. Thus we have characterized the original inverse limit construction in universal terms.

3. O-categories. In most of the categories used for the denotational semantics of programming languages, the hom-sets have a natural partial order structure. When solving recursive domain equations only the projections are considered, and they are easily defined in terms of the partial order. Wand [44] introduced **O**-categories to study such categories at a suitably abstract level. We now present a view of his work as providing theorems and definitions which facilitate the application of the basic lemma, as outlined in the introduction.

DEFINITION 5. A category, $\mathbf{K}$, is an **O**-category if and only if (i) every hom-set is a partial order in which every ascending ω-sequence has a l.u.b. and (ii) composition of morphisms is an ω-continuous operation with respect to this partial order.

Note that if $\mathbf{K}$ is an **O**-category, so is $\mathbf{K}^{\mathrm{op}}$, and if $\mathbf{L}$ is another so is $\mathbf{K}\times\mathbf{L}$. Here the orders are inherited, and in the case of $\mathbf{K}^{\mathrm{op}}$ we have $f^{\mathrm{op}}\sqsubseteq g^{\mathrm{op}}$ if and only if $f\sqsubseteq g$, for any morphisms f and g of $\mathbf{K}$ (in general we will omit the superscript when writing morphisms in opposite categories).

As it happens **O**-categories are a particular case of a general theory of **V**-categories where **V** is any closed category (MacLane [21]); here **O** is the category whose objects are those partial orders with l.u.b.s of all increasing ω-sequences and whose morphisms are the ω-continuous functions between the partial orders. We will not use any of the general theory but just take over the idea of endowing the morphism sets with extra structure—in this case that of being an object in **O**.

DEFINITION 6. Let K be an **O**-category and let $A\to^f B\to^g A$ be arrows such that $g\circ f=\mathrm{id}_A$ and $f\circ g\sqsubseteq\mathrm{id}_B$. Then we say that $\langle f, g\rangle$ is a *projection pair from A to B*, that g is a *projection* and that f is an *embedding*.

LEMMA 3. *Let $\langle f, g\rangle$ and $\langle f', g'\rangle$ both be projection pairs from A to B, in an* **O***-category K. Then $f\sqsubseteq f'$ if and only if $g\sqsupseteq g'$.*

Proof. If $f\sqsubseteq f'$ then $g\sqsupseteq g\circ f'\circ g'\sqsupseteq g\circ f\circ g'=g'$. Conversely, if $g\sqsupseteq g'$ then $f=f\circ g'\circ f'\sqsubseteq f\circ g\circ f'\sqsubseteq f'$. $\square$

So, in particular, it follows that one half of a projection pair determines the other; if f is an embedding we write f^R for the corresponding projection which we call the *right adjoint* of f; if g is a projection we write g^L for the corresponding embedding which we call the *left adjoint* of g. (Our use of the term "adjoint" is only a matter of convenience; when the **K**-objects are posets and the morphisms are monotonic maps, it agrees with a standard usage.)

Given any **O**-category **K** we can now form the subcategory, $\mathbf{K}^E$, of the embeddings. For the identity morphism $\mathrm{id}_A : A \to A$ is an embedding with $\mathrm{id}_A^R = \mathrm{id}_A$, and if $A \to^f B \to^{f'} C$ are embeddings, so is $(f' \circ f)$ with $(f' \circ f)^R = f^R \circ f'^R$. Equally, we can form $\mathbf{K}^P$, the subcategory of the projections. We do *not* try to take either of these as **O**-categories under the induced ordering; indeed they are, in general, not **O**-categories.

3.1. Duality for O-categories. Our discussion of adjoints shows that we have the duality $\mathbf{K}^E \cong (\mathbf{K}^P)^{\mathrm{op}}$ (and so too $\mathbf{K}^P \cong (\mathbf{K}^E)^{\mathrm{op}}$). There is another kind of duality arising from the fact that an embedding in $\mathbf{K}^{\mathrm{op}}$ is just a projection in **K**. Thus $(\mathbf{K}^{\mathrm{op}})^E \cong \mathbf{K}^P$ (and so $(\mathbf{K}^{\mathrm{op}})^P \cong \mathbf{K}^E$). Thus an $\boldsymbol{\omega}$-chain in $(\mathbf{K}^{\mathrm{op}})^E$ can be considered as an $\boldsymbol{\omega}^{\mathrm{op}}$-chain in $\mathbf{K}^P$, and a colimiting cone in $(\mathbf{K}^{\mathrm{op}})^E$ can be identified with a limiting cone in $\mathbf{K}^P$. We therefore have the following dualities (for an **O**-category **K**): embedding/projection, $\boldsymbol{\omega}$-chain in $\mathbf{K}^E$/$\boldsymbol{\omega}^{\mathrm{op}}$-chain in $\mathbf{K}^P$, colimiting cone in $\mathbf{K}^E$/limiting cone in $\mathbf{K}^P$. A further example, **O**-colimit/**O**-limit, is provided by Definition 7.

These observations will be used in the proof of Theorem 2.

Our first theorem is trivial but does illustrate the idea of transferring properties from **K** to $\mathbf{K}^E$.

THEOREM 1. *Let* **K** *be an* **O**-*category which has a terminal object,* $\perp$, *and in which every* $\hom(A, B)$ *has a least element,* $\perp_{A,B}$. *Suppose too that composition is left-strict in the sense that for any* $f : A \to B$ *we have* $\perp_{B,C} \circ f = \perp_{A,C}$. *Then* $\perp$ *is the initial object of* $\mathbf{K}^E$.

Proof. First, if $f, f' : \perp \to A$ are both embeddings, then they have a common right adjoint, as $\perp$ is terminal in K, and so, by Lemma 3, they are equal. Second, $\perp_{\perp,A} : \perp \to A$ is an embedding with right adjoint $\perp_{A,\perp}$. For $\perp_{A,\perp} \circ \perp_{\perp,A} : \perp \to \perp$ must be $\mathrm{id}_\perp$ the unique map from $\perp$ to $\perp$ and $\perp_{\perp,A} \circ \perp_{A,\perp} = \perp_{A,A}$ (by left-strictness)$\sqsubseteq \mathrm{id}_A$. $\square$

To make the connection with the basic lemma, we need to be able to relate **O**-notions (expressed in terms of the ordering of hom-sets) to ω-notions (expressed in terms of $\boldsymbol{\omega}^{\mathrm{op}}$-limits/$\boldsymbol{\omega}$-colimits). This is the main purpose of Theorem 2. Another way to view this result, exemplified further in the ensuing discussion, is that it is concerned with the correspondence between local properties (that is, properties local to particular hom-sets) and global properties of the category. Yet another way to regard Theorem 2 is to note that it contains the most complete and general formulation of the limit-colimit coincidence, remarked in Scott [33], that we have been able to develop.

DEFINITION 7. Let **K** be an **O**-category and $\mu : \Delta \to A$ a cone in $\mathbf{K}^E$, where Δ is the ω-chain $\langle A_n, f_n \rangle$. Then μ is an **O**-*colimit* of Δ provided that $\langle \mu_n \circ \mu_n^R \rangle_n$ is increasing with respect to the ordering of $\hom(A, A)$ and $\bigsqcup_n \mu_n \circ \mu_n^R = \mathrm{id}_A$. Dually an **O**-*limit* of an $\boldsymbol{\omega}^{\mathrm{op}}$-chain Γ in K^P is a cone $\nu : B \to \Gamma$ in **K** such that $\langle \nu_n^L \circ \nu_n \rangle_n$ is increasing and $\bigsqcup_n \nu_n^L \circ \nu_n = \mathrm{id}_B$.

Obviously, μ is an **O**-colimit of Δ if μ^R is an **O**-limit of Δ^R, where we define Δ^R to be $\langle A_n, f_n^R \rangle$ and μ^R to be $\langle \mu_n^R \rangle_{n \in \omega}$.

THEOREM 2. *Let* **K** *be an* **O**-*category and* Δ *be an* $\boldsymbol{\omega}$-*chain in* $\mathbf{K}^E$. *Consider the six properties*

(a) Δ *has a colimit in* **K**,

(b) Δ^R *has a limit in* $\mathbf{K}$,

(c) Δ *has an* $\mathbf{O}$-*colimit*,

(d) Δ^R *has an* $\mathbf{O}$-*limit*,

(e) Δ *has a colimit in* $\mathbf{K}^E$,

(f) Δ^R *has a limit in* $\mathbf{K}^P$.

