LAMBDA-DEFINABILITY IN THE FULL TYPE HIERARCHY

Gordon D. Plotkin

Department of Computer Science,
University of Edinburgh,
Edinburgh EH9 3JZ,
Scotland.

Dedicated to H.B. Curry on the occasion of his 80th Birthday

The completeness theorem for the first-order predicate calculus characterises provability by a semantic means which demonstrates the logical nature (validity) of theorems. Our aim here is to attempt something similar for definability in the full type hierarchy by terms of the typed λ-calculus. The obvious first try is invariance under permutations, but this fails. A first extension using hereditarily defined relations characterises λ-definability up to type level 2 (theorem 1); we do not know what happens at higher types. A second extension using a generalised kind of relation succeeds in characterising λ-definability at all types when the ground set is infinite (theorem 2). Along the way (theorem 3) we obtain a completeness theorem for βη-conversion. It would be interesting to investigate relative definability, to look at other models of the typed λ-calculus and to consider the untyped λ-calculus. Since the present work was completed, Statman has obtained other interesting results in the same area; see, especially, [Sta] where, among many other things, a stronger version of our theorem 3 is proved.

For information on the typed λ-calculus, consult [Hin]; here we briefly consider the necessary background material. The set of *types* is the least set containing ι and containing $(\sigma \to \tau)$ if it contains σ and τ ; $(\sigma_1,\ldots,\sigma_m,\tau)$ abbreviates $(\sigma_1 \to (\ldots.(\sigma_m \to \tau)\ldots))$ (for $m \geq 0$). The *rank* (= *order* = *level*) of a type is defined by induction on types: $r(\iota) = 0$ and $r(\sigma \to \tau) = \max(r(\tau), r(\sigma) + 1)$. We assume a denumerable set, Var_σ, of *variables* x^σ of each type σ, and put $Var = \bigcup_\sigma Var_\sigma$ (and often omit the superscripts on variables). The set of *terms* of the typed λ-calculus (as considered here) is the least set such that:

1. Each variable x^σ is a term of type σ.

2. If M, N are terms of types $(\sigma \to \tau),\sigma$ respectively then (MN) is a term of type τ (called a *combination*).

3. If M is a term of type τ then $(\lambda x^\sigma.M)$ is a term of type $(\sigma \to \tau)$ (called an *abstraction*).

The set of free variables of a term M is denoted by FV(M); we do not distinguish α-equivalent terms and often drop brackets (understood as associated to the left); we use $M =_{\beta,\eta} N$ to mean M and N are β,η-interconvertible.

We consider a fixed non-empty ground set D throughout and the *full type hierarchy* $\{D_\sigma\}$ is defined over D by: $D_\iota = D$ and $D_{\sigma \to \tau} = (D_\sigma \to D_\tau)$ the set of all functions from D_σ to D_τ. The set of *environments* is $Env = \{\rho : Var \to \cup D_\sigma \mid \forall x^\sigma . \rho x^\sigma \in D_\sigma\}$; $\rho[d/x^\sigma]$, where d is in D^σ has value ρy when $y \neq x$ and d if $y = x$. The *valuation* $[\![M]\!](\rho)$ of a term is defined by induction on terms:

1. $[\![x^\sigma]\!](\rho) = \rho x^\sigma$
2. $[\![MN]\!](\rho) = [\![M]\!](\rho)([\![N]\!](\rho))$
3. $[\![\lambda x^\sigma.M]\!](\rho)(d) = [\![M]\!](\rho[d/x^\sigma])$

If M has type σ then $[\![M]\!](\rho)$ is in D_σ. The value of $[\![M]\!](\rho)$ depends only on what values ρ assigns to the free variables of M; if M is closed we often omit reference to ρ. If $M =_{\beta,\eta} N$ then for all ρ, $[\![M]\!](\rho) = [\![N]\!](\rho)$. An element d in $\cup D_\sigma$ is λ-*definable* if there is a closed term M (one without free variables) such that $d = [\![M]\!]$; it is λ-*definable* from $X \subseteq \bigcup_\sigma D_\sigma$

if there is a closed term M and elements $d_1,\ldots,d_n$ of X so that $d = [[M]](d_1)\ldots(d_n)$.

