Cartesian Closed Categories of Separable Scott Domains

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Abstract

We classify all sub-cartesian closed categories of the category of separable Scott domains. The classification employs a notion of coherence degree determined by the possible inconsistency patterns of sets of finite elements of a domain. Using the classification, we determine all sub-cartesian closed categories of the category of separable Scott domains that contain a universal object. The separable Scott domain models of the $\lambda\beta$-calculus are then classified up to a retraction by their coherence degrees.

Keywords: Scott domain, cartesian closed category, lambda calculus

1. Introduction

We revisit some classical themes in domain theory: models of the λ-calculus, universal domains, and cartesian closed categories (ccc's) of domains. In [1], Scott showed that $\mathcal{P}(\omega)$, the partial order of all sets of natural

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numbers, is universal in the cartesian closed category of separable continuous lattices and continuous functions, by which is meant that $\mathcal{P}(\omega)$ is in this category and every object in the category is a retract of it. As it is universal in a cartesian closed category, $\mathcal{P}(\omega)$ is necessarily a model of the $\lambda\beta$-calculus because its function space is then a retract of it.

Next, in [2], Plotkin introduced the cartesian closed category of coherent separable continuous bounded complete pointed domains, and showed that it contains a universal object $\mathbb{T}_\perp^\omega$.

Then, in [3], Scott introduced the category of Scott domains, and gave a universal object for the full sub-cartesian closed category of the separable Scott domains. In unpublished work[1] he gave another universal object for this category: the partial order of all consistent propositional theories over a given countably infinite set of propositional letters. It is worth noting that in [1], Scott had already essentially considered the separable Scott domains in terms of closed subsets of retracts of $\mathcal{P}(\omega)$.

With the introduction of powerdomains [4, 5], attention was also paid to wider categories of separable algebraic domains, and eventually interest arose in finding characterisations of cartesian closed categories by maximality properties. The first of these was given by Smyth [6]; it characterises the separable bifinite domains as the largest full subcategory of the separable pointed algebraic domains that is a sub-ccc of the category of all directed complete partial orders (dcpo's). Later, Jung introduced his FS domains, and gave a characterisation [7, 8] of them as the largest full subcategory of the separable pointed (by definition continuous) domains that is a sub-ccc of the category of all pointed dcpo's.

We consider three natural classification questions concerning the category **Dom** of separable Scott domains. The first asks which such Scott domains are models of the $\lambda\beta$-calculus, up to retraction; the second asks which retract closed full sub-cartesian closed categories of **Dom** have universal objects; and the third asks, more broadly, for the classification of all retract closed full sub-cartesian closed categories of **Dom**. We answer all three questions. The answers to the first two are straightforward once the classification is available (although not all the classification is needed).

Regarding the third question, there is an obvious remark: as well as the three categories already mentioned, there are also their three full subcate-

[1] A space of retracts, Bremen talk, Nov. 1979; Manuscript, April, 1980

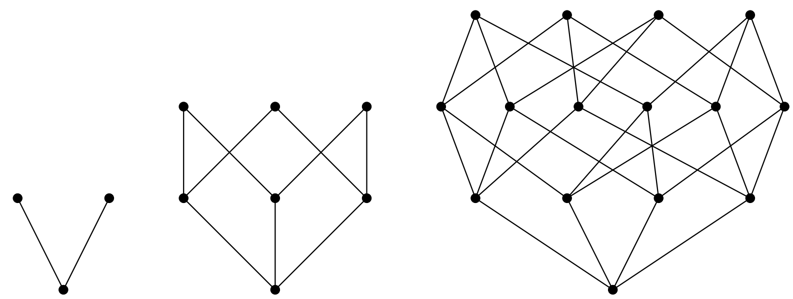

Figure 1: The first three truncated hypercubes $\mathbb{T}_2$, $\mathbb{T}_3$, and $\mathbb{T}_4$

gories of finite domains. Less obviously, the notion of coherence generalises: say that a Scott domain is *n-coherent* when any subset of size at least n is bounded if all its subsets of size n are bounded; the ω-algebraic lattices arise when $n = 0$ and when $n = 1$, and the coherent domains arise when $n = 2$. For $n \geq 2$, define the *truncated n-dimensional hypercube* $\mathbb{T}_n$ to be the n-dimensional hypercube ordered outwards from the origin and with the maximal point removed. Figure 1 depicts the first three truncated hypercubes. Then $\mathbb{T}_n$ is a simple example of a domain that is n-coherent, but not $(n-1)$-coherent. Note that $\mathbb{T}_2$ is $\mathbb{T}_\perp$.

It is not hard to show that the full subcategories of n-coherent separable domains provide further examples of retract closed full sub-ccc's of **Dom**, and so does their union, and all the full subcategories of domains with finitely many points. However this does not exhaust all the possibilities, as points can participate in truncated hypercubes of all dimensions. So we can classify points according to the pattern of their participation in truncated hypercubes and, in turn, we can classify other points according to their participation in such participations, and so on. This leads to well-founded countably branching trees of points and so to countable ordinals. These, in turn allow us to assign ordinal-valued invariants called *coherence degrees*, first to domains, and then to categories of them. Armed with these invariants and two other rather simpler ones (whether a domain is finite and the cardinality of its set of maximal points), one can give a complete classification of the retract closed full sub-ccc's of **Dom**.

After giving preliminary definitions and notation for domain theory in Section 2, we define coherence degrees of domains and categories in Section 3. Next, in Section 4, we calculate coherence degrees; this enables us to prove

Theorem 4.9 showing that categories defined in terms of their coherence degrees are sub-ccc's of **Dom**. In Section 5 we analyse domains in terms of retracts of coherent powers of standard domains determined by the coherence degrees of the domains under analysis. This enables us to prove Theorem 5.6 showing that retract closed full sub-ccc's of **Dom** containing the domain of natural numbers can be characterised in terms of their coherence degrees. Section 6 considers the remaining class of categories, those whose objects have finitely many maximal points. Putting all this together in Section 7, we prove our classification theorem, Theorem 7.1. We are then able to show which of the categories classified have universal domains, see Theorem 7.3, and to identify all retract closed full sub-ccc's of **Dom** generated by a model of the $\lambda\beta$-calculus, see Theorem 7.4. Finally, Theorem 7.6 gives a classification of all locally-monotone fully faithful functors between the ccc's given by the classification theorem.

Regarding future work, a question of immediate interest is extending our results to the, so-to-speak, continuous Scott domains, more precisely, to the separable consistently-complete pointed domains. Regarding algebraic domains, having Smyth's result on the maximal ccc of separable pointed algebraic domains, one can ask for a classification of all such categories; with Jung's result, there is an analogous question for the separable domains. In a different direction, having available a classification of all separable Scott domain models of the $\lambda\beta$-calculus, one could investigate the relations between the corresponding realisability models.

2. Domain Theory

We generally use the terminology of [9]. In particular, *Scott domains* are the algebraic bounded complete pointed domains. The *separable* algebraic domains are those with countably many finite elements, and we write D^{o} for the set of finite elements of an algebraic domain D. A subset of a boundedly complete partial order is *consistent* if it has an upper bound, equivalently if it has a least upper bound. A *minimally inconsistent (mic)* subset of a bounded complete partial order is one that is inconsistent, but all of whose proper subsets are consistent; every such set is finite. If $\{x_0, \ldots, x_{n-1}\}$ is a mic set in a Scott domain then there are finite $a_i \leq x_i$, such that $\{a_0, \ldots, a_{n-1}\}$ is also a mic set.

A *basis* of a Scott domain is a set of finite elements of the domain such that every finite element is a least upper bound of elements of the basis. The cartesian product $D \times E$ of two Scott domains is a Scott domain and its finite elements have the form $\langle a, b \rangle$ with a and b finite, and the elements of the forms $\langle a, \bot \rangle$ or $\langle \bot, b \rangle$ form a basis. We write both $D \Rightarrow E$ and E^D for the function space of all continuous functions from D to a dcpo E. Given $a \in D^{\mathrm{o}}$ and $y \in E$ we write $a \Rightarrow y$ for the *step function* from D to E given by:

$$(a \Rightarrow y)(x) = \begin{cases} y & \text{if } x \geq a, \\ \bot & \text{otherwise.} \end{cases}$$

If y is finite then so is $a \Rightarrow y$. The function space $D \Rightarrow E$ of two Scott domains is a Scott domain; it has a basis consisting of the step functions $a \Rightarrow b$ with $a \in D^{\mathrm{o}}$ and $b \in E^{\mathrm{o}}$.