We have: properties (a)–(d) *are equivalent* (*to each other*); (a) *implies* (e); *and* (e) *is equivalent to* (f). *Indeed, a cone* $\mu:\Delta\to A$ (*a cone* $\nu:A\to\Delta^R$) *is colimiting* (*limiting*) *in* $\mathbf{K}$ *if and only if* $\mu(\nu)$ *is an* $\mathbf{O}$-*colimit* ($\mathbf{O}$-*limit*); *any colimiting* (*limiting*) *cone of* $\Delta(\Delta^R)$ *in* $\mathbf{K}$ *is also a colimiting* (*limiting*) *cone of* $\Delta(\Delta^R)$ *in* $\mathbf{K}^E(\mathbf{K}^P)$; *and* $\mu:\Delta\to A$ *is colimiting in* $\mathbf{K}^E$ *if and only if* μ^R *is limiting in* $\mathbf{K}^P$.

Proof. We establish Propositions A–E which are jointly equivalent to the result. We always suppose that Δ has the form $\langle A_n, f_n\rangle$. Note that $\langle \mu'_n\circ\mu_n^R\rangle_{n\in\omega}$ is increasing for any cones $\mu:\Delta\to A$ and $\mu':\Delta\to A'$ *in* $\mathbf{K}$, as we have: $\mu'_n\circ\mu_n^R = (\mu'_{n+1}\circ f_n)\circ(\mu_{n+1}\circ f_n)^R = \mu'_{n+1}\circ(f_n\circ f_n^R)\circ\mu_{n+1}^R \sqsubseteq \mu'_{n+1}\circ\mu_{n+1}^R$.

PROPOSITION A. *If* $\mu:\Delta\to A$ *is an* $\mathbf{O}$-*colimit, then* μ *is colimiting in both* $\mathbf{K}$ *and* $\mathbf{K}^E$.

Proof of Proposition (A). Choose a cone $\mu':\Delta\to A$ in $\mathbf{K}$, and suppose $\theta:A\to A'$ is a morphism from μ to μ' (i.e., for all n, $\theta\circ\mu_n=\mu'_n$). Then θ is determined by:

$$\theta=\theta\circ\bigsqcup\mu_n\circ\mu_n^R=\bigsqcup(\theta\circ\mu_n)\circ\mu_n^R=\bigsqcup\mu'_n\circ\mu_n^R.$$

This proves uniqueness; for existence we can define θ as $\bigsqcup\mu'_n\circ\mu_n^R$, since the above remark shows that $\langle\mu'_n\circ\mu_n^R\rangle$ is increasing, and calculate:

$$\theta\circ\mu_m=\left(\bigsqcup_{n\geqq m}\mu'_n\circ\mu_n^R\right)\circ\mu_m=\bigsqcup_{n\geqq m}\mu'_n\circ\mu_n^R\circ\mu_n\circ f_{mn}=\bigsqcup_{n\geqq m}\mu'_n\circ f_{mn}=\mu'_m.$$

So μ is colimiting in $\mathbf{K}$. For $\mathbf{K}^E$ it only remains to show that if μ' is actually a cone in $\mathbf{K}^E$ then θ as defined above is an embedding. By the remark above, $\langle\mu_n\circ\mu_n'\rangle_{n\in\omega}$ is increasing; to show that θ is an embedding, we prove that it has the right adjoint $\theta^R=\bigsqcup\mu_n\circ\mu_n'^R$.

On the one hand we have:

$$(\bigsqcup\mu_n\circ\mu_n'^R)\circ(\bigsqcup\mu'_n\circ\mu_n^R)=\bigsqcup\mu_n\circ(\mu_n'^R\circ\mu'_n)\circ\mu_n^R=\bigsqcup\mu_n\circ\mu_n^R=\mathrm{id}_A;$$

on the other hand we have:

$$(\bigsqcup\mu'_n\circ\mu_n^R)\circ(\bigsqcup\mu_n\circ\mu_n'^R)=\bigsqcup\mu'_n\circ(\mu_n^R\circ\mu_n)\circ\mu_n'^R=\bigsqcup\mu'_n\circ\mu_n'^R\sqsubseteq\mathrm{id}_{A'}.\quad\square$$

Dually, we have Proposition B:

PROPOSITION B. *If* $\nu:A\to\Delta^R$ *is an* $\mathbf{O}$-*limit, then* ν *is limiting in both* $\mathbf{K}$ *and* $\mathbf{K}^P$.

PROPOSITION C. *If* $\nu:A\to\Delta^R$ *is limiting in* $\mathbf{K}$, *then each* ν_n *is a projection and* ν *is an* $\mathbf{O}$-*limit of* Δ^R.

Proof of Proposition (C). For each A_m we can define a cone $\nu^{(m)}:A_m\to\Delta^R$ in $\mathbf{K}$ by:

$$\nu_n^{(m)}=\begin{cases}f_{mn} & (m\leqq n),\\ (f_{nm})^R & (m>n).\end{cases}$$

To see that $\nu^{(m)}$ is a cone, we first check that if $r\geqq\max(m,n)$ then $\nu_n^{(m)}=f_{nr}^R\circ f_{mr}$. For if $m\leqq n$, then $f_{nr}^R\circ f_{mr}=f_{nr}^R\circ(f_{nr}\circ f_{mn})=\nu_n^{(m)}$; if $m>n$ then $f_{nr}^R\circ f_{mr}=$

$(f_{mr}\circ f_{nm})^R\circ f_{mr} = f^R_{nm} = \nu^{(m)}_n$. Now we see that $\nu^{(m)}$ is a cone, as follows:

$$\begin{aligned} f^R_n\circ\nu^{(m)}_{n+1} &= f^R_n\circ(f^R_{(n+1)r}\circ f_{mr}) \qquad (\text{by the above with } r=\max(m, n+1))\\ &= (f_{(n+1)r}\circ f_n)^R\circ f_{mr} = f^R_{nr}\circ f_{mr}\\ &= \nu^{(m)}_n \qquad (\text{by the above}).\end{aligned}$$

Now as $\nu: A\to\Delta^R$ is a limiting cone there is, for each m, a mediating morphism $\theta_m: A_m\to A$ from $\nu^{(m)}$ to ν. So we have for all m and n: $\nu_n\circ\theta_m = \nu^{(m)}_n$. Putting n equal to m we find that $\nu_m\circ\theta_m = \mathrm{id}_{A_m}$, which is half the proof that ν_m is a projection with $\nu^L_m = \theta_m$.

Next we connect up the θ_m's by showing that $\theta_m = \theta_{m+1}\circ f_m$, which holds since $\theta_{m+1}\circ f_m$ mediates between $\nu^{(m)}$ and ν as can be seen from:

$$\begin{aligned} \nu_n\circ(\theta_{m+1}\circ f_m) &= \nu^{(m+1)}_n\circ f_m \qquad (\theta_{m+1} \text{ mediates between } \nu^{(m+1)} \text{ and } \nu)\\ &= f^R_{nr}\circ f_{(m+1)r}\circ f_m \qquad (\text{with } r=\max(m+1, n))\\ &= f^R_{nr}\circ f_{mr} = \nu^{(m)}_n.\end{aligned}$$

This, in turn, enables us to show that $\langle\theta_m\circ\nu_m\rangle_{m\in\omega}$ is increasing: $\theta_m\circ\nu_m = \theta_{m+1}\circ f_m\circ f^R_m\circ\nu_{m+1}\sqsubseteq\theta_{m+1}\circ\nu_{m+1}$. Consequently, as $\mathbf{K}$ is an $\mathbf{O}$-category, we may define $\theta: A\to A$ by: $\theta = \bigsqcup_{m\in\omega}\theta_m\circ\nu_m$. To finish the proof, we show that $\theta = \mathrm{id}_A$ (as then we also have $\theta_m\circ\nu_m\sqsubseteq\theta = \mathrm{id}_A$, completing the proof that $\nu^L_m = \theta_m$). This follows from the fact that θ mediates between ν and itself as is shown by:

$$\nu_n\circ\theta = \nu_n\circ\bigsqcup_{m\geqq n}\theta_m\circ\nu_m = \bigsqcup_{m\geqq n}(\nu_n\circ\theta_m)\circ\nu_m = \bigsqcup_{m\geqq n}\nu^{(m)}_n\circ\nu_m = \bigsqcup_{m\geqq n} f^R_{nm}\circ\nu_m = \nu_n.$$

□

By duality:

PROPOSITION D. *If $\mu:\Delta\to A$ is colimiting in $\mathbf{K}$, then each μ_n is an embedding, and μ is an $\mathbf{O}$-colimit.*

PROPOSITION E. *$\mu:\Delta\to A$ is colimiting in $\mathbf{K}^E$ if and only if $\mu^R: A\to\Delta^R$ is limiting in $\mathbf{K}^P$.*

Proof of Proposition E. Obvious. □

This completes the proof of Theorem 2. □

Our first main use of Theorem 2 will be to establish the evident corollary that if $\mathbf{K}$ is an $\mathbf{O}$-category which has all $\boldsymbol{\omega}^{\mathrm{op}}$-limits, then $\mathbf{K}^E$ has all $\boldsymbol{\omega}$-colimits. The second main use concerns functors, but first a definition and another corollary prove convenient.