Because of the "logical" nature of the λ-definable elements, they should be invariant under permutations of D. Precisely, let $\pi: D \to D$ be a permutation and define $\pi_\sigma: D_\sigma \to D_\sigma$ by induction on types putting $\pi_\iota = \pi$ and for f in $D_{(\sigma \to \tau)}$,

$$\pi_{(\sigma \to \tau)}(f) = \pi_\tau \circ f \circ \pi_\sigma^{-1}.$$

Then we say an element d of D_σ is *invariant* if $\pi_\sigma(d) = d$ for all such permutations, π. It is easily shown [Läu] that all λ-definable elements are invariant but, as remarked by Läuchli, there are uncountably many invariant elements when D is infinite (even in $D_{((\iota \to \iota),\iota,\iota)}$). For example taking $\circ =_{def} (\iota,\iota,\iota)$ as a truthvalue type let tt and ff be, respectively, the terms λx. λy.x and λx.λy.y. The *ground equality* EQ: $D_{(\iota,\iota,\circ)}$ is invariant but not λ-definable if $|D| > 1$, where EQ(d)(d') is [[tt]] if d = d' and [[ff]] otherwise.

M. Gordon proposed, as a possible remedy, that relations rather than just permutations should be extended to higher types; this idea was also used by Howard for defining his hereditarily majorisable functionals [Tro]. Specifically suppose $R \subseteq D^\kappa$ (κ any ordinal) and define $R_\sigma \subseteq D_\sigma^\kappa$ by induction on types putting $R_\iota = R$ and for f in $D^\kappa_{(\sigma \to \tau)}$,

$$R_{(\sigma \to \tau)}(f) \equiv \forall d \in D_\sigma^\kappa .(R_\sigma(d) \supset R_\tau(f(d))).$$

Here f(d) is $\langle f_\lambda(d_\lambda)\rangle_{\lambda<\kappa}$. Then an element d of D_σ *satisfies* R if $R_\sigma(\langle d\rangle_{\lambda<\kappa})$ holds.

PROPOSITION 1. Suppose $R \subseteq D^\kappa$. Then every λ-definable element satisfies R and every element λ-definable from a set of elements satisfying R itself satisfies R.

Proof. We demonstrate by induction on terms M that:

$$\forall \rho \in Env^\kappa.(\forall x^\tau \in FV(M).R_\tau(\rho(x^\tau))) \supset R_\sigma([[M]](\rho))$$

where σ is the type of M. Here $\rho(x^\tau)$ is $\langle \rho_\lambda(x^\tau)\rangle_{\lambda<\kappa}$ and $[[M]](\rho)$

is $\langle [\![M]\!](\rho_\lambda) \rangle_{\lambda<\kappa}$.

In case M is a variable, x^τ, $[\![M]\!](\rho) = \rho(x^\tau)$ which satisfies R_σ by assumption. In case M is a combination $(M_1 M_2)$, $[\![M_1 M_2]\!](\rho) = [\![M_1]\!](\rho)([\![M_2]\!](\rho))$ and this satisfies R_σ by the definition of $R_{(\sigma \to \tau)}$ using the induction hypothesis for M_1 and M_2. In case M is an abstraction $(\lambda x^\sigma . M_1)$ let d satisfy R_σ. Then $[\![\lambda x^\sigma . M_1]\!](\rho)(d) = [\![M_1]\!](\rho')$, where $\rho' = \langle \rho_\lambda [d_\lambda / x^\sigma] \rangle$, and we can apply the induction hypothesis to M_1, concluding the inductive proof. The first part of the proposition then follows applying the above to closed M. The second part is then immediate. ☒