A *retraction pair* between two dcpos D and E consists of two continuous functions

$$D \xrightarrow{e} E \xrightarrow{r} D$$

such that $r \circ e = \mathrm{id}_E$. We write $D \lhd E$ to show that such a retraction pair exists; this relation is evidently a preorder.

From now on we only consider separable Scott domains, and take the liberty of referring to them simply as "domains" and, as above, write **Dom** to refer to the category of these domains and the continuous functions between them.

3. Coherence degrees

We begin by recording some equivalent characterisations of the n-coherent domains.

Proposition 3.1. *The following are equivalent for a domain D and $n \geq 1$:*

1. *A subset $X \subseteq D$ with at least n elements is consistent, if every subset of X with n elements is consistent.*
2. *Any mic set in D has size at most n.*
3. *Any mic set in D^{o} has size at most n.*
4. *The truncated hypercube, $\mathbb{T}_{n+1}$, is not a retract of D.*

When these hold, we say that D is n-coherent*.*

Proof. We omit the straightforward proofs, except to remark that for the equivalences involving (4), one shows, for $n \geq 2$, that D has a mic set of size n if, and only if, $\mathbb{T}_n$ is a retract of D. □

To obtain a handle on more involved cases, we consider trees of points in domains, as discussed above. We work with countable rooted trees. A *rooted tree* is a directed graph whose underlying undirected graph is connected and acyclic, together with a distinguished node, called the *root*, with the graph oriented so that all edges point away from the root; below we often just say "tree" rather than"rooted tree." The *child* relation of such a graph is given by the set of its edges; we write the child relation between nodes n and n' as $n \to n'$.

A *branch* of such a rooted tree is a linear subset b of its nodes, by which we mean that, given any two distinct nodes in the subset, one is an ancestor of the other; a *complete* branch is a maximal such subset. A rooted tree is *linear* if it has just one branch, *well-founded* if all of its complete branches are finite, and *proper* if it has at least two nodes and every finite incomplete branch of the tree extends to a finite complete branch. Well-founded rooted trees are automatically proper if they have at least two nodes. The linear trees $l_n = 0 \to \ldots \to n$, where $n > 0$, provide simple examples of proper rooted trees; we also use the non-well-founded proper tree $\mathbb{N}^\infty$ which has elements n and n^+ for every $n \in \mathbb{N}$, with $n \to n+1$ and $n \to n^+$.

A *morphism* between two rooted trees is a morphism of their underlying directed graphs that preserves the roots. In case it is an inclusion, we say that the first tree is a *subtree* of the other one.

Definition 3.2. *A* mic tree *in a domain D is a proper countable rooted tree τ together with a* labelling *map $\ell : \tau \to D^{\mathrm{o}}$ such that:*

1. *For any finite complete branch b of the tree, the set $\ell(b) \subseteq D^{\mathrm{o}}$ is minimally inconsistent.*
2. *The labelling map ℓ is 1-1 on branches.*

Note that, for any infinite or incomplete finite branch b of the tree, the set $\ell(b) \subseteq D^{\mathrm{o}}$ is consistent. Note too that mic trees in a domain D whose trees are one of the l_n correspond to mic sets of size $n+1$ with a chosen linear order. Below we often confuse mic trees with their underlying trees: for example we say that a mic tree is complete when we mean its underlying tree is, or refer to a branch of the mic tree when we mean a branch of its underlying tree. If B is a basis of a domain D, and τ is a proper countable rooted tree, we say that $\ell : \tau \to B$ is a mic tree in D if its composition with the inclusion $\iota : B \subseteq D^{\mathrm{o}}$ is.

There is a standard assignment of ordinals to nodes in well-founded countable rooted trees and then to the trees themselves. We assign ordinals $\|v\|$ to nodes v of such a tree τ by:

$$\|v\| =_{\mathrm{def}} \sup\{\|v'\| + 1 \mid v \to v'\}.$$

So zero is assigned to a leaf, and every node is assigned the least ordinal strictly greater than any assigned to any of its children.

The ordinal $\|\tau\|$ of such a tree τ is defined to be the ordinal of its root. The ordinals obtained in this way are exactly the countable ones. If τ is a proper such tree then $\|\tau\| > 0$, as τ must then have at least two nodes. As is evident, $\|l_n\| = n$. It is not hard to see that there is a morphism $\tau \to \tau'$ between two such trees if, and only if, $\|\tau\| \leq \|\tau'\|$.

Combining these ordinals with mic trees we can first define coherence degrees of domains and then of categories of domains. The *coherence degree* $\|D\|$ of a domain D is defined to be the least ordinal not accessed by a mic tree:

$$\|D\| =_{\mathrm{def}} \sup\{\|\tau\| + 1 \mid \ell : \tau \to D^{\mathrm{o}} \text{ is a well-founded mic tree}\}.$$

Note that $\|D\|$ is an isomorphism invariant of D. We always have $\|D\| \leq \omega_1$, but never equal to 1, and $\|D\| = 0$ if, and only if, D is a lattice. It is not hard to see directly from the definition and the above remarks that, for $n \geq 1$, a domain is n-coherent precisely when its coherence degree is less than or equal to n. So, in particular, $\|\mathbb{T}_n\| = n$, for $n \geq 2$.

Next, the coherence degree $\|\mathbf{C}\|$ of a full subcategory $\mathbf{C}$ of $\mathbf{Dom}$ is defined to be the least ordinal which is not a coherence degree of one of its objects:

$$\|\mathbf{C}\| =_{\mathrm{def}} \sup\{\|D\| + 1 \mid D \in \mathrm{Ob}(\mathbf{C})\}.$$

We have $\|\mathbf{C}\| \leq \omega_1 + 1$, but never equal to 2. The case $\|\mathbf{C}\| = 0$ is where $\mathbf{C}$ has no objects.

We define $\mathbf{Dom}_\alpha$ to be the full subcategory of all domains whose coherence degree is smaller than α, for $\alpha \leq \omega_1 + 1$, but $\alpha \neq 0$ and $\alpha \neq 2$; we define $\mathbf{Dom}^{\mathrm{f}}_\alpha$ to be the full subcategory of $\mathbf{Dom}_\alpha$ of domains with finitely many points; and we define $\mathbf{Dom}^{\mathrm{fm}}_\alpha$ to be the full subcategory of $\mathbf{Dom}_\alpha$ of domains with finitely many maximal points. Note that $\mathbf{Dom} = \mathbf{Dom}_{\omega_1+1}$, that $\mathbf{Dom}_1 = \mathbf{Dom}^{\mathrm{fm}}_1$ is the category of ω-algebraic lattices, equivalently the full subcategory of domains with one maximal point, and that $\mathbf{Dom}_{n+1}$ is the category of n-coherent domains.

We need a supply of "standard" domains. Given any rooted tree τ, define $\mathbb{T}_\tau$ to be the pointed dcpo of all subsets of τ not containing any complete finite branch, partially ordered by subset; we call such subsets *configurations*. If τ is countable and non-empty then $\mathbb{T}_\tau$ is a domain whose finite elements are the finite configurations. If, further, τ is proper then the assignment $\ell(v) = \{v\}$, for $v \in \tau$, is a mic tree in $\mathbb{T}_\tau$, which we call the *standard* mic tree for $\mathbb{T}_\tau$. As an example, note that $\mathbb{T}_{l_n} \cong \mathbb{T}_{n+1}$, for $n > 0$.

4. Calculation of coherence degrees

In this section we relate coherence degrees with various domain-theoretic constructions. Throughout, let D and E be domains.

Lemma 4.1. *If $D \lhd E$ then $\|D\| \leq \|E\|$.*

Proof. By assumption we have a retraction $D \xrightarrow{e} E \xrightarrow{r} D$. There is therefore a function $f : D^o \to E^o$ such that $f(a) \leq e(a)$ and $r(f(a)) = a$ for all $a \in D^o$. Then, if $\ell : \tau \to D^o$ is a mic tree in D, $f \circ \ell : \tau \to E^o$ is a mic tree in E. □

Lemma 4.2. *Let τ be a well-founded countable tree, and suppose every leaf occurs in one of finitely many, possibly overlapping, classes of leafs. Then there is a subtree of τ with the same root and ordinal as τ, and with all its leafs in the same class.*

Proof. The proof is straightforward, by induction on $\|\tau\|$. □

Proposition 4.3.