DEFINITION 8. An $\mathbf{O}$-category $\mathbf{K}$ is said to have *locally determined ω-colimits of embeddings* provided that, whenever Δ is an $\boldsymbol{\omega}$-chain in $\mathbf{K}^E$ and $\mu:\Delta\to A$ is a cone in $\mathbf{K}^E$, μ is colimiting in $\mathbf{K}^E$ if and only if μ is an $\mathbf{O}$-colimit. (Note that, by Theorem 2, only half of the implication can ever be in doubt.)

COROLLARY (to Theorem 2). *Suppose that the $\mathbf{O}$-category $\mathbf{K}$ has all $\boldsymbol{\omega}^{\mathrm{op}}$-limits (i.e., every $\boldsymbol{\omega}^{\mathrm{op}}$-chain in $\mathbf{K}$ has a limit). Then $\mathbf{K}$ has locally determined $\boldsymbol{\omega}$-colimits of embeddings.*

Proof. Suppose that $\mu:\Delta\to A$ is colimiting in $\mathbf{K}^E$. Let $\nu': A'\to\Delta^R$ be a limit with respect to $\mathbf{K}$ for Δ^R. By Theorem 2, ν' is an $\mathbf{O}$-limit for Δ^R. Thus, ν'^L is an $\mathbf{O}$-colimit for Δ, so that by Theorem 2, ν'^L is colimiting in $\mathbf{K}^E$. Hence μ is isomorphic with ν'^L and must itself be an $\mathbf{O}$-colimit (the property of being an $\mathbf{O}$-colimit is trivially invariant under isomorphism of cones). □

By duality, the conclusion of the corollary also follows from the assumption that $\mathbf{K}$ has all $\boldsymbol{\omega}$-colimits, but that is not so useful. At present we lack an example of an $\mathbf{O}$-category which does not have locally determined $\boldsymbol{\omega}$-colimits of embeddings.

As mentioned in the Introduction, a major reason for introducing $\mathbf{K}^E$ is to enable us to consider contravariant functors on $\mathbf{K}$ as covariant ones on $\mathbf{K}^E$; the remainder of this section develops the idea. We consider throughout (that is, to the end of the section) three $\mathbf{O}$-categories $\mathbf{K}$, $\mathbf{L}$, $\mathbf{M}$ and a covariant functor $T:\mathbf{K}^{\mathrm{op}}\times\mathbf{L}\to\mathbf{M}$. Purely covariant functors are included by suppressing $\mathbf{K}$ (i.e., taking it to be the trivial one-object category), contravariant ones by suppressing $\mathbf{L}$, and mixed ones by taking $\mathbf{K}$ and $\mathbf{L}$ to be product categories as required.

DEFINITION 9. The functor T is *locally monotonic* if and only if it is monotonic on the hom-sets; that is, for $f, f':A\to B$ in $\mathbf{K}^{\mathrm{op}}$, $g, g':C\to D$ in $\mathbf{L}$, if $f\sqsubseteq f'$ and $g\sqsubseteq g'$, then $T(f, g)\sqsubseteq T(f', g')$.

LEMMA 4. *If T is locally monotonic, a covariant functor $T^E:\mathbf{K}^E\times\mathbf{L}^E\to\mathbf{M}^E$ can be defined by putting, for objects A, $B: T^E(A,B)=T(A,B)$ and for morphisms $f, g: T^E(f, g)=T((f^R)^{\mathrm{op}}, g)$.*

Proof. First if $f:A\to B$ in $\mathbf{K}^E$ and $g:C\to D$ in $\mathbf{L}^E$, then $T(f^R, g)$ is an embedding with right adjoint $T(f, g^R)$ as: $T(f, g)^R\circ T(f^R, g)=T(f^R\circ f, g^R\circ g)=T(\mathrm{id}_A, \mathrm{id}_C)=\mathrm{id}_{T(A,C)}$ and also: $T(f^R, g)\circ T(f, g^R)=T(f\circ f^R, g\circ g^R)\sqsubseteq T(\mathrm{id}_B, \mathrm{id}_D)$ (by local monotonicity) $=\mathrm{id}_{T(B,D)}$.

Secondly, $T^E(\mathrm{id}_A, \mathrm{id}_C)=T(\mathrm{id}_A^R, \mathrm{id}_C)=T(\mathrm{id}_A, \mathrm{id}_C)=\mathrm{id}_{T(A,C)}$.

Thirdly if $A\to^f A'\to^{f'} A'$ in $\mathbf{K}^E$ and $B\to^g B'\to^{g'} B''$ in $\mathbf{L}^E$ then

$$\begin{aligned} T^E(f', g')\circ T^E(f, g) &= T(f'^R, g')\circ T(f^R, g)=T(f^R\circ f'^R, g'\circ g)\\ &= T((f'\circ f)^R, g'\circ g)=T^E(f'\circ f, g'\circ g). \qquad \square \end{aligned}$$

Under some assumptions on $\mathbf{K}$ and $\mathbf{L}$, we can transfer a local continuity property of T to the ω-continuity of T^E.

DEFINITION 10. The functor T is *locally continuous* (equivalently, is an $\mathbf{O}$-functor) if and only if it is ω-continuous on the hom-sets—that is, if $f_n:A\to B$ is an increasing ω-sequence in $\mathbf{K}^{\mathrm{op}}$ and $g_n:C\to D$ is one in $\mathbf{L}$, then $T(\bigsqcup_{n\in\omega} f_n, \bigsqcup_{n\in\omega} g_n)=\bigsqcup_{n\in\omega} T(f_n, g_n)$.

Note that the constant and projection functors are locally continuous, and that the locally continuous functors are closed under composition, tupling and taking opposite functors.

THEOREM 3. *Suppose T is locally continuous and both $\mathbf{K}$ and $\mathbf{L}$ have locally determined $\boldsymbol{\omega}$-colimits of embeddings. Then T^E is ω-continuous.*

Proof. Let $\Delta=\langle\langle A_n, B_n\rangle, \langle f_n, g_n\rangle\rangle$ be an ω-chain in $\mathbf{K}^E\times\mathbf{L}^E$ and let $\mu:\Delta\to\langle A, B\rangle$ be colimiting, where $\mu=\langle\sigma_n, \tau_n\rangle_{n\in\omega}$. Then $\langle\sigma_n\rangle:\langle A_n, f_n\rangle\to A$ is colimiting in $\mathbf{K}^E$ and $\langle\tau_n\rangle:\langle B_n, g_n\rangle\to B$ is colimiting in $\mathbf{L}^E$. It follows by the assumptions on $\mathbf{K}$ and $\mathbf{L}$ that $\mathrm{id}_A=\bigsqcup\sigma_n\circ\sigma_n^R$ and $\mathrm{id}_B=\bigsqcup\tau_n\circ\tau_n^R$ with the right-hand sides increasing.

