Under consideration for publication in Math. Struct. in Comp. Science

Predicate Transformers for Convex Powerdomains

KLAUS KEIMEL and GORDON D. PLOTKIN [†]

Received ????

We investigate laws for predicate transformers for the combination of nondeterministic choice and (extended) probabilistic choice, where predicates are taken to be functions to the extended nonnegative reals, or to closed intervals of such reals. These predicate transformers correspond to state transformers, which are functions to convex powerdomains, the appropriate powerdomains for the combined forms of nondeterminism. As with standard powerdomains for nondeterministic choice, these come in three flavours: lower, upper and order-convex, and so there are also three kinds of predicate transformers. In order to make the connection, the powerdomains are first characterised in terms of relevant classes of functionals.
Much of the development is carried out at an abstract level, a kind of domain-theoretic functional analysis: one considers d-cones, which are dcpos equipped with a module structure over the nonnegative extended reals, in place of topological vector spaces. It remains to carry out such a development for probabilistic choice per se; it would presumably be necessary to work with a notion of convex space rather than of a cone.

1. Introduction

In this paper we characterise predicate transformers combining nondeterminism and (general) valuations, as a contribution to the programme of a domain-theoretic account of the combination of ordinary and probabilistic nondeterminism. The problem of finding such a characterisation was raised, but left open, in (Tix, Keimel, Plotkin 2005). The problem is, in fact, threefold as there are three such natural combinations, corresponding to the three classical domain-theoretic powerdomains: lower, upper and (order-)convex.

It would be more natural, from the point of view of computer science applications, to restrict to subprobability valuations, rather than allowing all of them. There has already been work along these lines for discrete domains (McIver and Morgan 2001a; McIver and Morgan 2001b; McIver, Morgan and Seidel 1996; Ying Minsheng 2003), and there

[†] This work was done with the support of EPSRC grant GR/S86372/01, a Royal Society-Wolfson Research Merit Award and APPSEM II.

has been interest in statistics in using spaces of sets of probability measures in the area of 'imprecise probabilities': see Huber's book (Huber 1981) for early work and Walley's text (Walley 1991) for more recent work.

However the mathematics seems to be more natural if we take all the valuations, since one can then work with notions of linearity rather than convexity. Indeed, in (Tix, Keimel, Plotkin 2005) it was possible to work in a rather abstract way considering d-cones and lower, upper and convex powercone constructions. The unrestricted valuations on a domain form the free d-cone over it, and the required combined powerdomains of a domain can be found by taking the powercones of the d-cone of its valuations, restricting to coherent domains in the convex case.

We therefore first consider predicate transformers for powercones and then specialise to the powerdomains. We would certainly also like to have corresponding results for the probabilistic case, and we hope that the present work, together with that of (Tix, Keimel, Plotkin 2005) will prove helpful to that end.

There is an illuminating relationship between predicate transformers and functional representations of monads. Dijkstra's classical 'healthy' predicate transformers (Dijkstra 1976) on a given set of states S are strict, continuous, binary meet-preserving maps:

$$\mathcal{P}(S) \longrightarrow \mathcal{P}(S)$$

This generalises to strict, continuous, binary meet-preserving maps:

$$\mathcal{O}(Q) \longrightarrow \mathcal{O}(P)$$

where, for any dcpo P, $\mathcal{O}(P)$ is the dcpo of open subsets of P, and, provided that Q is a domain, such maps are in bijective correspondence with continuous functions:

$$P \longrightarrow \mathcal{S}(Q_\perp)$$

where $\mathcal{S}$ is the upper powerdomain monad, and $(-)_\perp$ is the lifting construction. (One can show that for any domain Q, $\mathcal{S}(Q_\perp)$ is the free lower semilattice over Q with a least element.) The connection between Dijkstra's predicate transformers and Smyth's powerdomains of flat dcpos was given in (Plotkin 1980); the above generalisation to arbitrary domains was, essentially, first given in (Smyth 1983), in the even more general setting of sober spaces. The relationship between suitable notions of predicate transformer for the lower and order-convex powerdomains was considered in (Bonsangue 1998).