As an example of non-definability suppose 0,1 are distinct elements of D and take $R = \{\langle 0,0\rangle, \langle 0,1\rangle, \langle 1,0\rangle\}$. Then $R_o([\![tt]\!],[\![ff]\!])$ does not hold as $R_\iota(1,0)$ and $R_\iota(0,1)$ but not $R_\iota([\![tt]\!](1)(0),[\![ff]\!](0)(1))$; so EQ does not satisfy R as $R_\iota(0,0)$ and $R_\iota(0,1)$ but not $R_\iota(EQ(0)(0),EQ(0)(1))$. This shows EQ is not λ-definable when $|D| > 1$.

As an example of non-relative definability consider the "universal quantification" functional, $F: D_{(\iota \to o)} \to D_o$ where:

$$F(f) = \begin{cases} [\![tt]\!] & (\text{if } f(d) = [\![tt]\!] \text{ for all } d \text{ in } D) \\ [\![ff]\!] & (\text{otherwise}) \end{cases}$$

Now F is invariant but not λ-definable from EQ if $|D| > 2$. For let $R = \{\langle 0,0\rangle, \langle 1,1\rangle\}$ where $0 \neq 1$. Then EQ satisfies R but with $f = [\![\lambda x^\iota . tt]\!]$ and $g(d) = [\![tt]\!]$ if d is 0 or 1 and $g(d) = [\![ff]\!]$ otherwise we have $R_{(\iota \to o)}(f,g)$ but not $R_o(Ff,Fg)$.

(Incidentally F is λ-definable from EQ if $|D| \leq 2$.)

THEOREM 1. Suppose $r(\sigma) \leq 2$. Then if D is infinite and $f \in D_\sigma$ satisfies every $R \subseteq D^2$, f is λ-definable.

Proof. We just consider two cases to give the idea without too much detail. The first case is $\sigma = (\iota,\iota,\iota)$. Let d,e,0,1 be elements of D with $0 \neq 1$ and put $R = \{\langle d,0\rangle, \langle e,1\rangle\}$. Then

R (fde,f01) and so for all d,e in D either fde = d and f01 = 0 or else fde = e and f01 = 1. So either f01 = 0 or f01 = 1 ; in the first case f = [[tt]] , in the second case f = [[ff]].

The second case is $\sigma = ((\iota,\iota),\iota,\iota)$. We can suppose $\omega \subseteq D_\iota$ and choose s: $D_\iota \to D_\iota$ to act as the successor on the integers. For g in $D_{(\iota \to \iota)}$ and d in D let $R = \{<g^n d, s^n 0> \mid n \geq 0\}$. As $R_\iota(d,0)$ and $R_{(\iota \to \iota)}(g,s)$ and f satisfies R there is an n such that fgd = $g^n d$ and fs0 = $s^n 0$. As the $s^n(0)$ are all different we see that for some n,f = $[[\lambda x.\ \lambda y.x^n(y)]]$, in an obvious notation. ⊠

We believe this theorem holds without the restriction on D; we know nothing about what happens at higher types.

To proceed further we try to interpret the implication sign in the definition of the $R_{\sigma \to \tau}$ in an intuitionistic way, hoping thereby to make any f satisfying $R_{\sigma \to \tau}$ more likely to be constructive and therefore λ-definable. In order to do this we use Kripke's ideas [Kri] on the interpretation of intuitionistic logic.

Specifically suppose $<W,\leq>$ is a quasiorder (i.e. a reflexive transitive relation), where we interpret W as a set of worlds and $\leqslant$ as an alternativeness relation over W and suppose too that $R \subseteq D^\kappa \times W$ is a relation such that for all d in D^κ, w in W:

$$R(d,w) \supset \forall w' \geq w.\ R(d,w')$$

We call such an R an I-*relation* and now define $R_\sigma \subseteq D^\kappa_\sigma \times W$ by putting $R_\iota = R$ and for any f in $D^\kappa_{\sigma \to \tau}$ and w in W:

$$R_{\sigma \to \tau}(f,w) \equiv \forall w' \geq w.\ \forall d \in D^\kappa_\sigma.(R_\sigma(d,w') \supset R_\sigma(fd,w')).$$

Then an element d of D_σ I-*satisfies* R if $R(<d>_{\lambda<\kappa},w)$ holds for all w in W. It is clear (taking W to be a singleton) how this generalises the previous idea of satisfaction.