We have to show that $T^E(\mu):T^E(\Delta)\to T^E(A, B)$ is a colimiting cone in $\mathbf{M}^E$, which we do by showing that it is an $\mathbf{O}$-limit (and then applying Theorem 2). First

$$\begin{aligned} \langle T^E(\mu_n)\circ T^E(\mu_n)^R\rangle_{n\in\omega} &= \langle T(\sigma_n^R, \tau_n)\circ T(\sigma_n^R, \tau_n)^R\rangle_{n\in\omega}\\ &= \langle T(\sigma_n^R, \tau_n)\circ T(\sigma_n, \tau_n^R)\rangle_{n\in\omega}\\ &= \langle T(\sigma_n\circ\sigma_n^R, \tau_n\circ\tau_n^R)\rangle_{n\in\omega}, \end{aligned}$$

which is increasing as $\langle\sigma_n \circ \sigma_n^R\rangle_{n\in\omega}$ and $\langle\tau_n \circ \tau_n^R\rangle_{n\in\omega}$ are, and as T is locally monotonic. Next,

$$\begin{aligned}\bigsqcup_{n\in\omega} T^E(\mu_n)\circ T^E(\mu_n)^R &= \bigsqcup_{n\in\omega} T(\sigma_n\circ\sigma_n^R, \tau_n\circ\tau_n^R) \qquad \text{(by the above)}\\ &= T\Big(\bigsqcup_{n\in\omega}\sigma_n\circ\sigma_n^R, \bigsqcup_{n\in\omega}\tau_n\circ\tau_n^R\Big) \qquad \text{(by local continuity)}\\ &= T(\mathrm{id}_A, \mathrm{id}_B) \qquad \text{(by the above)}\\ &= \mathrm{id}_{T(A,B)}.\end{aligned}$$

□

4. Examples. In this section we present several useful **O**-categories where our general theory can be applied. In general we only sketch proofs and even omit them when they are either evident or not directly relevant to the main line of the argument (but for Example 2, see [22]). The first example is elementary, being little more than an illustration of the ideas. The second example is the category of cpos where all the needed domains for denotational semantics can be constructed. This is an approach where as few axioms as possible are imposed. That makes the axioms very easy to understand but admits domains of little computational interest. The third example illustrates various completeness conditions that are weaker than Scott's original requirement of complete lattices, which rules out some natural and useful domains. The fourth example considers axioms of algebraicity and continuity which attempt to force the domains to be computationally realistic. One use of the completeness axioms (cf. Example 3) is that when combined with algebraicity (or continuity), function spaces exist although they need not otherwise do so. Example 5 turns in a different direction, suggesting a certain category of continuous algebras as an appropriate place for the semantics of programming languages with nondeterministic constructs. Example 6 considers a category of relations over cpos where it is possible to construct a wide variety of recursively specified relations; these are useful when relating different semantics.

Example 1. *Partial functions.* We consider the category **Pfn** of sets and the partial functions between them. The partial order relation between partial functions is just set inclusion and can also be defined for any $f, g: A \to B$ by

$$f \sqsubseteq g \equiv \forall a \in A . f(a)\downarrow \supset f(a) = g(a)$$

(where $f(a)\downarrow$ means that $f(a)$ is defined). Clearly limits of increasing ω-sequences $\langle f_n\rangle_{n\in\omega}$ exist being just the set-union $\bigcup f_n$ so that

$$(\bigsqcup f_n)(a) = \begin{cases} b & (\exists n . f_n(a) = b),\\ \text{undefined} & (\forall n . f_n(a) \text{ is undefined}).\end{cases}$$

It is easy to see that $f: A \to B$ is an embedding if and only if it is total and one-to-one; in that case $f^R = f^{-1}$. Thus to within isomorphism, embeddings are just inclusions. Note that the totally undefined function $\varnothing: A \to B$ is the least element of hom (A, B) and that composition is left strict in the sense of Theorem 1. The empty set is the terminal object, the unique mapping being $\varnothing: A \to \varnothing$ and so the conditions of Theorem 1 apply, and we see that $\varnothing$ is the initial object in $\mathbf{Pfn}^E$ (and of course that is trivial anyway).

Turning to $\boldsymbol{\omega}$-colimits in $\mathbf{Pfn}^E$, it is obvious that they exist, as $\boldsymbol{\omega}$-chains $\Delta = \langle A_n, f_n\rangle$ are, to within isomorphism, just increasing sequences $A_0 \subseteq A_1 \subseteq \cdots$, and so $A = \bigcup A_n$

is the colimiting object with the colimiting cone of inclusions $\mu_n : A_n \subseteq A$. This also follows from Theorem 2, since **Pfn** has $\boldsymbol{\omega}^{\mathrm{op}}$-limits. These are constructed as in **Set**: let $\Delta = \langle A_n, f_n \rangle$ be an $\boldsymbol{\omega}^{\mathrm{op}}$-chain; put $A = \{a \in \Pi A_n \mid \forall n . a_n = f_n(a_{n+1})\}$ and define $\nu_n : A \to A_n$ to be the "nth projection" $a \mapsto a_n$. Then A is the limit of Δ and ν is the colimiting cone.

Turning to functors, we define the Cartesian product on morphisms $f : A \to A'$ and $g : B \to B'$ by:

$$(f \times g)(a, b) = \begin{cases} \langle fa, gb \rangle & (\text{if } f(a) \text{ and } g(b)\downarrow), \\ \text{undefined} & (\text{otherwise}). \end{cases}$$

This makes the Cartesian product locally continuous and so, by Theorem 3, ω-continuous. Amusingly, the categorical product exists but is different from Cartesian product and is not even locally monotonic. On the other hand, the categorical sum exists and is locally continuous. On objects it is disjoint union:

$$A + B = (\{0\} \times A) \cup (\{1\} \times B)$$

and for morphisms $f : A \to A'$ and $g : B \to B'$,

$$(f+g)(c) = \begin{cases} \langle 0, f(a) \rangle & (\exists a \in A . c = \langle 0, a \rangle \text{ and } f(a)\downarrow), \\ \langle 1, g(b) \rangle & (\exists b \in B . c = \langle 1, b \rangle \text{ and } g(b)\downarrow), \\ \text{undefined} & (\text{otherwise}). \end{cases}$$

Finally, there is a natural function-space construction defined by

$$A \to B = \hom(A, B),$$

and for $f : A' \to A$ and $g : B \to B'$

$$(f \to g)(h) = g \circ h \circ f.$$

This is not even locally monotonic, and so Theorem 3 does not apply. This is as expected, as the recursive domain equation (1) *cannot* have solutions in **Pfn** (for nontrivial At) by evident cardinality considerations. For another example of an elementary **O**-category, the reader can consider the category **Rel** of binary relations between sets, with the subset ordering on relations.

Example 2. *Complete partial orders.* We consider the category **CPO** (essentially introduced in § 2) of complete partial orders and ω-continuous functions. It is an **O**-category when we define the order between morphisms $f, g : A \to B$ in the natural pointwise fashion

$$f \sqsubseteq g \equiv \forall a \in A . f(a) \sqsubseteq g(a).$$

Limits of increasing ω-sequences of morphisms are defined pointwise. The conditions of Theorem 1 are satisfied, as the trivial one-point partial order is the terminal object and any given hom (A, B) has least element $a \mapsto \perp_B$ and composition is left-strict.

Turning to $\boldsymbol{\omega}$-limits (to apply Theorem 3), let $\Delta = \langle A_n, f_n \rangle$ be an $\boldsymbol{\omega}$-chain and construct $\nu : A \to \Delta$ as in the case of **Pfn** taking the partial order on A componentwise so that for any a, a' in A.

$$a \sqsubseteq a' \equiv \forall n . a_n \sqsubseteq a'_n.$$

This makes A a cpo with least upper bounds of increasing ω-sequences, $\langle a^{(n)} \rangle_{n \in \omega}$ taken componentwise

$$\bigsqcup_n a^{(n)} = \left\langle \bigsqcup_m a_m^{(n)} \right\rangle_{m \in \omega}$$

and with least element $\langle \bigsqcup_{n \geq m} f_{nm}(\perp_{A_n})\rangle$. Further ν is a cone of continuous functions and it is limiting, as if $\nu': A' \to \Delta$ is any other, then if θ is a mediating morphism we have, for all n, that $\theta(a')_n = \nu'_n(a')$, determining θ as a continuous function. Thus we have sketched the proof that **CPO** has all $\boldsymbol{\omega}^{\mathrm{op}}$-limits.