1. $\|D \times E\| = \max\{\|D\|, \|E\|\}$.
2. *For any countable collection of domains D_i $(i \in I)$, we have:*

$$\| \textstyle\prod_i D_i\| = \sup_i \|D_i\|.$$

Proof.

1. As $\|D \times E\| \geq \max\{\|D\|, \|E\|\}$ follows from Lemma 4.1, it is enough to show that if there is a well-founded mic tree τ in $D \times E$ then there is a well-founded mic tree τ' in either D or E such that $\|\tau'\| = \|\tau\|$. To this end, one applies Lemma 4.2, taking two classes for leafs. The first class contains those leafs v where the first projections of the labels of the branch leading to v form a mic set in D, and the second class is the same, but for E and the second projections. Note that either the first or the second projection of a mic set in $D \times E$ must be a mic set.

2. As one inequality again follows from Lemma 4.1, we show the other one. Suppose that we have a well-founded mic tree τ in $\prod_i D_i$. Then as the label of the root of τ is finite, it is bottom in all but finite many coordinates $i_1, \ldots, i_k$. We can then obtain a mic tree $\tau \to D_{i_1} \times \ldots \times D_{i_k}$ by projecting all the labels of τ nodes down to the coordinates $i_1, \ldots, i_k$. Then, applying the first part of the proposition, we find a well-founded mic tree with the same ordinal as τ in one of the $D_{i_1}, \ldots, D_{i_k}$. $\square$

Using Lemma 4.1 and Proposition 4.3 we can calculate the coherence degrees associated to other constructions. For example, consider the lifting construction $D_\perp$ which adds a new least element to a domain D. One checks that $D \lhd D_\perp \lhd \mathbb{O} \times D$, where $\mathbb{O}$ is Sierpinski space, i.e., the two-point lattice $\{\perp, \top\}$. So we have $\|D_\perp\| = \|D\|$. The coalesced sum $D + E$ of two domains is a separated sum, except that the two bottom elements are identified. Here we have $\mathbb{T}_\perp, D, E \lhd D + E \lhd \mathbb{T}_\perp \times D \times E$ and so we see that $\|D + E\| = \max\{\|D\|, \|E\|, 2\}$. Regarding bilimits one can check that **Dom** itself has them, and then that the $\mathbf{Dom}_\alpha$ do too, for $\alpha < \omega$. However that is as far as it goes, as every domain (n-coherent domain) is a bilimit of finite domains (finite n-coherent domains). (If $a_0, a_1, \ldots$ is an enumeration of the finite elements of a given domain D then D is a bilimit of the D_i where D_i is the finite sub-domain of D generated by $a_0, \ldots, a_{i-1}$.)

Lemma 4.4. *Let B be a basis for the finite elements of a domain D. Then if there is a mic tree $\ell : \tau \to D$, with τ well-founded, there is also a mic tree $\ell' : \tau' \to B \subseteq D^{\mathrm{o}}$, with τ' well-founded, and with $\|\tau'\| \geq \|\tau\|$.*

Proof. We formulate a more general statement and prove it by induction. For any non-repeating sequence $w = a_1, \ldots, a_n$ of finite elements of D, with $n \geq 0$, say that a *w-mic tree* in D is a well-founded countable tree τ, together with a *labelling* map $\ell : \tau \to D^{\mathrm{o}}$ such that:

1. For any finite complete branch b of τ, the set $\{a_1, \ldots, a_n\} \cup \ell(b) \subseteq D^{\mathrm{o}}$ is minimally inconsistent.
2. For any incomplete branch b of τ, the set $\{a_1, \ldots, a_n\} \cup \ell(b) \subseteq D^{\mathrm{o}}$ is consistent.
3. The labelling map ℓ is 1-1 on branches, and its range excludes $a_1, \ldots, a_n$.

We prove for all well-founded countable trees τ that, for every sequence $w = a_1, \ldots, a_n$ of finite elements of D, if there is a w-mic tree $\ell : \tau \to D^{\mathrm{o}}$, then there is also a w-mic tree $\ell' : \tau' \to B$, with $\|\tau'\| \geq \|\tau\|$ (we mean that

$\ell' : \tau' \to D^{\mathrm{o}}$ is a w-mic tree such that $\ell'(\tau') \subseteq B$). The proof is by induction on $\|\tau\|$.

So, given τ and $w = a_1, \ldots, a_n$, suppose there is a w-mic tree $\ell : \tau \to D^{\mathrm{o}}$. Let r be the root of τ and τ_i ($i \in I$) be the (possibly empty) collection of its immediate subtrees. Then, for any $i \in I$ we have a $w, \ell(r)$-mic tree $\ell_i : \tau_i \to D^{\mathrm{o}}$, where ℓ_i is the restriction of ℓ to τ_i. By induction hypothesis there are then $w, \ell(r)$-mic trees $\ell'_i : \tau'_i \to B$, with $\|\tau'_i\| \geq \|\tau_i\|$.

Next, since B is a basis we have $\ell(r) = \bigvee_{j=1}^{m} a'_j$ where $m > 0$ and the a'_j are different elements of B. Let τ'' be the tree with the same root r and with the τ'_i as subtrees (we can assume without loss of generality that r is not a node of any of the τ'_i and that no two of the τ'_i have a common node); note that $\|\tau''\| \geq \|\tau\|$. We know that for any complete branch b of any τ'_i that $\{a_1, \ldots, a_n\} \cup \{\ell(r)\} \cup \ell'_i(b)$ is minimally inconsistent. It follows that for some nonempty $\beta \subseteq \{1, \ldots, m\}$ the set $\{a_1, \ldots, a_n\} \cup \{a'_j \mid j \in \beta\} \cup \ell'_i(b)$ is also minimally inconsistent.

So, for each such β we can consider the class L_β of all those leafs of τ'' such that $\{a_1, \ldots, a_n\} \cup \{a'_j \mid j \in \beta\} \cup \ell'_i(b)$ is minimally inconsistent where b is the branch of τ'_i leading to that leaf (and where τ'_i is the unique subtree of τ'' containing the leaf). By Lemma 4.2 there is a $\beta \subseteq \{1, \ldots, m\}$ and a subtree τ''' of τ'' with the same root and ordinal as τ'' with all its leafs in L_β. Let j_k ($k = 1, \ldots, p$) be an enumeration without repetitions of β.

Now consider the tree τ' that is the same as τ''' except that the root r is expanded to a sequence $r_{j_1} \to \ldots \to r_{j_p}$ (we can assume, without loss of generality, that none of the r_{j_k} are nodes of any τ'_i). We evidently have that $\|\tau'\| \geq \|\tau'''\| = \|\tau''\| \geq \|\tau\|$. We obtain a w-mic tree $\ell' : \tau' \to B$ if we set ℓ' to be the same as ℓ'_i on τ'_i, and set $\ell'(r_{j_k}) = a'_{j_k}$, for $k = 1, \ldots, p$. This concludes the proof. □

Proposition 4.5. $\|D \Rightarrow E\| = \|E\|$.

Proof. It follows from Lemma 4.1 that $\|E\| \leq \|D \Rightarrow E\|$. For the converse direction we use the basis of $D \Rightarrow E$ of step functions $a \Rightarrow b$, where $a \in D^{\mathrm{o}}$ and $b \in E^{\mathrm{o}}$. It is easy to see that a finite set $\{a_i \Rightarrow b_i \mid i = 1, \ldots, n\}$ of such functions is minimally inconsistent iff $\{a_i \mid i = 1, \ldots, n\}$ is consistent and $\{b_i \mid i = 1, \ldots, n\}$ is minimally inconsistent.

So suppose we have a well-founded mic tree $\ell : \tau \to (D \Rightarrow E)^{\mathrm{o}}$. Then, by Lemma 4.4, there is a well-founded mic tree $\ell' : \tau' \to B$ where B is the basis of step functions, and $\|\tau'\| \geq \|\tau\|$. By the remark on minimally inconsistent sets of step functions we then have a well-founded mic tree $\ell'' : \tau' \to E^{\mathrm{o}}$

where $\ell''(n) = b$ if $\ell'(n) = a \Rightarrow b$. This shows that $\|E\| \geq \|D \Rightarrow E\|$, and completes the proof. □

Given a morphism between proper trees τ and τ', we obtain a mic tree $\ell : \tau \to \mathbb{T}_{\tau'}$ by composing the morphism with the standard mic tree in $\mathbb{T}_{\tau'}$.