To see the relationship with a functional representation of the upper powerdomain, note first that, by transposition, one has a bijective correspondence of continuous functions:

$$\frac{P \longrightarrow \mathbb{O}^{\mathbb{O}^Q}}{\mathcal{O}(Q) \longrightarrow \mathcal{O}(P)}$$

as $\mathcal{O}(P)$ is isomorphic to the dcpo $\mathbb{O}^P$ of all continuous functions from P to Sierpinski space. This correspondence evidently cuts down to one between predicate transformers, as defined above, and continuous functions to the sub-dcpo of $\mathbb{O}^{\mathbb{O}^Q}$ of those functionals which are strict and preserve binary meets. However, if Q is a domain then $\mathcal{S}(Q_\perp)$ is isomorphic to the dcpo of these functionals, and this gives us the above general characterisation. This functional characterisation of $\mathcal{S}(Q_\perp)$ was, essentially, given in (Heckmann 1993), and it follows from the Hofmann-Mislove theorem (Gierz *et al.* 2003); the relation between this theorem, functional representations and continuous universal quantifiers was presented in (Escardo 2004, Chapter 11).

Let us take another example, closer to our present concerns and which illustrates that the notion of predicate will, in general, vary. There is a 'Riesz' representation theorem (Kirch 1993; Tix 1995), and see (Tix, Keimel, Plotkin 2005, Chapter 2), for the dcpo of all valuations $\mathcal{V}(P)$ of a dcpo P:

$$\mathcal{V}(P) \cong \mathcal{L}(P)^*$$

Here $\mathcal{L}(P)$ is the collection of all continuous functions to $\overline{\mathbb{R}}_+$ the dcpo of the nonnegative reals extended by a point at infinity, which latter has an evident semiring structure, and, then, $\mathcal{L}(P)^*$ consists of the *linear* functionals in $\mathcal{L}(\mathcal{L}(P))$. (We say that a functional is linear if it preserves the operations of addition and multiplication by a positive real (*scalar* multiplication), with these operations being defined in the natural pointwise way on $\mathcal{L}(P)$.) We therefore have a bijective correspondence between continuous functions:

$$P \longrightarrow \mathcal{V}(Q)$$

and predicate transformers, if we now take these to be linear continuous functions:

$$\mathcal{L}(Q) \longrightarrow \mathcal{L}(P)$$

In both examples a functional representation theorem gives rise to a predicate transformer characterisation. Notice that the converse also holds: the characterisation implies the representation (take $P = \mathbb{1}$ in the above). Our strategy is to find the functional representation first as that seems simpler and more direct than beginning with the predicate transformer characterisation.

The two examples follow a certain pattern, with the 'object of truthvalues' $\mathbb{O}$, $\overline{\mathbb{R}}_+$ being, respectively, the free $\mathcal{S}(-)_\perp$ or $\mathcal{V}$-algebra on 1, the terminal object of the category of dcpos; further, the requirement to be strict and to preserve binary meets or to preserve addition and scalar multiplication is equivalent to requiring the relevant algebra structure to be preserved. Our case will be similar, but we will not be able to require all the algebra structure to be preserved, the essential obstacle being that the monads we consider are not commutative. However, there are more subtle requirements, such as sublinearity, that do allow functional representation theorems and consequent notions of healthy predicate transformer.

All our functional representation theorems deal with the representation of certain convex sets by functionals with characteristic properties. Functional representations have a long history in convex analysis. They go back to the seminal paper (H. Minkowski 1903) where an order-preserving bijection between compact convex subsets of $\mathbb{R}^3$ and their *support functionals* is established; the latter are characterized as sublinear functionals on $\mathbb{R}^3$. The book (Bonnesen and Fenchel 1934) contains an extension of these results to $\mathbb{R}^n$. There the bijection is further shown to be an isomorphism of topological cones, where the compact convex subsets are endowed with the Hausdorff metric and the Minkowski sum and the sublinear functionals with the compact-open topology and pointwise addition. In Chapter 13 of (Rockafellar 1972) these results were extended to possibly unbounded closed convex sets and sublinear functionals that admit the value $+\infty$.

The literature contains many generalisations to topological vector spaces. The paper (Hörmander 1955) is noteworthy. There, among other results, a bijection was established between the closed convex subsets of a locally convex topological vector space V and those sublinear functionals on its topological dual V^* which are lower semicontinuous for the weak* topology. The more subtle question of characterizing the support functionals of compact convex sets in this generality was treated in (Tolstogonov 1976). The survey paper (Kutateladze and Rubinov 1972) gives a very complete account of the classical theory. To some extent, our representation theorems follow the classical patterns although in the quite different domain theoretical setting. We do not know of any classical functional representation results corresponding to those for our convex lenses in Section 6.