Turning to functors, we have categorical product and function space functors. The product of two cpos A and B is their Cartesian product with the componentwise ordering; it is easily verified to be the categorical product. Its action on morphisms $f: A \to A'$ and $g: B \to B'$ is also defined as usual:

$$(f \times g)(a, b) = \langle fa, gb \rangle.$$

Clearly, product is locally continuous. The function-space functor has the same (formal) definition as in **Pfn**; however, this time it is easily seen to be locally monotonic and, indeed, continuous. The function-space functor is the categorical one and **CPO** is Cartesian closed.

Unfortunately, **CPO** does not have categorical sums. It is therefore better to consider the category $\mathbf{CPO}_\perp$ of cpos and strict continuous functions (where for any cpos A and B a function $f: A \to B$ is *strict* if $f(\perp_A) = \perp_B$). This has a categorical sum which is defined on cpos A and B by putting

$$A + B = [\{0\} \times (A \setminus \{\perp\})] \cup [\{1\} \times (B \setminus \{\perp\})] \cup \{\perp\},$$

with the partial order defined by

$$\begin{aligned} c \sqsubseteq c' \equiv & [\exists a, a' \in A . a \sqsubseteq a' \wedge c = \langle 0, a \rangle \wedge c' = \langle 0, a' \rangle] \\ & \vee [\exists b, b' \in B . b \sqsubseteq b' \wedge c = \langle 1, b \rangle \wedge c' = \langle 1, b' \rangle] \\ & \vee c = \perp \end{aligned}$$

In other words, $A + B$ is the *coalesced* sum, that is, it is the disjoint union of A and B, but with least elements identified. The action of sum on morphisms turns out to be given by putting for $f: A \to A'$ and $g: B \to B'$

$$(f+g)(c) = \begin{cases} \langle 0, f(a) \rangle & (\exists a \in A . f(a) \neq \perp \wedge c = \langle 0, a \rangle), \\ \langle 1, g(b) \rangle & (\exists b \in B . g(b) \neq \perp \wedge c = \langle 1, b \rangle), \\ \perp & (\text{otherwise}), \end{cases}$$

and this shows that the sum is a locally continuous functor. Now we know that $+^E$ and, for example $\times^E$ are covariant $\boldsymbol{\omega}$-continuous bifunctors on $\mathbf{CPO}_\perp^E$ and $\mathbf{CPO}^E$ respectively. Luckily however these latter categories are the same, as both embeddings and projections in **CPO** are strict. (To see this let $f: A \to B$ be an embedding in **CPO**. Then $f(\perp) \sqsubseteq f(f^R(\perp)) \sqsubseteq \perp$ and so $f(\perp) = \perp$; also $f^R(\perp) = f^R(f(\perp)) = \perp$.).

In the same vein we can consider the *smash product* $A \otimes B$ in $\mathbf{CPO}_\perp$ defined as $\{\langle a, b \rangle \in A \times B \mid a \neq \perp \equiv b \neq \perp\}$ with the componentwise ordering inherited from the product $A \times B$ (which happens also to be the categorical product in $\mathbf{CPO}_\perp$). On morphisms $f: A \to A'$ and $f: B \to B'$ the functor acts as follows:

$$(f \otimes g)(a, b) = \begin{cases} \langle f(a), g(b) \rangle & (\text{if } f(a) \neq \perp \text{ and } g(b) \neq \perp), \\ \langle \perp, \perp \rangle & (\text{otherwise}). \end{cases}$$

This definition shows that the smash product is locally continuous. It can be characterized categorically. Say that a function $f: A \times B \to C$ of cpos is *bistrict* if and only if for any b in B, we have $f(\perp, b) = \perp$ (*left-strictness*) and also for any a in A we have $f(a, \perp) = \perp$ (*right-strictness*). Then the evident *bistrict* function $\otimes: A \times B \to A \otimes B$ is the *universal* bistrict continuous function from $A \times B$ in $\mathbf{CPO}_\perp$.

The *strict function-space* functor is defined as before (formally) but this time in $\mathbf{CPO}_\perp$ and we denote it by $\to_\perp$. It also is easily seen to be locally continuous. From a categorical point of view (see [21]), smash product makes $\mathbf{CPO}_\perp$ a symmetric monoidal category. Further, as we have a natural bijection

$$\hom(A \otimes B, C) \cong \hom(A, B \to_\perp C),$$

$\mathbf{CPO}_\perp$ is even a closed category. This to some extent repairs the fact that it is not Cartesian closed and explains the appearance of the smash product.

Finally we note a useful functor $(\cdot)_\perp : \mathbf{CPO} \to \mathbf{CPO}_\perp$ called *lifting*. For any cpo D,

$$(D)_\perp = (\{0\} \times D) \cup \perp,$$

with the partial order defined for any d, d' in $(D)_\perp$ by:

$$d \sqsubseteq d' \equiv (\exists c, c' \in D . c \sqsubseteq c' \wedge d = \langle 0, c\rangle \wedge d' = \langle 0, c'\rangle) \vee d = \perp.$$

On morphisms $f : D \to E$ we have for any d in $(D)_\perp$

$$(f)_\perp(d) = \begin{cases} \langle 0, f(c)\rangle & (\text{if } d = \langle 0, c\rangle), \\ \perp & (\text{if } d = \perp). \end{cases}$$

From a categorical point of view, lifting is the left adjoint to the forgetful functor from $\mathbf{CPO}_\perp$ to $\mathbf{CPO}$. We now have a wide variety of covariant continuous functors over $\mathbf{CPO}^E = \mathbf{CPO}_\perp^E$; Lehmann and Smyth discuss many of their uses in [20]. What is more, all of them arise naturally from a categorical point of view.

It does not appear to be useful to use $\mathbf{O}$ itself (a variant of Reynolds' predomains [32]) as $\mathbf{O}^E$ has no initial object. (To see this, suppose to the contrary that D is initial. Then there is an embedding $f : D \to \varnothing$, and we must therefore have $D = \varnothing$; but clearly $\varnothing$ is not initial, as there is no embedding $g : \varnothing \to X$ for any nonempty X.) On the other hand, one often sees an alternative definition of cpo where it is assumed that all *directed* sets have l.u.b.s rather than just increasing ω-sequences. (A subset X of a partial order P is directed if it is nonempty and any two elements of X have an upper bound in X.) Let us call these partial orders dcpos (*directed complete partial orders*). One easily adapts the above discussion to this case. Which definition to take is not a choice of great significance. On the one hand, the restriction to ω-sequences gives a larger category and is also computationally natural, as they arise when taking least fixed points; on the other hand the directed sets are natural mathematically. The following fact shows the difference is essentially one of cardinality.

FACT 1.a. *A partial order with a least element is a* cpo *if and only if it has all l.u.b.s of countable directed sets.*

b. *A function $f : A \to B$ between* cpos *is ω-continuous if and only if it preserves all l.u.b.s of countable directed sets.*

Proof. First note that for any countable directed set X there is an increasing sequence $\langle x_n\rangle_{n\in\omega}$ of elements of X such that any element of X is less than some x_n. Then we have $\bigsqcup X = \bigsqcup x_n$ and part a easily follows. For part b, suppose $f : A \to B$ is ω-continuous and let $X \subseteq A$ be directed. With $\langle x_n\rangle_{n\in\omega}$ as above, we calculate

$$f(\bigsqcup X) = f(\bigsqcup x_n) = \bigsqcup f(x_n) = \bigsqcup_{x\in X} f(x),$$

and this finishes the proof, as the other direction is immediate. $\square$

Another way to look at these matters was discussed by Markowsky [22], who noted that a partial order is a dcpo if and only if it has l.u.b.s of all (well-ordered) chains, and a function between dcpos preserves all l.u.b.s of directed sets if and only if it preserves all l.u.b.s of (well-ordered) chains.

Example 3. *Completeness.* We consider some full subcategories of **CPO** defined by imposing various completeness conditions.

DEFINITION 11. Let D be a partial order. A subset, X, of D is κ-consistent if and only if whenever $Y \subseteq X$ and $\|Y\| < \kappa$, Y has an upper bound in D.

Any subset, X, of a nonempty partial order is 0-, 1- and 2-consistent; it is 3-consistent if and only if it is *pairwise consistent* in the sense that any pair of its elements has an upper bound in D; it is ω-consistent if and only if any finite subset of its elements has an upper bound in D. Clearly if $\kappa \leqq \kappa'$ then every κ'-consistent subset is also ω-consistent; clearly too, any directed subset is ω-consistent.