LEMMA 1. With R_σ as above and for any d in D^κ_σ, w in W:

$$R(d,w) \supset \forall\ w' \geq w.R(d,w')$$

Proof. The proof is an easy induction on σ, using the transitivity of $\leq$. ⊠

PROPOSITION 2 Suppose $R \subseteq D^{\kappa} \times W$ is an I-relation. Then every λ-definable element I-satisfies R and every element λ-definable from a set of elements I-satisfying R itself I-satisfies R.

Proof. We demonstrate by induction on terms M that:

$$\forall w \in W. \forall \rho \in Env^{\kappa}.((\forall x^{\tau} \in FV(M). R_{\tau}(\rho\, x^{\tau}, w)) \supset R_{\sigma}([\![M]\!](\rho), w))$$

where σ is the type of M. The proposition follows.

The cases where M is a variable or a combination are easy - the latter uses the reflexivity of $\leq$. In case M is an abstraction $(\lambda x^{\sigma}.M_1)$ suppose $R_{\sigma}(d, w')$ where $w' \geq w$. Then $[\![\lambda x^{\sigma}.M_1]\!](\rho)(d) = [\![M_1]\!](\rho')$ where $\rho' = \langle \rho_{\lambda}[d_{\lambda}/x^{\sigma}]\rangle_{\lambda<\kappa}$. Now by lemma 1 and the assumption on $d, R_{\tau}(\rho' x^{\tau}, w')$ holds for all x^{τ} in $FV(M_1)$ and we can apply the induction hypothesis to M_1. ⊠

THEOREM 2 (Completeness Theorem) Suppose D is infinite. Then an element d of D_{σ} is λ-definable iff it I-satisfies every I-relation $R \subseteq D^3 \times W$.

We do not know if the restriction on D can be dropped or if 3 can be reduced to 2 - it cannot be reduced to 1 because, for example, if $D = \omega$ and $F: D_{((\iota \to \iota),\iota,\iota)}$ is defined by:

$$F(g)(d) = g^{g(d)}(d)$$

then it I-satisfies every I-relation $R \subseteq D \times W$ but is not λ-definable.

The consistency half (definability implies I-satisfaction) of theorem 2 is given by proposition 2; the rest of this paper is devoted to proving the other half. The intention is to construct a suitable W and R. We begin with some notation for vectors. If $d = \langle d_1, \ldots, d_m\rangle$ in $\prod_{1\leq i\leq m} D_{\sigma_i}$ is a vector (= finite sequence) of elements and f is in $D_{(\sigma_1,\ldots,\sigma_m,\tau)}$ then fd is $fd_1 \ldots d_m$ $(m \geq 0)$; if $v = \langle x_1, \ldots, x_m\rangle$ is a vector of variables then v is *non-repeating* if the x_i are all different;

for ρ in Env, ρv is $\langle \rho x_1, \ldots, \rho x_m \rangle$; for a term M, Mv is $Mx_1 \ldots x_m$ and $\lambda v.M$ is $\lambda x_1 \ldots \lambda x_m.M$. A property P holds for *essentially all* vectors of variables if there is a finite set $F \subseteq$ Var such that whenever no component of v is in F then P(v) holds. Concatenation of vectors is indicated by juxtaposition; note the essential unambiguity of the notation Mvv'.