Lemma 4.6.

1. *Let τ and τ' be countable well-founded trees with at least two nodes such that $\|\tau\| \leq \|\tau'\|$. Then there is a mic tree $\tau \to (\mathbb{T}_{\tau'})^{\mathrm{o}}$.*
2. *Let $f : \tau \to \tau'$ be a function between the nodes of two well-founded proper trees such that, for any complete branch b of τ, $f(b)$ is a complete branch of τ' and f, restricted to b, is 1-1. Then there is a morphism of rooted trees $h : \tau \to \tau'$.*
3. *Let τ be a countable tree with at least two nodes. Then there is a morphism of rooted trees $\tau \to \mathbb{N}^\infty$.*
4. *Let τ be a countable, non-well-founded, proper tree. Then there is a mic tree $\mathbb{N}^\infty \to (\mathbb{T}_\tau)^{\mathrm{o}}$.*

Proof.

1. As τ and τ' are countable well-founded trees with $\|\tau\| \leq \|\tau'\|$ there is a morphism of rooted trees $f : \tau \to \tau'$. Next, for any leaf v of τ, let $b(v)$ be a (necessarily incomplete) branch of τ' starting at $f(v)$ and ending at a leaf of τ'. Then one can define a mic tree $\ell : \tau \to (\mathbb{T}_{\tau'})^{\mathrm{o}}$ by putting $\ell(v) = \{f(v)\}$ when v is not a leaf of τ, and $\ell(v) = b(v)$ when it is.
2. We proceed by induction on $\|\tau\|$, dividing the proof into two cases. In the first, suppose that all complete branches of τ have the same length m. Then there is a complete branch of τ' of length m and the result is immediate.
   Otherwise there are two complete branches b and b' of τ of different lengths. So $f(b)$ and $f(b')$ are two complete branches of τ' of different lengths. Let r be the root of τ. As $f(r)$ is on both the different complete branches $f(b)$ and $f(b')$, it must be the root of τ'. By the 1-1 assumption, no other node of τ maps to the root of τ'.
   Let τ_i $(i \in I)$ be the immediate subtrees of τ. Any node of a given τ_i is on a branch beginning with r and r_i, the root of τ_i. As the image of this branch under f is a branch of τ', the node is mapped by f to the same subtree of τ' as r_i is. So for every $i \in I$ there is an immediate subtree τ_i' such that f maps all of τ_i to τ_i'.

If τ_i has at least two nodes then τ_i' also has at least two nodes, and the restriction of f to τ_i clearly satisfies the conditions of the lemma. So, using the induction hypothesis, we see that there is a morphism of rooted trees $h_i : \tau_i \to \tau_i'$. If τ_i has only a single node, then, by the conditions of the lemma, so too must τ_i'. In this case set h_i to be the pair $r_i \mapsto f(r_i)$. The union of the h_i, together with the pair $r \mapsto f(r)$, forms the required morphism.

3. Label the root of τ by 0; label those children of the root which are leafs by 0^+, and all others by 1; and proceed down τ analogously.
4. Since τ is non-well-founded it has an infinite branch $n_0 \to n_1 \to \ldots$. Since τ is proper, there is an infinite sequence of natural numbers $0 = i_0 < i_1 < \cdots$ such that, for all $j \geq 1$, there are non-empty finite sets of nodes a_j' not intersecting this complete branch, such that $\{n_{i_0}, \ldots, n_{i_j - 1}\} \cup a_j'$ is a complete finite branch. We can then define a mic tree $\ell : \mathbb{N}^\infty \to (\mathbb{T}_\tau)^{\mathrm{o}}$ by setting for $j \geq 0$:

$$\ell(j) = \{n_{i_j}, \ldots, n_{i_{j+1}-1}\} \qquad \text{and} \qquad \ell(j^+) = b_{j+1}. \qquad \square$$

Proposition 4.7.

1. *For any countable, well-founded tree τ with at least two elements:*

$$\|\mathbb{T}_\tau\| = \|\tau\| + 1.$$

2. *For any countable, non-well-founded, proper tree τ:*

$$\|\mathbb{T}_\tau\| = \omega_1.$$

Proof.

1. As there is a mic tree $\tau \to (\mathbb{T}_\tau)^{\mathrm{o}}$, we have $\|\mathbb{T}_\tau\| \geq \|\tau\| + 1$. So assume that we have a well-founded mic tree $\tau' \to (\mathbb{T}_\tau)^{\mathrm{o}}$. The singleton configurations form a basis B for $\mathbb{T}_\tau$, so by Lemma 4.4 there is a well-founded mic tree $\ell : \tau'' \to B$ with $\|\tau''\| \geq \|\tau'\|$. Identifying singletons with their elements, we can regard ℓ as a function from τ'' to τ. As ℓ is a mic tree, this function obeys the conditions of Lemma 4.6(2) and so there is a morphism $\tau'' \to \tau$ of rooted trees. Therefore $\|\tau''\| \leq \|\tau\|$ and we conclude that $\|\tau'\| \leq \|\tau\|$.
2. This is immediate, using the last two parts of Lemma 4.6. $\square$

It follows from Proposition 4.3.2 and Proposition 4.7, that there are finite and infinite domains of all possible coherence degrees.

Lemma 4.8.

1. *For $\alpha \leq \omega_1 + 1$, but not $\alpha = 0$ or $\alpha = 2$, we have:*

$$\|\mathbf{Dom}_\alpha\| = \alpha.$$

2. *For $\alpha \leq \omega$, but not $\alpha = 0$ or $\alpha = 2$, we have:*

$$\|\mathbf{Dom}^{\mathrm{f}}_\alpha\| = \alpha.$$

Proof. This is an immediate consequence of the fact that there are finite and infinite domains of all possible coherence degrees. □

Putting this together with Lemma 4.1 and Propositions 4.3 and 4.5 we obtain:

Theorem 4.9.

1. *For $\alpha \leq \omega_1 + 1$, but not $\alpha = 0$ or $\alpha = 2$, the category $\mathbf{Dom}_\alpha$ is a retract closed full sub-ccc of $\mathbf{Dom}$, and $\|\mathbf{Dom}_\alpha\| = \alpha$.*
2. *For $\alpha \leq \omega$, but not $\alpha = 0$ or $\alpha = 2$, the category $\mathbf{Dom}^{\mathrm{f}}_\alpha$ is a retract closed full sub-ccc of $\mathbf{Dom}$, and $\|\mathbf{Dom}^{\mathrm{f}}_\alpha\| = \alpha$.*

Therefore, no two of these categories, or the trivial categories, are the same.

5. Tree-theoretic domain analysis

A *coherent power* of a domain E is one of the form E^D where D is a 2-coherent domain. We write $D \lhd_{\mathrm{c}} E$ to mean that D is a retract of a coherent power of E. We have:

Lemma 5.1. *If $\ell : \tau \to D^{\mathrm{o}}$ is a mic tree then $\mathbb{T}_\tau \lhd_{\mathrm{c}} D$.*

Proof. Every non-empty countable tree τ can be regarded as a coherent domain, if we order it with its root as bottom and add a "point-at-infinity" above every infinite branch. With this understanding, we can define a retraction $\mathbb{T}_\tau \xrightarrow{e} D^\tau \xrightarrow{r} \mathbb{T}_\tau$ by:

$$e(u) = \bigvee_{n \in u} (n \Rightarrow \ell(n)) \qquad \text{and} \qquad r(f) = \{n \mid f(n) \geq \ell(n)\}.$$

□

Lemma 5.2. *There is a mic tree* $\mathbb{N}^\infty \to D^{\mathrm{o}}$ *in any domain* D *of coherence degree* ω_1.