DEFINITION 12. A partial order, D, is κ-complete if and only if it is nonempty and every κ-consistent subset has a least upper bound.

It follows from the above remarks that if $\kappa \leqq \kappa'$, then every κ-complete partial order is κ'-complete; also every ω-complete partial order is a cpo (and even a dcpo). Clearly for $0 \leqq n < 3$ the n-complete partial orders are the complete lattices and the 3-complete partial orders are the *coherent* cpos, in the sense of [24], [29] and, essentially, [11]. We now see that a partial order is ω-complete if and only if it is consistently complete in the sense of [29], [36] and, essentially, [11], [24].

FACT 2. *Let D be a partial order. It is ω-complete if and only if it is a dcpo with l.u.b.s of all subsets with upper bounds in D.*

Proof. Let D be ω-complete. We have already noted that it is a dcpo. Also any subset with an upper bound in D is ω-consistent and so has a least upper bound.

With the converse hypotheses, let X be an ω-consistent subset. Then every finite subset has an upper bound in X and so has a least upper bound. The set of such l.u.b.s is then directed and so must itself have a l.u.b. which is also the l.u.b. of X. □

Turning to the properties of the full subcategory of κ-complete partial orders, we see that Theorem 1 may be applied, as the one-point cpo is a complete lattice. To see that $\boldsymbol{\omega}^{\mathrm{op}}$-limits exist, let $\Delta = \langle A_n, f_n \rangle$ be an $\boldsymbol{\omega}$-chain and define $\nu: A \to \Delta$ as before. As this defines a limiting cone in **CPO**, it only remains to show that A is κ-complete. The proof employs an idea of Scott, for the case of complete lattices (we have already employed it to show that A has a least element).

FACT 3. *A is κ-complete.*

Proof. Suppose $X \subseteq A$ is κ-consistent. Then for every m, so is $\{x_m | x \in X\}$, and then the least upper bound of X is $\langle \bigsqcup_{m \geqq n} f_{mn}(\bigsqcup \{x_m | x \in X\})\rangle_{n \in \omega}$. □

So the category of embeddings is a full subcategory of $\mathbf{CPO}^E$ with the same colimiting cones of $\boldsymbol{\omega}$-chains. It follows that any ω-continuous functors over $\mathbf{CPO}^E$ which preserve ω-completeness cut down to ω-continuous functors on the subcategory. This remark applies to all the functors discussed in Example 2 except the sum functor, which only preserves κ-completeness for $\kappa \geqq 3$. Sums of lattices can be defined by adding a new top element or by equating top elements as in [34], [30], and can be dealt with by local continuity. General completeness concepts have been considered in [3]; it would be interesting to see how they fit into the present considerations. One approach to handling nondeterminism and concurrency is to use one of several available powerdomain functors. These are available over **CPO** (see [15]), and the Smyth powerdomain [37] is available over the ω-complete cpos. However, the Plotkin powerdomain [27] does not preserve ω-completeness; a very weak notion of completeness was needed, leading to the so-called **SFP** objects (briefly considered in § 5).

Example 4. *Continuity and algebraicity.* Now we consider the ω-continuous and the ω-algebraic cpos. Our main definitions (13 and 15) are formulated entirely in "countable" terms, but we pause to show that one could just as well start from definitions (14 and 16) formulated without any countability restrictions.

DEFINITION 13. Let D be a cpo. The *countable way-below* (=*relative compactness*) relation is defined by: $x \ll_\omega y$ if and only if for every countable directed subset Z of D, if $y \sqsubseteq \bigsqcup Z$ then $x \sqsubseteq z$ for some z in Z.

A countable subset, B, of D is an ω-basis of D if and only if for every element x of D the set $B_\omega(x) =_{\text{def}} \{b \in B \mid b \ll_\omega x\}$ is directed with l.u.b. x.

The cpo D is *ω-continuous* if and only if it has an ω-basis.

This definition is not quite the very similar one considered in the case of complete lattices by Scott in [33] and more generally for dcpos in [36], [23] and several papers in [7].

DEFINITION 14. Let D be a dcpo. The *way-below* (=*relative compactness*) relation is defined by: $x \ll y$ if and only if for every directed subset Z of D, if $y \sqsubseteq \bigsqcup Z$ then $y \sqsubseteq z$ for some z in Z.

A subset B of D is a *basis* of D if and only if for every element x of D the set $B(x) =_{\text{def}} \{b \in B \mid b \ll x\}$ is directed with l.u.b. x.

The dcpo D is *continuous* if and only if it has a basis.

To relate these two notions, we first note a few useful and easily proved facts. In a cpo we have that $x \ll_\omega y$ implies $x \sqsubseteq y$ and $x \ll_\omega y \sqsubseteq z$ implies $x \ll_\omega z$; analogous facts with $\ll$ replacing $\ll_\omega$ hold in a dcpo; for any elements x, y of a dcpo, if $x \ll y$ then $x \ll_\omega y$.

FACT. 4. A partial order is an ω-continuous cpo if and only if it is a continuous dcpo with a countable basis.

Proof. Let D be an ω-continuous cpo with ω-basis B. First, we see it is a dcpo. For if X is any directed set then $A = \bigcup_{x \in X} B_\omega(x)$ is countable and directed (by the above facts), and so its l.u.b. exists and is the l.u.b. of X. Next we show for any elements x and y of D that $x \ll y$ if and only if $x \ll_\omega y$. Suppose $x \ll_\omega y$ and $y \sqsubseteq \bigsqcup Z$ where Z is directed. Then $y \sqsubseteq \bigsqcup A$ as before, and so for some z in Z and a in $B_\omega(z)$ we have $x \sqsubseteq a \ll_\omega z$, and so $x \sqsubseteq z$. This establishes $x \ll y$. We have already noted the converse, that $x \ll y$ implies $x \ll_\omega y$. Therefore $B_\omega(x) = B(x)$ for any x, and so B is a countable basis. The converse assertion—that any continuous dcpo with a countable basis is ω-continuous—is proved along the same lines. $\square$

The ω-algebraic cpos are a subclass of the ω-continuous ones and can also be presented in two ways.

DEFINITION 15. Let D be a cpo. An element x is ω-finite (=ω-compact) if and only if $x \ll_\omega x$. The cpo is ω-algebraic if and only if there is an ω-basis of finite elements.

DEFINITION 16. Let D be a dcpo. An element x is *finite* (=*compact*) if and only if $x \ll x$. The cpo is *algebraic* if and only if there is a basis of finite elements.

One then sees that D is an ω-algebraic cpo if and only if it is an algebraic dcpo with a countable basis of finite elements. Also in any (ω-) algebraic dcpo (cpo), there is only one (ω-) basis, namely the set of all (ω-) finite elements.

Turning to the full subcategory of the ω-continuous cpos, we note that it contains the one-point cpo, so Theorem 1 applies; however, it does not have all $\boldsymbol{\omega}^{\text{op}}$-limits, and we conjecture it does not have $\boldsymbol{\omega}$-colimits. The same remarks apply to the ω-algebraic cpos. Fortunately, however, the embedding subcategories inherit $\boldsymbol{\omega}$-colimits from $\mathbf{CPO}_\perp$. We need a preliminary lemma.