Functional representation theorems have also been given by workers in the area of imprecise probabilities. Huber (Huber 1981, Prop. 10.2.1) gives a theorem characterising the functionals generated by non-empty closed convex sets of probability measures over a finite set; Maaß (Maaß 2002) gives a theorem generalising both that and Walley's (Walley 1991, Theorem 3.6.1), and refers to (Bonsall 1954) for a yet more general functional analytic theorem.

In Section 2, below, we give the needed technical background for our results and introduce a useful general notion, that of a d-cone semilattice. This is followed, in Section 3, by some further development of the powercones introduced in (Tix, Keimel, Plotkin 2005), including some abstract discussion of powercones at the level of d-cone semilattices. In Section 4 we consider generalities on functional representations of powercones and powerdomains. This enables the efficient presentation of the form and elementary properties of these representations in the cases of the lower and upper powercones. However deriving the corresponding information for the convex powercone is a rather complex affair, involving, among other things, the crucial condition (*) introduced below to define the so-called canonically $\subseteq$-sublinear maps. An analogous condition arises in the case of order-convex powerdomains in the treatment of predicate transformers in (Bonsangue 1998) and, less directly, in the definition of the basis of the Vietoris locale (Johnstone 1985).

Section 5 gives theorems on sublinear and superlinear functions as sups or infs of linear ones, concluding with Theorem 1 characterising canonically $\subseteq$-sublinear maps as unions of linear ones and thereby casting some light on property (*). These results enable us to prove our functional representation theorems in the following Section 6. At the level of d-cones these are Theorems 2, 3 and 4; at the level of convex powerdomains these are Corollaries 2, 3 and 4. It is worth noting that, in the upper and order-convex cases, we make use of the domain-theoretic Banach-Alaoglu theorem established in (Plotkin 2006); indeed that theorem was proved in order to make such representation theorems possible. Finally, in Section 7, we use our representation theorems to characterise the predicate transformers corresponding to state transformers. We again begin the development at a suitably general level. At the level of d-cones the predicate transformer characterisations are given by Theorems 5, 6 and 7; at the level of convex powerdomains these are Corollaries 5, 6 and 7.

We remark, finally, that in (Tix, Keimel, Plotkin 2005) a small imperative language was given with both ordinary and probabilistic nondeterminism, together with three semantics, using the three convex powerdomains. It is straightforward using our results to further give this language three corresponding predicate transformer semantics and to show each pair of semantics isomorphic, with the isomorphism being given by the appropriate functor W of Section 7.

2. Technical preliminaries

We refer to (Gierz *et al.* 2003) for a detailed discussion of *dcpos* (directed complete partially ordered sets) and *domains* (continuous dcpos), but recall some notation and definitions here. Let X be a subset of a dcpo. If it is directed we write $\bigvee^{\uparrow} X$ for its least upper bound. We write $\uparrow X$ for the set of all elements of the dcpo dominating some element of X; $\downarrow X$ is defined dually. Upper sets, also called *saturated*, are characterised by the property that $\uparrow X = X$. We say that X is *order-convex* iff $X = \downarrow X \cap \uparrow X$; we write $\mathrm{conv}_{\leq}(X)$ for the least order-convex set containing X, viz $\downarrow X \cap \uparrow X$. The way-below relation is written $\ll$, and $\twoheaduparrow X$ is the set of all elements of the dcpo way-above some element of X. Topological notions on dcpos like continuity, open, closed, compact, etc., always refer to the Scott topology, unless indicated otherwise; we write $\overline{X}$ for the closure of a subset of a topological space. A domain is *coherent* if the intersection of any two compact saturated subsets is compact too. The product of two coherent domains is again coherent: this can be shown using, e.g., Lemma 18 of (Jung and Tix 1998). A continuous map $f\colon P \to Q$ is an *order-embedding* if $fx \leq fy$ implies $x \leq y$, for all x, y in P. Finally, we write Dom for the category of domains and continuous maps, and Dom^c for the full subcategory of the coherent domains.

The central concept in this paper is that of a *d-cone*. This concept has been introduced by Kirch and by Tix (Kirch 1993; Tix 1995) as a slight modification of Claire Jones' *abstract probabilistic domains* (Jones 1990; Jones and Plotkin 1989; Heckmann 1994). We refer to (Tix, Keimel, Plotkin 2005) for information on d-cones, but give all the required definitions here.

A d-cone C has an order structure and an algebraic structure. The order structure is that of a dcpo. The algebraic structure is that of a *cone*, that is, there is an addition $(x, y) \mapsto x + y\colon C \times C \to C$, which is required to be associative and commutative and to have a neutral element 0, and a scalar multiplication $(r, x) \mapsto rx\colon \mathbb{R}_+ \times C \to C$, which satisfies the same equational laws as in vector spaces except that the scalars are restricted to the set $\mathbb{R}_+$ of nonnegative reals.