From now on we assume D is infinite. Let $||\cdot||$ be a map from terms of type ι to D such that:

$$|| M || = || N || \text{ iff } M =_{\beta,\eta} N.$$

Define $d \sim_\rho M$ for any environment ρ, element d of D_σ and term M of type σ (where $\sigma = (\sigma_1, \ldots, \sigma_m, \iota)$) by:

$$d \sim_\rho M \equiv \text{For essentially all non-repeating } v \text{ in } \prod_{1\le i\le m} \mathrm{Var}_{\sigma_i}, \quad d(\rho v) = ||Mv||.$$

Note that this relation depends only on the value of ρ at variables of types strictly smaller than σ. Also if $d \sim_\rho M_i$ $(i = 0,1)$ then $M_0 =_{\beta,\eta} M_1$. Now for d in D_σ let $M(d,\rho)$ be a term of type σ such that $d \sim_\rho M(d,\rho)$ if one exists, and an arbitrary term (say x^σ) of that type otherwise.

Now we can define an environment ρ_s by putting for x^σ where $\sigma = (\sigma_1, \ldots, \sigma_m, \iota)$, and d in $\prod_{1\le i\le m} D_{\sigma_i}$

$$\rho_s(x^\sigma)(d) = ||x^\sigma M(d_1,\rho_s) \ldots M(d_m,\rho_s)||$$

The above remarks show, by structural induction on σ, that this is a good definition.

LEMMA 2. For all terms M $[[M]](\rho_s) \sim_{\rho_s} M$.

Proof. Without loss of generality we can just prove the proposition by induction on terms in long $\beta\eta$-normal form, see [Jen]. So assume M has the form $\lambda x_1 \ldots \lambda x_m.xM_1 \ldots M_n$ where $xM_1 \ldots M_n$ has type ι and the M_j $(1\le j\le n)$ are in long $\beta\eta$-normal form. To show $[[M]](\rho_s) \sim_{\rho_s} M$ it is enough to consider only vectors, v, in $\prod_{1\le i\le m} Var_{\sigma_i}$ none of whose components are free variables of M. By taking α-conversions of M we see that the case $v = \langle x_1,\ldots,x_m\rangle$ is typical. So we calculate:

$$\begin{aligned}[\![M]\!](\rho_s)(\rho_s x_1)\ldots(\rho_s x_m) &= [\![M_1\ldots x_m]\!](\rho_s)\\ &= [\![xM_1\ldots M_n]\!](\rho_s)\\ &= \rho_s(x)[\![M_1]\!](\rho_s)\ldots[\![M_n]\!](\rho_s)\\ &= ||xM([\![M_1]\!](\rho_s),\rho_s)\ldots M([\![M_n]\!](\rho_s),\rho_s)||\\ &= ||xM_1\ldots M_n|| \text{ (by induction}\end{aligned}$$

hypothesis, the definition of $||\cdot||$ and the above remark on $\underset{\rho}{\sim}$)

$$= ||Mx_1\ldots x_m||. \quad ⊠$$

This gives a completeness theorem for βη-conversion (cf. [Böh]). From now on we generally omit the reference to ρ_s in $[\![M]\!](\rho_s)$.

THEOREM 3. For any term M of type ι, $[\![M]\!] = ||M||$. Further for any terms M and N of the same type σ:

$$M =_{\beta,\eta} N \text{ iff } \forall\rho\in Env.[\![M]\!](\rho) = [\![N]\!](\rho) \text{ iff } [\![M]\!](\rho_s) = [\![N]\!](\rho_s)$$

Proof. The first part is immediate from lemma 2. For the second part the implications from left to right are well-known; for the converses suppose $[\![M]\!](\rho_s) = [\![N]\!](\rho_s)$, $\sigma = (\sigma_1,\ldots,\sigma_m,\iota)$ and let x_i be a variable of type σ_i not free in either M or N ($1\le i\le m$). Then

$$\begin{aligned}||Mx_1\ldots x_m|| &= [\![Mx_1\ldots x_m]\!] \text{ (by the first part)}\\ &= [\![Nx_1\ldots x_m]\!] \text{ (by assumption)}\\ &= ||Nx_1\ldots x_m|| \text{ (by the first part).}\end{aligned}$$

So $Mx_1\ldots x_m =_{\beta,\eta} Nx_1\ldots x_m$ and so taking the x_i to be all different we find that $M =_{\beta,\eta} N$. ⊠

The second part of this theorem fails if D is finite; for example, there are only finitely many elements in $D_{((\iota\to\iota),\iota,\iota)}$ but infinitely many closed normal terms of that type.