LEMMA 5a. *Embeddings (in* **CPO***) preserve the countable relative compactness relation.*

b. *Let E be a* cpo *and B be a countable subset of E. If x is an element of E and there is a directed subset C of $B_\omega(x)$ with l.u.b. x, then $B_\omega(x)$ is directed, with l.u.b. x.*

c. *Let E be a* cpo *and B and C be subsets of E, with B countable. Suppose that for every element y of C, $B_\omega(y)$ is directed with l.u.b. y and suppose too that for every element x of E there is a countable directed subset, C_x, of C such that $x = \bigsqcup C_x$. Then B is a basis for E.*

Proof. a. Let $f: D \to E$ be an embedding in **CPO** and let x, y be elements of D where $x \ll_\omega y$. If $f(y) \sqsubseteq \bigsqcup Z$ where Z is a countable directed subset of E, then $y = f^L(f(y)) \sqsubseteq \bigsqcup f^L(Z)$. So for some z in Z we have $x \sqsubseteq f^L(z)$, and so $f(x) \sqsubseteq f(f^L(z)) \sqsubseteq z$, showing that $f(x) \ll_\omega f(y)$.

b. If $u \ll_\omega x$ and $v \ll_\omega x$, then there are u', v' in C such that $u \sqsubseteq u'$, $v \sqsubseteq v'$. But as C is directed, this shows that $B_\omega(x)$ is directed too.

c. Take x in E and consider $\{B_\omega(y) | \in C_X\}$. This is a directed set, with respect to $\subseteq$, of directed sets as C_x is directed; its union is therefore directed and is clearly a subset of $B_\omega(x)$ with l.u.b. x. So by part b, $B_\omega(x)$ is directed with l.u.b. x. □

FACT 5. *Let $\Delta = \langle D_n, f_n \rangle$ be an* **ω**-*chain in* $\mathbf{CPO}^E$ *of ω-continuous (ω-algebraic)* cpos. *Suppose $\mu: \Delta \to D$ is colimiting. Then D is ω-continuous (ω-algebraic).*

Proof. We use Lemma 5c to show that D is ω-continuous when the D_n are. Let $B^{(n)}$ be an ω-basis for D_n ($n \in \omega$); we claim $B \stackrel{\text{def}}{=} \bigcup_n \mu_n(B^{(n)})$ is an ω-basis for D. Let C be $\bigcup_n \mu_n(D_n)$. By Theorem 2 applied to **CPO**, we can take $C_x = \{\mu_n(\mu_n^R(x)) | n \in \omega\}$. Now (Lemma 5a) for each y in D_n, $\mu_n(B_\omega^{(n)}(y))$ is a directed subset of $B_\omega(\mu_n(y))$ with l.u.b. y. So by Lemma 5b $B_\omega(\mu_n(y))$ is directed with l.u.b. $\mu_n(y)$. Thus Lemma 5c applies. In the case where the D_n are all ω-algebraic, we take $B^{(n)}$ to be the ω-finite elements of D_n and find a basis of ω-finite elements of D. □

So the full subcategory of $\mathbf{CPO}^E$ of the ω-continuous (ω-algebraic) cpos is an ω-category that inherits **ω**-colimits, from $\mathbf{CPO}^E$. It follows that any ω-continuous functor over $\mathbf{CPO}^E$ that preserves ω-continuity (ω-algebraicity) cuts down to an ω-continuous functor over the subcategory. This enables all the functors discussed above for $\mathbf{CPO}^E$ to be handled except the function space functors which preserve neither ω-continuity nor ω-algebraicity (see [24], [23] for a counterexample). Here completeness considerations help. The full subcategory of $\mathbf{CPO}^E$ of the κ-complete and ω-continuous (ω-algebraic) cpos is clearly an ω-category that inherits **ω**-colimits from $\mathbf{CPO}^E$ (for $\kappa \leqq \omega$).

Now all the functors discussed above preserve the property of being both κ-complete and ω-continuous (ω-algebraic). We have already noted this for all except the function-space functors. For these, one notes that the proofs in [24], [23] for ω-complete and continuous (algebraic) dcpos adapt easily to ω-complete and ω-continuous (algebraic) cpos whether we consider all continuous functions or only the strict ones. The general case then follows from the facts that κ-complete implies ω-complete for $\kappa \leqq \omega$ and that κ-completeness is preserved.

Example 5. *Nondeterministic domains.* The category **NDO** was found useful for the semantics of nondeterministic and parallel programs in [15]. Its objects are the *nondeterministic* cpos $\langle D, \sqsubseteq, \cup \rangle$ where $\langle D, \sqsubseteq \rangle$ is a cpo and $\cup : D^2 \to D$ is an associative, commutative absorptive ω-continuous binary function (called *union*); the morphisms $f: D \to E$ are those ω-continuous functions which preserve union.

The trivial one-point object is terminal in $\mathbf{NDO}_\perp$ and the conditions of Theorem 1 are satisfied. Further, **NDO** has $\boldsymbol{\omega}^{\text{op}}$-limits. Indeed, the forgetful functor $U: \mathbf{NDO} \to \mathbf{CPO}$ creates them. Let $\Delta = \langle D_n, f_n \rangle$ be an ω-chain in **NDO** and suppose $\nu: E \to U\Delta$ is universal in **CPO**, being constructed as shown above. Then if we want a union on E so that the ν_n are **NDO** morphisms, we have for elements x, y, of E:

$$(x \cup y)_n = \nu_n(x \cup y) = \nu_n(x) \cup \nu_n(y) = x_n \cup y_n.$$

So this determines union, and it is easily seen that with this definition we obtain a universal cone in **NDO**. One interesting locally continuous function is $\to_{\subseteq}$, where on objects D, E:

$$D \underset{\subseteq}{\to} E = \{f: D \to E | f \text{ is } \omega\text{-continuous and preserves } \subseteq\},$$

(where $x \subseteq y \equiv_{\text{def}} x \cup y = y$), with the pointwise order and union, and defined as usual on morphisms. Other examples can be found in [15].

Of course, there are many interesting varieties (or pseudovarieties of one kind or another) subject to similar considerations [9], [25]. However, we have no clear idea of the possible applications.

Example 6. *ω-complete relations.* This category (or rather a slight variation of it) has been found to be useful for relating different semantics by Reynolds [31] (see also [12], [26]). It has as objects structures $\langle D, E, R\rangle$ where D and E are cpos and $R \subseteq D \times E$ is a binary relation which is *ω-complete* in the sense that if $\langle d_n\rangle, \langle e_n\rangle$ are increasing sequences in D and E, respectively, such that $R(d_n, e_n)$ holds for all integers n, then $R(\bigsqcup d_n, \bigsqcup e_n)$ holds too; the morphisms are pairs $\langle f, g\rangle : \langle D, E, R\rangle \to \langle D', E', R'\rangle$ where $f: D \to D'$, $g: E \to E'$ are morphisms in **CPO**, and for all x in D and y in E if $R(x, y)$ holds then so does $R'(fx, gy)$.

The terminal object is $\langle \perp, \perp, R_\perp\rangle$ where $\perp$ is the one-point cpo and $R_\perp$ is the complete binary relation over $\perp$; clearly, too, all the other conditions of Theorem 1 apply. Next **ω**-limits exist. To see this, let $\Delta = \langle\langle D_n, E_n, R_n\rangle, \langle f_n, g_n\rangle\rangle$ be an **ω**-chain. Let $\nu': D \to \langle D_n, f_n\rangle$ and $\nu'': E \to \langle E_n, g_n\rangle$ be limiting cones in **CPO**, constructed as above. Then $\nu: \langle D, E, R\rangle \to \Delta$ is limiting where $\nu_n = \langle \nu'_n, \nu''_n\rangle$ and $R(d, e)$ holds if and only if for all n the relation $R_n(d_n, e_n)$ holds.

A useful function space functor is given on objects by putting $\langle D, E, R\rangle \to \langle D', E', R'\rangle =^{\text{def}} \langle D \to D', E \to E', R \to R'\rangle$ where $D \to D'$ and $E \to E'$ are the cpos of all ω-continuous functions, and where

$$(R \to R')(f, g) \equiv \forall x \in D, y, \in E. R(x, y) \supset R'(f(x), g(y))$$

The action of the functor on morphisms is defined analogously to the case of **CPO**. Other examples can be found in [31]. This idea can be extended to several relations and to relations of any denumerable degree. It can also be combined with the ideas of Example 4 to consider continuous structures of various kinds. However, we must point out that the scope and usefulness of these mathematical possibilities is not known. We do not have a nice language for functors which permits a uniform treatment of the examples in [31], [12], [26]; we do not know why these relations seem to be needed only when function-spaces arise, for in other cases structural induction [20] seems to be sufficient; we wonder if the continuous structures should be accompanied by suitable logics along the lines of LCF [14] but possibly intuitionistic [35].

5. Computability. The approach to domain equation theory presented above may be seen as an abstraction from Scott's "D_∞" method [33]. As we have said, the "universal domain" method (Scott [34]), and its relation to the above theory, are to be treated in a separate paper. However, there is an aspect of universal domain theory, stressed by Scott, which must be mentioned here: computability. Starting from a suitable universal domain, it is possible to provide a smooth treatment of effective computability for all the constructs of interest, generalizing the relevant parts of classical recursion theory (Scott [34]). Lacking anything like this, the theory presented above has to be considered as seriously defective.