Proof. There is a set of well-founded mic trees of uncountably many ordinals in D. There is a subset of these, also of uncountably many ordinals, sharing a common labelling $a_0 \in D^{\mathrm{o}}$ of the root. There is then a subset of the latter, again of uncountably many different ordinals, sharing a common labelling $a_1 \in D^{\mathrm{o}}$ of a child of the root, and so on. This way we obtain an infinite sequence $a_0, a_1, a_2, \ldots$ in D^{o} such that each initial segment $a_0, \ldots, a_n$ is the labelling of an incomplete branch of a well-founded mic tree (in fact, of uncountably many of them). By completing such a branch, we may find, taking a few sups if need be, an a_n^+ such that $\{a_0, \ldots, a_n, a_n^+\}$ is a mic set. A labelling $\ell : \mathbb{N}^\infty \to D^{\mathrm{o}}$ is now obtained by setting $\ell(n) = a_n$ and setting $\ell(n^+) = a_n^+$. $\square$

Theorem 5.3. *Let* D *be a domain. Then there are countably many mic trees* $\ell_k : \tau_k \to D^{\mathrm{o}}$ $(k \in K)$ *such that* $D \lhd \mathcal{P}(\omega) \times \prod_{k\in K} \mathbb{T}_{\tau_k}$.

Proof. Call a finite element of D *neutral* if it is consistent with every other element, i.e., it does not participate in any mic set, and *sharp* otherwise. Let a_i $(i \in I)$ be an enumeration without repetition of the neutral elements, and b_j $(j \in J)$ of the sharp elements. Here I and J are initial segments of ω, possibly empty.

Consider the forest whose nodes are the strictly increasing sequences $[j_0, \ldots, j_m]$, $j_0 < \cdots < j_m$, of elements of J, and whose child relation is that of "extension by one element." Define the *terminal nodes* to be those nodes $[j_0, \ldots, j_m]$ for which $\{b_{j_0}, \ldots, b_{j_m}\}$ is a mic set, and set

$$K =_{\mathrm{def}} \{k \in J \mid [k] \text{ has a descendant that is a terminal node}\}.$$

For $k \in K$, define τ_k to be the rooted tree whose nodes $[j_0, \ldots, j_m]$ have $j_0 = k$ and have a descendant that is a terminal node, and with child relation that inherited from the forest. Note that every finite branch of τ_k extends to a complete finite branch, and so, in particular, τ_k has at least one other node apart from $[k]$. The required labelling $\ell_k : \tau_k \to D^{\mathrm{o}}$ is defined by $\ell_k([j_0, \ldots, j_m]) = b_{j_m}$.

We construct a retract $D \xrightarrow{e} \mathcal{P}(\omega) \times \prod_{k\in K} \mathbb{T}_{\tau_k} \xrightarrow{r} D$. In one direction, set:

$$e(x) = \langle \{i \in I \mid a_i \le x\}, \langle \{[j_0, \ldots, j_m] \in \tau_k \mid b_{j_m} \le x\}\rangle_{k\in K}\rangle.$$

Note that every $\pi_n(\pi_2(e(x)))$ is consistent as if a complete finite branch were included and $[j_0, \dots, j_m]$ were its leaf, then $\{b_{j_0}, \dots, b_{j_m}\}$ would be a mic set bounded by x, which is impossible.

In the other direction set

$$r(u, y) = \bigvee\{a_i \mid i \in u\} \vee \bigvee\{b_j \mid j \in L_{u,y}\}$$

where

$$L_{u,y} = \{j \in J \mid \forall k \in K.\ \text{every } \tau_k \text{ node ending in } j \text{ is contained in } \pi_k(y)\}.$$

We need to show that $\{b_j \mid j \in L_{u,y}\}$ is consistent. If not, then it includes a mic set $\{b_{j_0}, \dots, b_{j_m}\}$ with $j_0 < \cdots < j_m$. But then all the nodes of the complete τ_{j_0} branch ending in $j_0, \dots, j_m$ are in $\pi_{j_0}(y)$, and that is a contradiction as $\pi_{j_0}(y)$ is a τ_{j_0} configuration.

Next, r is continuous as, for any $k \leq j$ there are only finitely many τ_k-nodes ending in j, and none at all if $k > j$.

It is easy to see that $r(e(x)) = x$ for any $x \in D$, using the fact that every finite element $\leq x$ is either neutral or sharp, and so appears either as some a_i or else as some b_j, when $j \in L_{u,y}$ where $(u, y) = e(x)$. □

Lemma 5.4. *Suppose that $D \lhd E$, that D has only finitely many elements, and that E is a bilimit of domains E_n. Then, for some n, $D \lhd E_n$.*

We write $D \lhd_{\text{fc}} E$ to mean that D is a retract of a finite coherent power of E.

Theorem 5.5. *The following are equivalent for any domains D and E, with E non-trivial:*

1. $\|D\| \leq \|E\|$,
2. $D \lhd_{\text{c}} E$.

Further, if D has only finitely many elements, then they are also equivalent to:

3. $D \lhd_{\text{fc}} E$.

Proof. That (2) implies (1) follows from Lemma 4.1 and Proposition 4.5. That (2) implies (3) follows from Lemma 5.4. So we are left with the implication from (1) to (2).

We know from Theorem 5.3 that there are countably many mic trees $\ell_k : \tau_k \to D^{\mathrm{o}}$ $(k \in K)$ such that $D \lhd \mathcal{P}(\omega) \times \prod_{k\in K} \mathbb{T}_{\tau_k}$. Because E is non-trivial we have $\mathcal{P}(\omega) \lhd_{\mathrm{c}} E$, and so it suffices to show that $\mathbb{T}_{\tau_k} \lhd_{\mathrm{c}} E$ for each $k \in K$. There are two cases.

If $\|E\| = \omega_1$ then by Lemma 5.2 there is a mic tree $\mathbb{N}^\infty \to E^{\mathrm{o}}$, and so $\mathbb{T}_{\tau_k} \lhd_{\mathrm{c}} \mathbb{T}_{\mathbb{N}^\infty} \lhd_{\mathrm{c}} E$ by Lemmas 4.6(3) and 5.1.

In the other case $\|E\| < \omega_1$, and so also $\|D\| < \omega_1$. Because τ_k is a mic tree in D we see that $\|\tau_k\| < \omega_1$, hence τ_k is well-founded by Proposition 4.7(2). By Theorem 5.3 there are countably many mic trees $\ell'_m : \tau'_m \to E^{\mathrm{o}}$ $(m \in M)$ such that $E \lhd \mathcal{P}(\omega) \times \prod_{m\in M} \mathbb{T}_{\tau'_m}$. By Lemmas 4.1 and 5.1 and Proposition 4.3(2) we obtain

$$\|E\| = \sup_{m\in M} \|\mathbb{T}_{\tau'_m}\|.$$

Therefore, for some $m \in M$ we get $\|\mathbb{T}_{\tau_k}\| \leq \|\mathbb{T}_{\tau'_m}\|$. By Proposition 4.7(1) we then obtain that $\|\tau_k\| \leq \|\tau'_m\|$, and so, by Lemmas 4.6(1) and 5.1, that $\mathbb{T}_{\tau_i} \lhd_{\mathrm{c}} \mathbb{T}_{\tau_m} \lhd_{\mathrm{c}} E$, as required. □

We can now analyse some possible cartesian closed categories of domains.

Theorem 5.6. *Let* **C** *be a full subcategory of* **Dom** *closed under retracts and function spaces.*

1. *If $\mathbb{N}_\perp$ is an object of* **C***, then* **C** *is one of the categories* $\mathbf{Dom}_\kappa$*, where* $2 < \kappa \leq \omega_1 + 1$.
2. *If all the objects of* **C** *have only finitely many points, then* **C** *is one of the categories* $\mathbf{Dom}^{\mathrm{f}}_\kappa$*, for* $\kappa \leq \omega$*, but not* $= 0, 2$*, or a trivial category.*

Proof.