We are now in a position to define $\langle W,\leq\rangle$ and R. First:

$W = \{\langle d,v,\bar{v}\rangle \mid d$ is a vector of elements of $\cup_\sigma D_\sigma$, v and $\bar{v}$ are non-repeating vectors of variables, all three have the same length and corresponding components have the same type$\}$.
For i = 1,2,3 the ith component of any w in W is written as w_i; for any w,w^+ in w the concatenation ww', is defined to be the componentwise concatenation $\langle w_1w_1^+, w_2w_2^+, w_3w_3^+\rangle$ and we define $w\leq w'$ to mean $w' = ww^+$ for some w^+. Finally, $R \subseteq D^3 \times W$ is defined by:

$R(d,w) \equiv$ There is a closed term M such that $d_1 = [\![M]\!](w_1)$, and $d_i = [\![Mw_i]\!]$ for i=2,3.

Clearly $\leq$ is reflexive and transitive and R is an I-relation.

A term, M, is <u>head-λ -free</u> iff it has the form $xM_1 \ldots M_k$.

LEMMA 3. Let M be a head-λ-free term and let d,d' be elements of D_σ. Then if $d\,\rho_s(v) = d'\,\rho_s(v)$ for essentially all non-repeating vectors, v, of variables of the appropriate types such that $d_\rho(v)$ has type ι, then $[\![M]\!](d) = [\![M]\!](d')$.

Proof. As $M(d,\rho_s) =_{\beta,\eta} M(d',\rho_s)$ by assumption, the conclusion is immediate from the definition of ρ_s. ☒

LEMMA 4. 1. Suppose $R_\sigma(f,g,h,w)$ holds where $w = \langle d,v,\bar{v}\rangle$. Then there is a closed term M such that $f = [\![M]\!]d$, $g(\rho v^+) = [\![Mvv^+]\!]$ and $h(\rho\bar{v}^+) = [\![M\overline{vv}^+]\!]$ whenever vv^+, $\overline{vv}^+$ are non-repeating vectors of variables of the appropriate type such that $[\![Mvv^+]\!]$ is of type ι.

2. Suppose f,g,h are of type σ and w is a world. If g,h are denotations of head-λ-free terms and there is a closed term M such that $f = [\![M]\!]w_1$, $g = [\![Mw_2]\!]$ and $h = [\![Mw_3]\!]$, then $R_\sigma(f,g,h,w)$ holds.

Proof. Both parts are proved together by induction on σ.

1. For ι the result is immediate from the definition of R.
For the case $\sigma \to \tau$ suppose $R_{\sigma\to\tau}(f,g,h,w)$ holds where $w = \langle d,v,\bar{v}\rangle$. Let $w' = w\langle e,x,\bar{x}\rangle$ be a world with e in D_σ. Then by induction hypothesis, using part 2 we see that $R_\sigma(e,\rho_s(x),\rho_s(\bar{x}),w')$ (take

$M = \lambda v.\lambda x.x$). Therefore, by the definition of $R_{\sigma \to \tau}$ we have $R_\tau(fe, g\rho_s(x), h\rho_s(\bar{x}), w')$. Therefore by induction hypothesis, using part 1, there is a closed term M such that $fe = [\![M]\!]de, (g\rho_s(x))\rho_s(v^+) = [\![Mvxv^+]\!]$ and $(h\rho_s(\bar{x}))\rho_s(\bar{v}^+) = [\![M\bar{v}\bar{x}\bar{v}^+]\!]$ whenever vxv^+, $\bar{v}\bar{x}\bar{v}^+$ are non-repeating vectors of variables of the appropriate type such that $[\![Mvxv^+]\!]$ is of type ι.