Fortunately, however, the deficiency can be remedied. Effectiveness can be built into **O**-categories in a satisfactory way. Here we will simply indicate some of the main points; for a fuller and more accurate treatment, see Smyth [38]. We again work by lifting suitable properties from domains to categories. *Algebroidal* categories are introduced as a generalization of algebraic cpos; they are categories with a (countable) "basis of finite objects". Then we get a handle on computability by requiring that bases be effectively presented.

Approach A. *Algebroidal categories.* These are the same as what Smyth previously called "algebraic categories" [37]. It has been brought to our attention that closely related notions have been discussed quite extensively in the literature of category theory, and this is what has prompted the change in nomenclature. Our algebroidal categories are essentially the "strongly ω-algebroidal categories" in (a slight extension of) the terminology of Banaschewski and Herrlich [6].

DEFINITION 17. An object A of a category $\mathbf{K}$ is *finite* in $\mathbf{K}$ provided that, for any $\boldsymbol{\omega}$-chain $\Delta = \langle V_n, f_n \rangle_{n\in\omega}$ in $\mathbf{K}$ with colimit $\mu : \Delta \to V$, the following holds: for any morphism $v : A \to V$, and for any sufficiently large n, there is a unique morphism $u : A \to V_n$ such that $v = \mu_n \circ u$. We say that $\mathbf{K}$ is *algebroidal* provided (i) $\mathbf{K}$ has an initial object and at most countably many finite objects, (ii) every object of $\mathbf{K}$ is a colimit of an $\boldsymbol{\omega}$-chain of finite objects, and (iii) every $\boldsymbol{\omega}$-chain of finite objects has a colimit in $\mathbf{K}$.

Notation. If $\mathbf{K}$ is algebroidal, we denote by $\mathbf{K}_0$ the full subcategory of $\mathbf{K}$ with objects the finite objects of $\mathbf{K}$.

The principal examples of interest to us are $\mathbf{SFP}^E$ (the category of **SFP** objects and embeddings [27]) and various of its subcategories, for example the category of bounded complete ω-algebraic ω-cpos and embeddings. The finite objects are in each case the finite domains.

THEOREM 4. *Every algebroidal category has all ω-colimits.*

Proof. See Smyth [37]. □

THEOREM 5. *Let $\mathbf{K}$ be an algebroidal category, and let $\mathbf{L}$ be an ω-category. Any functor F_0 from $\mathbf{K}_0$ into $\mathbf{L}$ extends uniquely* (*up to natural isomorphism of functors*) *to an ω-functor from $\mathbf{K}$ into $\mathbf{L}$.*

Outline of proof. For each nonfinite object D of $\mathbf{K}$, choose a particular colimiting cone $\mu D : \Delta_D \to D$, with Δ_D an $\boldsymbol{\omega}$-chain in $\mathbf{K}_0$; and for each $\boldsymbol{\omega}$-chain Δ in $\mathbf{L}$ choose a particular colimiting cone $\mu_\Delta : \Delta \to D_\Delta$ in $\mathbf{L}$. The extension of F_0 to all objects of $\mathbf{K}$ is immediate (via the chosen colimiting cones in $\mathbf{K}$, $\mathbf{L}$). To define the extension F of F_0 to morphisms, consider first morphisms $v : A \to V$ where A is finite and V nonfinite. Since A is finite, v factorizes as $v = (\mu_V)_n \circ u$. Then we put $Fv = (\mu_{F_0(\Delta_V)})_n \circ F_0 u$. Next, for morphisms $h : V \to W$, where V is nonfinite, define Fh as the mediating morphism from the colimiting cone $F(\mu_V)(= \mu_{F_0(\Delta_V)})$ to $F(h \circ \mu_V)$. Of course, it has to be checked that F so defined preserves composition of morphisms, and so is a functor (this is nontrivial).

Now suppose that $F, F' : \mathbf{K} \to \mathbf{L}$ are two ω-continuous functors which extend F_0. For each object V of $\mathbf{K}$ we have a canonical isomorphism $\tau_V : FV \to F'V$, namely the mediating morphism from $F(\mu_V)$ to $F'(\mu_V)$. Naturality of τ means that for $h : V \to W$, $F'h \circ \tau_V = \tau_W \circ Fh$; this is established by showing that $\tau_W \circ Fh$ mediates between the colimiting cone $F(\mu_V)$ and $F'h \circ \tau_V \circ F(\mu_V)$. □

Theorem 4 yields at once that $\mathbf{SFP}^E$ (for example) is an ω-category. Theorem 5 can be useful, at least heuristically, in setting up the definitions of appropriate ω-functors. Under these circumstances, the solution of typical domain equations, via the basic lemma, is unproblematic. More interesting is the question of effectiveness. The following definition seems natural:

DEFINITION 18. Let $\langle A_n\rangle_{n\in\omega}$, $\langle f_n\rangle_{n\in\omega}$ be enumerations of the objects and morphisms, respectively, of $\mathbf{K}_0$, where $\mathbf{K}$ is an algebroidal category. We say that $\mathbf{K}$ is *effectively given*, relative to these enumerations, provided that the following predicates are recursive in the indices:

i) $A_i = A_j$; $f_i = f_j$

ii) dom $(f_k) = A_i$; cod $(f_k) = A_i$

iii) f_k is an identity

iv) $f_i \circ f_j = f_k$.

This enables us to define an *effectively given object* (of $\mathbf{K}$) as an object that is given as the colimit of an effective $\boldsymbol{\omega}$-chain of finite objects, that is, as the colimit of a chain of the form

$$A_{r(0)} \xrightarrow{f_{s(0)}} A_{r(1)} \xrightarrow{f_{s(1)}} \cdots$$

(r, s recursive). One will naturally try to define a *computable morphism*, similarly, as the colimit of an effective $\boldsymbol{\omega}$-chain of finite morphisms (that is, morphisms of $\mathbf{K}_0$). Actually, such a characterization would be inadequate. The definitions given so far are, strictly speaking, appropriate only for categories of the form $\mathbf{K}^E$, whereas we are certainly interested in computability of morphisms other than embeddings. For an adequate treatment, we have to reformulate the definitions so as to apply to $\mathbf{O}$-categories; this is done in Smyth [38] where, for example, we find that an "admissible" $\mathbf{O}$-category $\mathbf{K}$ is, roughly speaking, one for which $\mathbf{K}^E$ is algebroidal. We can then define a *computable* functor, roughly, as a continuous functor F for which we can effectively assign to each finite object (morphism) $A(f)$ an effective $\boldsymbol{\omega}$-chain having $F(A)$ $(F(f))$ as colimit. A basic result, in terms of these definitions, will be that the initial fixpoint of a computable functor is computable.

Approach B. *Effective domains/categories.* Kanda [17] proposes that only computable items should be admitted to the domains and categories which we study—in contrast to the usual practice of first building all the continuous/countably-based items and then picking out the computable items from among these. This entails a modification of the closure properties required of the domains and categories: we now demand closure of domains with respect to sups only of *effective* $\boldsymbol{\omega}$-chains, and of categories with respect to effective colimits of effective $\boldsymbol{\omega}$-cochains. This approach works quite smoothly, and indeed yields a theory which is formally very close to Smyth [36] as far as concerns effective domains. In regard to the theory of effective categories (as developed by Kanda), perhaps the most striking feature of this theory is the very simple definition of *computable functor* (Kanda has "effective functor") in terms of indexings of hom-sets, which it permits.

Unlike Approach A, however, Kanda's theory does not pretend to give a general account of effectiveness in domains. In his theory, the definitions of an effective domain and of an effective category are quite independent. In order to apply the theory, we first define a particular category of "effective domains", and then show that this category satisfies the axioms for an "effective category". The definition is ad hoc, in the sense that no general or uniform notion of effective domains is proposed: we cannot, for example, define an effective domain to be an object of an "effective category of domains" (in contrast with our Approach A).

We incline to the view that these problems can best be attacked by means of the ideas mentioned in Approach A (finite objects in categories, etc.); but that it may be worthwhile to develop the argument in accordance also with the main ideas of Approach B, namely, that only computable items should be admitted to the field of discourse.

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