1. As C contains $\mathbb{N}_\perp$ and is closed under retracts and function spaces, it contains the universal coherent domain, $\mathbb{T}^\omega_\perp$. It is therefore closed under $\lhd_{\mathrm{c}}$. So if it contains a domain of a given coherence degree, we see by Theorem 5.5 that it contains every domain of equal, or smaller coherence degree, and the conclusion follows.
2. Assume **C** is non-trivial. Suppose first that it contains a domain that is not a lattice. Then, as it is closed under retracts, it contains $\mathbb{T}_\perp$. So as it is also closed under function spaces, it contains $\mathbb{T}_\perp^{\mathbb{T}^n_\perp}$ and so $\mathbb{T}^n_\perp$, for every $n \geq 0$. As every finite coherent domain is a retract of some $\mathbb{T}^n_\perp$, it follows from Theorem 5.5 that if **C** contains a domain of a given coherence degree then it contains every finite domain of equal,

or smaller, coherence degree. It follows that $\mathbf{C}$ is one of the categories $\mathbf{Dom}^{\mathrm{f}}_{\kappa}$, for $2 < \kappa \leq \omega$.
Suppose instead that $\mathbf{C}$ contains only lattices. Then, as it is non-trivial, it contains the two-point lattice $\mathbb{O}$; so, for any $n \geq 0$, it contains $\mathbb{O}^{\mathbb{O}^n}$, and so, also, $\mathbb{O}^n$. As every lattice with only finitely many points is a retract of some $\mathbb{O}^n$, it follows that $\mathbf{C}$ is $\mathbf{Dom}^{\mathrm{f}}_1$. □

The condition in the first part of the theorem that $\mathbf{C}$ contains $\mathbb{N}_\perp$ can be replaced by the condition that $\mathbf{C}$ contains an object with infinitely many maximal elements, as we have:

Lemma 5.7. *The following are equivalent for any domain D:*

1. $\mathbb{N}_\perp$ *is a retract of D.*
2. *D has infinitely many maximal elements.*

Proof. First suppose we have a retraction

$$\mathbb{N}_\perp \xrightarrow{e} D \xrightarrow{r} \mathbb{N}_\perp$$

Then $e(\mathbb{N})$ is a countably infinite set of mutually inconsistent elements of D. Taking a maximal element greater than each of them, we obtain a countably infinite set of (mutually inconsistent) maximal elements.

Conversely, suppose that we have a countably infinite set $X \subseteq D$ of maximal elements. They are necessarily mutually inconsistent. We first construct two sequences, one, y_n $(n \geq 0)$, a sequence of mutually inconsistent elements, the other, b_n $(n \geq 0)$, a sequence of finite elements, such that $b_n \leq y_n$, and b_n is inconsistent with y_m for every $m \geq n$.

Let $x_0, x_1 \in X$ be any two elements. There are inconsistent finite elements $a_0 \leq x_0$ and $a_1 \leq x_1$. Any $x \in X$ must be inconsistent either with a_0 or with a_1, otherwise we would have $a_0 \leq x$ and $a_1 \leq x$ by the maximality of x. If there are infinitely many elements of X inconsistent with a_0 we set $y_0 = x_0$, $b_0 = a_0$, and $X' = \{x \in X \mid x \text{ inconsistent with } a_0\}$. Otherwise there are infinitely many elements of X which are inconsistent with a_1, in which case we set $y_0 = x_1$, $b_0 = a_1$ and $X' = \{x \in X \mid x \text{ is inconsistent with } a_1\}$. We now proceed in the same way with X' in place of X to construct b_1 and y_1, and so on.

For any $n \geq 0$, since y_n is inconsistent with b_m for $m < n$, and $y_n \geq b_n$, we can obtain a finite $c_n \geq b_n$ which is inconsistent with b_m for every $m < n$.

But then the c_n form an infinite set of mutually inconsistent finite elements of D, and so we can define the required retraction

$$\mathbb{N}_\perp \xrightarrow{e} D \xrightarrow{r} \mathbb{N}_\perp$$

by setting $e(\perp) = \perp$ and $e(n) = c_n$, and $r(x) = n$ if $x \geq c_n$ for some n, and $r(x) = \perp$, otherwise. □

6. Domains with finitely many maximal elements

For any domain D we define $\max(D)$ to be the (necessarily non-empty) subset of its maximal elements, and $\max_\wedge(D)$ to be the sub-partial order of D of all meets of maximal elements of D. As this partial order is closed under non-empty meets—they are inherited from D—it is consistently-complete and has a least element.

There is a natural monotone closure operation c_D on D with fixed points $\max_\wedge(D)$, defined by:

$$c_D(x) = \bigwedge\{m \in \max(D) \mid x \leq m\}.$$

It has a pleasant characterisation as the greatest increasing monotone function over D.

In the case that D has finitely many maximal elements, $\max_\wedge(D)$ is a finite domain. Further, c_D is then continuous as, for every $x \in D$, there is a finite $a \leq x$ such that if $x \not\leq m \in \max(D)$ then $a \not\leq m$, and so then, for all $m \in \max(D)$, we have $x \leq m$ if, and only if, $a \leq m$, and so $c_D(a) = c_D(x)$.

Lemma 6.1. *Suppose D and E are domains with finitely many maximal elements. Then so are $D \times E$ and $D \Rightarrow E$.*

Proof. This is obvious for $D \times E$. For $D \Rightarrow E$, suppose that $f : D \to E$. Then we have that $f \leq c_D \circ f \circ c_E$. But functions of this latter form are in 1-1 correspondence with the monotone functions between the finite partial orders $\max_\wedge(D)$ and $\max_\wedge(E)$, and there are only finitely many of those. □

This lemma immediately yields:

Proposition 6.2. *For $\alpha \leq \omega$, but not $\alpha = 0$ or $\alpha = 2$, the category $\mathbf{Dom}^{\mathrm{fm}}_\alpha$ of domains D with finitely many maximal elements and $\|D\| < \alpha$ is a full sub-cartesian closed category of* **Dom**.

Lemma 6.3. *Let D be a domain with finitely many maximal elements. Then*

$$D \lhd \max_\wedge(D) \times \mathcal{P}(\omega)$$

and, further, $\|D\| = \|\max_\wedge(D)\|$.

Proof. For the first part, it suffices to define a retraction

$$D \xrightarrow{\quad e \quad} \max_\wedge(D) \times \prod_{u\in\max_\wedge(D)} D_u \xrightarrow{\quad r \quad} D$$

where D_u is the lattice of all elements of D below u. This is so because the product of D_u's is a retract of $\mathcal{P}(\omega)$ by the universality of $\mathcal{P}(\omega)$. We define e by:

$$e(x) = \langle c_D(x), \langle u \wedge x\rangle_{u\in\max_\wedge(D)}\rangle.$$

Clearly e is well-defined and continuous. We define r by:

$$r(u, \langle x_v\rangle_{v\in\max_\wedge(D)}) = \bigvee_{v\leq u} x_v.$$

To see that the supremum on the right-hand side exists, note that all x_v such that $v \leq u$ are contained in the lattice D_u. To see that r is continuous, it suffices to note that it is monotone in its first argument.

Finally we need to show that r is left inverse to e. We have:

$$\begin{aligned} r(e(x)) &= r(c_D(x), \langle v \wedge x\rangle_{v\in\max_\wedge(D)}) \\ &= r((\bigwedge\{m \in \max(D) \mid x \leq m\}), \langle v \wedge x\rangle_{v\in\max_\wedge(D)}) \\ &= \bigvee\{v \wedge x \mid v \leq \bigwedge\{m \in \max(D) \mid x \leq m\}\}. \end{aligned}$$

So we evidently have that $r(e(x)) \leq x$. Taking v to be the necessarily non-empty infimum $v = \bigwedge\{m \in \max(D) \mid x \leq m\}$, we have $v \wedge x = x$ and so we also have that $r(e(x)) \geq x$.

For the coherence degree calculation, we already have the inequality $\|\max_\wedge(D)\| \leq \|D\|$ as we know that $\max_\wedge(D) \lhd D$. Further, from the retraction $D \lhd \max_\wedge(D) \times \mathcal{P}(\omega)$ we have the converse inequality as then:

$$\|D\| \leq \|\max_\wedge(D) \times \mathcal{P}(\omega)\| = \max(\|\max_\wedge(D)\|, \|\mathcal{P}(\omega)\|) = \|\max_\wedge(D)\|.$$

□

Lemma 6.4. *If D is an infinite domain then $\mathcal{P}(\omega)$ is a retract of $D \Rightarrow D$.*

Proof. There is an countably infinite set S of distinct finite elements in D. The step functions $a \Rightarrow a$ ($a \in S$) form an antichain of finite elements in $D \Rightarrow D$. $\square$

Theorem 6.5. *Let* $\mathbf{C}$ *be a full subcategory of* $\mathbf{Dom}$*, closed under retracts and function spaces, all of whose objects have finitely many maximal elements, and containing an infinite object. Then* $\mathbf{C}$ *is one of the categories* $\mathbf{Dom}^{\mathrm{fm}}_{\alpha}$*, where* $\alpha \leq \omega$*, but not* $\alpha = 0$ *or* $\alpha = 2$*.*

Proof. By Lemma 6.4, $\mathbf{C}$ contains $\mathcal{P}(\omega)$. In the case that all its objects are lattices it must therefore be $\mathbf{Dom}^{\mathrm{fm}}_{1}$ the full sub-ccc of $\mathbf{Dom}$ of all lattices.