Clearly M may depend on e, x and $\bar{x}$. As $g\rho_s(x)\rho_s(v^+) = [\![Mvxv^+]\!]$ and neither side of the equation mentions e or $\bar{x}$ and as vxv^+ is non-repeating and M is closed it follows by Theorem 3 that M is independent of e or $\bar{x}$; similarly, using the equation for h it is independent of x. Therefore we have a closed term M such that $fe = [\![M]\!]\, de$, $g(\rho_s x)(\rho_s v^+) = [\![Mvxv^+]\!]$, $h(\rho_s \bar{x})(\rho_s \bar{v}^+) = [\![M\bar{v}\bar{x}\bar{v}^+]\!]$ whenever e is in D_σ and vxv^+ and $\bar{v}\bar{x}\bar{v}^+$ are non-repeating vectors of variables of the appropriate type such that $[\![Mvxv+]\!]$ is of type ι, which finishes the proof of part 1.

2. For ι the result is immediate from the definition of R. For the case $(\sigma \to \tau)$ suppose f,g,h are of type $(\sigma \to \tau)$, $w = \langle d, v, \bar{v}\rangle$ is a world, that g,h are denotations of head-λ-free terms and that there is a closed term M such that $f = [\![M]\!]d$, $g = [\![Mv]\!]$ and $h = [\![M\bar{v}]\!]$. Let $w' = w\langle d^+, v^+, \bar{v}^+\rangle$ be a world and suppose that $R_\sigma(e, a, b, w')$. Then by induction hypothesis using part 1, there is a closed term M_1 such that $e = [\![M_1]\!]dd^+$, $a(\rho(v^{++})) = [\![M_1 vv^+v^{++}]\!]$ and $b(\rho(\bar{v}^{++})) = [\![M_1 \bar{v}\bar{v}^+\bar{v}^{++}]\!]$ whenever vv^+v^{++}, $\bar{v}\bar{v}^+\bar{v}^{++}$ are non-repeating vectors of variables of the appropriate type such that $[\![M_1 vv^+v^{++}]\!]$ is of type ι.

Now we have, $f(e) = [\![M]\!]d([\![M_1]\!]dd^+) = [\![M_2]\!]dd^+$, where $M_2 = \lambda v.\lambda v^+.Mv(M_1 vv^+)$. Since $a(\rho_s(v^{++})) = [\![M_1 vv^+]\!]\rho_s(v^{++})$ for essentially all non-repeating vectors of variables of the appropriate types such that $a(\rho_s(v^{++}))$ has type ι and since g is the denotation of a head-λ-free term, we can apply lemma 3 to see that $g(a) = g[\![M_1 vv^+]\!]$. Therefore $g(a) = [\![Mv]\!]\,[\![M_1 vv^+]\!] = [\![M_2 vv^+]\!]$ and similarly $h(b) = [\![M_2 vv^+]\!]$. As g(d) and h(b) are clearly, therefore, denotations of head-λ-free terms and as we have already shown that $f(e) = [[M_2]]\, dd^+$ it follows by the induction hypotheses, using part 2, that $R_\tau(fe, ga, hb, w')$, showing $R_{\sigma \to \tau}(f,g,h,w)$ and concluding the inductive proof. ☒

The proof of the rest of theorem 2 is now immediate. For suppose an element d in D_σ I-satisfies every I-relation $R \subseteq D^3 \times W$. Then with R as defined above and taking w_o as the world all of whose components are empty we have $R_\sigma(d,d,d,w_o)$. Then by lemma 4.1 there is a closed term M such that $d = [[M]]$.

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