Otherwise it contains an object with at least two maximal elements, and so it contains $\mathbb{T}_{\perp}$, which has coherence degree 2. We know from Lemma 6.3 that every object D in $\mathbf{C}$ has finite coherence degree, and, indeed, that $\|D\| = \|\max_{\wedge}(D)\|$. So $\mathbf{C}$ must be a subcategory of one of the $\mathbf{Dom}^{\mathrm{fm}}_{\alpha}$, with $2 < \alpha \leq \omega$.

Let D be an object in $\mathbf{C}$ of coherence degree $n > 1$. Then $\mathbb{T}_n \lhd_{\mathrm{fc}} \max_{\wedge}(D)$, by Theorem 5.5(3), and so $\mathbb{T}_n$ is in $\mathbf{C}$, as $\mathbb{T}_{\perp}$ is in $\mathbf{C}$. So $\mathbb{T}_{\perp}$ and $\mathcal{P}(\omega)$ are in $\mathbf{C}$ and so is $\mathbb{T}_n \times \mathcal{P}(\omega)$, as we have the following chain of retractions:

$$\mathbb{T}_n \times \mathcal{P}(\omega) \lhd \mathbb{T}_n \times \mathbb{O}^{\mathcal{P}(\omega)} \lhd \mathbb{T}_n \times \mathbb{T}_n^{\mathcal{P}(\omega)} \lhd \mathbb{T}_n^{\mathbb{O}_{\perp} + \mathcal{P}(\omega)_{\perp}} \lhd \mathbb{T}_n^{\mathcal{P}(\omega) + \mathcal{P}(\omega)} \lhd \mathbb{T}_n^{\mathbb{T}_{\perp} \times \mathcal{P}(\omega) \times \mathcal{P}(\omega)} \cong (\mathbb{T}_n^{\mathbb{T}_{\perp}})^{\mathcal{P}(\omega)}.$$

Now, if E is any domain of coherence degree $\leq n$ with finitely many maximal elements, we have, using Theorem 5.5(3) and Lemma 6.3, that:

$$E \lhd \max_{\wedge}(E) \times \mathcal{P}(\omega) \lhd_{\mathrm{fc}} \mathbb{T}_n \times \mathcal{P}(\omega).$$

So E is in $\mathbf{C}$, and the conclusion follows. $\square$

7. The classification theorem and its consequences

Combining Theorems 4.9, 5.6, and 6.5, and Proposition 6.2, we obtain our classification theorem.

Theorem 7.1 (Classification Theorem)**.** *The non-trivial retract closed full sub-ccc's of* $\mathbf{Dom}$ *are given by the following distinct families of different such categories:*

1. $\mathbf{Dom}_{\alpha}$*, for* $2 < \alpha \leq \omega_1 + 1$*. These are the categories that contain an object with infinitely many maximal elements.*

2. $\mathbf{Dom}_{\alpha}^{\mathrm{fm}}$*, for* $2 < \alpha \leq \omega$*. These are the categories all of whose objects have only finitely many maximal elements, and that contain an object with more than one maximal element and an object with infinitely many elements.*
3. $\mathbf{Dom}_1 = \mathbf{Dom}_1^{\mathrm{fm}}$*, the category of lattices. This is the unique category all of whose objects have one maximal element and that contains an object with infinitely many elements.*
4. $\mathbf{Dom}_{\alpha}^{\mathrm{f}}$*, for* $2 < \alpha \leq \omega$*. These are the categories all of whose objects have only finitely many elements, but contain an object with more than one maximal element.*
5. $\mathbf{Dom}_1^{\mathrm{f}}$*, the category of finite lattices. This is the unique category all of whose objects are finite and have one maximal element.*

There is a difference between cartesian closed subcategories and sub-cartesian closed categories. Specifically, while we have classified the non-trivial retract closed full sub-ccc's of **Dom**, it is conceivable that there are other non-trivial retract closed full subcategories of **Dom** which are cartesian closed, but are not sub-ccc's of **Dom**, that is, they do not inherit their cartesian closed structure from **Dom**. Fortunately this is not the case, as all full subcategories of **Dom** which are ccc's are sub-ccc's of **Dom**; indeed this holds more generally, for the category of pointed dcpos, as detailed in [10].

We pause to consider the closure of these categories under various constructions. From the remarks made above, we see that all of them are closed under lifting and binary products, and all of them except for the two categories of lattices are closed under separated sums. We also see that the ones closed under bilimits are **Dom** and the $\mathbf{Dom}_n$, for $n \geq 0$.

We also consider the standard three powerdomains [9]. Every domain D has a lower (Hoare) powerdomain $\mathcal{P}_{\mathrm{L}}(D)$, an upper (Smyth) powerdomain $\mathcal{P}_{\mathrm{U}}(D)$, and a convex (Plotkin) powerdomain $\mathcal{P}_{\mathrm{C}}(D)$. As $\mathcal{P}_{\mathrm{L}}(D)$ is always a lattice and $\mathcal{P}_{\mathrm{C}}(D)$ need not be a domain, e.g., if D is $\mathbb{T}_{\perp} \times \mathbb{T}_{\perp}$, we only consider the upper powerdomain further.

It turns out that $\mathcal{P}_{\mathrm{U}}(\mathbb{T}_{\perp}^{\omega})$ is universal in **Dom**. As mentioned in the Introduction, the domain of all consistent propositional theories over countably many letters is universal in **Dom**. But this domain is isomorphic to the partial order (and therefore domain) $\mathcal{C}_{\neq\emptyset}(\mathbb{T}^{\omega})$ of all non-empty compact subsets of $\mathbb{T}^{\omega}$, ordered by reverse inclusion and there is a retraction pair:

$$\mathcal{C}_{\neq\emptyset}(\mathbb{T}^{\omega}) \xrightarrow{\iota} \mathcal{P}_{\mathrm{U}}(\mathbb{T}_{\perp}^{\omega}) \xrightarrow{r} \mathcal{C}_{\neq\emptyset}(\mathbb{T}^{\omega})$$

where $r(x) =_{\mathrm{def}} x \cap \mathbb{T}^\omega$. So $\mathcal{P}_{\mathrm{U}}(\mathbb{T}_\perp^\omega)$ is indeed universal in **Dom**. It follows from Lemma 5.4 that every finite domain is a retract of some $\mathcal{P}_{\mathrm{U}}(\mathbb{T}_\perp^n)$, and it is not hard to see that if D has finitely many maximal elements, so does $\mathcal{P}_{\mathrm{U}}(D)$. As $\mathcal{P}_{\mathrm{U}}(D)$ is a lattice if D is, we see that the categories closed under the upper powerdomain are: **Dom**, $\mathbf{Dom}_\omega^{\mathrm{fm}}$, $\mathbf{Dom}_1$, $\mathbf{Dom}_\omega^{\mathrm{f}}$, and $\mathbf{Dom}_1^{\mathrm{f}}$.

We now consider universal domains and models of the $\lambda\beta$-calculus. Using the classification theorem, and some of our other results, we can classify all the full sub-ccc's of **Dom** that have a universal object and we can characterise all models of the $\lambda\beta$-calculus in **Dom**, up to a retraction.

Lemma 7.2. *For a non-trivial domain D:*

$$D^D \lhd D \textit{ implies } D \times D \lhd D \textit{ implies } D^\omega \lhd D \textit{ implies } \mathcal{P}(\omega) \lhd D.$$

Theorem 7.3. *The full sub-ccc's of* **Dom** *that contain a non-trivial universal object are the* $\mathbf{Dom}_{\alpha+1}$*, for* $\alpha \leq \omega_1$*, but not* $\alpha = 1$*.*

Proof. Let $\mathbf{C}$ be a full sub-ccc of **Dom** containing a non-trivial universal object U. Then U is a non-trivial model of the $\lambda\beta$-calculus. There are two cases. The first is where U is a lattice, and so $\mathbf{C}$ is a category of lattices. As U is a non-trivial model of the $\lambda\beta$-calculus, by Lemma 7.2 we have $\mathcal{P}(\omega) \lhd U$ and so $\mathbf{C}$ must be $\mathbf{Dom}_1$, the category of ω-algebraic lattices.

The second is where U is not a lattice, and so $\mathbb{T}_\perp \lhd U$, and we find that $\mathbf{C}$ contains $\mathbb{N}_\perp$, as $\mathbb{N}_\perp \lhd \mathbb{T}_\perp^\omega \lhd U^\omega \lhd U$, using Lemma 7.2 for the last retraction. So by the classification theorem we have that $\mathbf{C}$ must be some $\mathbf{Dom}_\alpha$. Since U is universal, for any D in $\mathbf{C}$ we have $D \lhd U$ and so $\|D\| \leq \|U\|$. It follows that $\mathbf{C}$ must be $\mathbf{Dom}_{\|U\|+1}$.

So we know that the $\mathbf{Dom}_{\alpha+1}$ are the only possible cases. However all these categories possess a universal object:

1. $\mathcal{P}(\omega)$ is universal for $\mathbf{Dom}_1$,
2. for $\alpha > 1$, we first note that, by the remark after Proposition 4.7, there is a domain D of coherence degree α. We then see, using Proposition 4.5 and Theorem 5.5, that $\mathbb{T}_\perp^\omega \Rightarrow D$ is universal for $\mathbf{Dom}_{\alpha+1}$,

and this concludes the proof. □

We next see that the non-trivial models of the $\lambda\beta$-calculus can be classified in terms of their coherence degrees, and, further, that whether or not one is a retract of another depends only on these degrees.

Theorem 7.4.

1. *There is a non-trivial model of the $\lambda\beta$-calculus of every coherence degree.*
2. *For any non-trivial models D and E of the $\lambda\beta$-calculus, $D \lhd E$ holds if, and only if, $\|D\| \leq \|E\|$.*

Proof. The first part is immediate from Theorem 7.3 as every universal object is a model of the $\lambda\beta$-calculus.

For the second part, it follows from Lemma 7.2 that a model E of the $\lambda\beta$-calculus is universal in the full sub-ccc of **Dom** generated by it. As E is non-trivial, that category must therefore be some $\mathbf{Dom}_{\alpha+1}$. It then follows from Lemma 4.1 that $\alpha = \|E\|$, and so, as $\|D\| \leq \|E\|$, that $D \lhd E$. □

Finally, returning to the classification theorem, we consider a natural categorical question concerning the relations between the different sub-ccc's arising in the theorem. Say that a functor

$$F : \mathbf{B} \to \mathbf{C}$$

is an *(order) embedding* if, and only if, it is full and faithful (and locally monotone). There are order embeddings within each of the families of sub-ccc's, namely the evident inclusions, and between them for $2 < \alpha < \omega$, we have the other evident ones:

$$\begin{array}{ccccc}
\mathbf{Dom}_1^{\mathrm{f}} & \longrightarrow & \mathbf{Dom}_\alpha^{\mathrm{f}} & & \\
\downarrow & & \downarrow & & \\
\mathbf{Dom}_1^{\mathrm{fm}} & \longrightarrow & \mathbf{Dom}_\alpha^{\mathrm{fm}} & \longrightarrow & \mathbf{Dom}_\alpha
\end{array}$$

and their compositions with the inclusions within the families. The question is whether there are other such order embeddings. We will see that, up to natural isomorphism, these are all there are.

Suppose that $F : \mathbf{B} \to \mathbf{C}$ is an embedding where $\mathbf{B}$ and $\mathbf{C}$ are full subcategories of **Dom**. Suppose too that $\mathbf{C}$ contains the one-point domain $\mathbb{1}$. Then as $\mathbf{B}(\mathbb{1}, \mathbb{1})$ and $\mathbf{C}(F(\mathbb{1}), F(\mathbb{1}))$ are in bijection, the latter is a singleton, and so $F(\mathbb{1})$ must be (isomorphic to) $\mathbb{1}$. Then, for any $D \in \mathbf{B}$, the bijection $F : \mathbf{B}(\mathbb{1}, D) \cong \mathbf{C}(F(\mathbb{1}), F(D))$ induces a bijection $F_D : D \cong F(D)$, natural in D.

Lemma 7.5. *Let $F : \mathbf{B} \to \mathbf{C}$ be an embedding of full subcategories of* **Dom**. *If* $\mathbf{B}$ *contains Sierpinski space $\mathbb{O}$ then either every F_D is an isomorphism of partial orders (when F is locally an isomorphism of partial orders), or else every F_D is an anti-isomorphism of partial orders (when F is locally an anti-isomorphism of partial orders). If, further,* $\mathbf{B}$ *contains $\mathbb{T}_\perp$ or $\mathcal{P}(\omega)$, then F_D is always an isomorphism of partial orders.*

Proof. As $F_{\mathbb{O}} : \mathbb{O} \cong F(\mathbb{O})$ is a bijection, $F(\mathbb{O})$ must have two elements, as $\mathbb{O}$ does; so $F(\mathbb{O})$ must itself be (isomorphic to) Sierpinski space, as there is just one domain with two elements. There are two cases according to whether $F_{\mathbb{O}} : \mathbb{O} \cong F(\mathbb{O})$ is an isomorphism or an anti-isomorphism of partial orders.

In the first case, choose D to show that F_D is an iso. For monotonicity, suppose that $x \leq y \in D$. Then there is a monotone $f : \mathbb{O} \to D$ such that $f(\perp) = x$ and $f(\top) = y$, and we calculate:

$$\begin{aligned} F_D(x) &= F_D(f(\perp)) & \\ &= F(f)(F_{\mathbb{O}}(\perp) & \text{(by naturality)} \\ &\leq F(f)(F_{\mathbb{O}}(\top)) & (F_{\mathbb{O}} \text{ is monotone}) \\ &= F_D(y) & \end{aligned}$$

We see that ${F_D}^{-1}$ is monotone similarly, as ${F_{\mathbb{O}}}^{-1}$ is. That F is locally an isomorphism of partial orders follows using naturality and the fact that F_D is such an isomorphism.

In the second case one proceeds similarly to the first case, but now using that $F_{\mathbb{O}}$ and ${F_{\mathbb{O}}}^{-1}$ are both anti-monotone.

Suppose now that $\mathbf{B}$ contains $\mathbb{T}_\perp$. Then $F(\mathbb{T}_\perp)$ has three elements and a non-identity isomorphism, as $\mathbb{T}_\perp$ does, and so must itself be $\mathbb{T}_\perp$ (up to isomorphism). As $\mathbb{T}_\perp$ has no anti-isomorphism of partial orders, the first case above, where $F_{\mathbb{O}}$ is an isomorphism of partial orders, holds.

Suppose instead that $\mathbf{B}$ contains $\mathcal{P}(\omega)$. Suppose too, for the sake of contradiction, that $F_{\mathbb{O}} : \mathbb{O} \cong F(\mathbb{O})$ is an anti-isomorphism of partial orders. Then $F_{\mathcal{P}(\omega)} : \mathcal{P}(\omega) \cong F(\mathcal{P}(\omega))$ is also an anti-isomorphism of partial orders; one can then show, using the naturality of $F_{\mathcal{P}(\omega)}$ at $\mathcal{P}(\omega)$ that any continuous $f : \mathcal{P}(\omega) \to \mathcal{P}(\omega)$ is also anti-continuous, meaning that it preserves meets of decreasing ω-chains. As this is false, we have the required contradiction, and the proof concludes. □

Theorem 7.6. *The order embeddings between retract closed full sub-ccc's of* **Dom** *are those enumerated above, up to a natural isomorphism.*

Proof. Let $F : \mathbf{B} \to \mathbf{C}$ be such an embedding. Using Lemma 7.5 we see that $F_D : D \cong F(D)$ is an isomorphism of partial orders, natural in D. Our classification of sub-ccc's is entirely in terms of order invariants; using those it is straightforward to check that the only possible such order-embeddings F are in the cases enumerated above. In those cases, the naturality of F_D provides a natural isomorphism with the corresponding inclusion functor. □

Using Lemma 7.5 one can further show that the embeddings between retract closed full sub-ccc's of **Dom** are those enumerated above, up to a natural isomorphism, except that one can also compose with the functor

$$(-)^{\mathrm{op}} : \mathbf{Dom}_1^{\mathrm{f}} \to \mathbf{Dom}_1^{\mathrm{f}}$$

which sends every finite lattice to its opposite. This functor is evidently an equivalence of categories and locally an anti-isomorphism of partial orders.

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