The order and the algebraic structure are linked by the requirement that addition and scalar multiplication are continuous in both variables. The notion of continuity employed here is that of Scott continuity in *bounded* directed complete partially ordered sets (bdcpos for short), which are defined to be those partial orders with lubs of bounded directed sets. A function between bdcpos is Scott continuous if it is monotonic and preserves suprema of bounded directed sets; this reduces to the usual notion of Scott continuity in the case of dcpos. Note that the nonnegative reals $\mathbb{R}_+$ endowed with the usual order form a bdcpo rather than a dcpo; adding an element $+\infty$ to $\mathbb{R}_+$, we obtain the extended nonnegative reals $\overline{\mathbb{R}}_+$, which form a dcpo, even a d-cone.

Let X be a subset of a d-cone. It is *convex* if $rx + (1 - r)y \in X$ whenever $x, y \in X$ and $r \in [0, 1]$. We write $\mathrm{conv}(X)$ for the least convex set containing X. Consider a function $f\colon C \to D$ between d-cones. If it is always true that:

$$f(rx) = rf(x),\ \ f(x + y) \leq f(x) + f(y),\ \ f(x + y) \geq f(x) + f(y)$$

then f is said to be *homogeneous*, *subadditive* and *superadditive*, respectively. We say that f is *sublinear* (*superlinear*), if it is homogeneous and subadditive (superadditive). A *linear function* is one that is both sublinear and superlinear.

We will work in the category Cone of d-cones and linear continuous maps. We will use two full subcategories CCone and CCone^c the objects of which are the *continuous* d-cones and the *coherent* continuous d-cones, respectively. The way-below relation on a continuous d-cone is *additive* if whenever $a \ll b$ and $a' \ll b'$ hold then $a + a' \ll b + b'$ does too.

Given dcpos P and Q, we write Q^P for the dcpo of all continuous maps from P to Q. If

D is a d-cone, D^P is also one when endowed with the pointwise operations. A special case was mentioned in the introduction: $\mathcal{L}(P) =_{def} \overline{\mathbb{R}}_+^P$ denotes the d-cone of all continuous functionals $f\colon P \to \overline{\mathbb{R}}_+$ (functions with range $\overline{\mathbb{R}}_+$ are often termed 'functionals'); $\mathcal{L}(P)$ is a domain if P is and then its way-below relation is additive if, and only if, P is coherent, see (Tix, Keimel, Plotkin 2005, Proposition 2.28). Recall here that we are using the Scott topology on $\overline{\mathbb{R}}_+$ the only open sets of which are the intervals $]r, +\infty]$, not the usual Hausdorff topology.

Given d-cones C and D, we write $[C, D]$ for the sub-d-cone of D^C of linear continuous maps from C to D. It can be shown that Cone is a symmetric monoidal closed category with unit $\overline{\mathbb{R}}_+$ and exponential $[-, -]$ (the tensor is less easy to describe). A special case of this function space was mentioned in the introduction: $C^* =_{def} [C, \overline{\mathbb{R}}_+]$ denotes the d-cone of all linear continuous functionals on C; it is called the *dual* d-cone of C. Every element a of a d-cone C defines a linear continuous functional $a^{**} = (f \mapsto f(a))$ on C^*, yielding a natural linear continuous map $a \mapsto a^{**} : C \to C^{**}$. If C is a continuous d-cone, this map is an order-embedding, see (Tix, Keimel, Plotkin 2005, Corollary 3.5). In case it is also surjective, and so an isomorphism of d-cones, we say that C is *reflexive*.

The evaluation functional $\mathrm{ev} : C^* \times C \to \overline{\mathbb{R}}_+$ gives rise to two topologies of interest. The *weak* Scott topology* on C^* has all sets of the form $W_{x,r} =_{def} \{f \in C^* \mid f(x) > r\}$ as a subbasis, where $x \in C$ and $r \in \mathbb{R}_+$; the *weak Scott topology* on C has all sets of the form $W_{f,r} =_{def} \{x \in C \mid f(x) > r\}$ as a subbasis, where $f \in C^*$ and $r \in \mathbb{R}_+$.

We will be particularly interested in the *extended probabilistic powerdomain*, i.e., the d-cone $\mathcal{V}(P)$ of all continuous valuations of a dcpo P mentioned in the introduction. A *valuation* on P is a strict monotonic modular function $\mu : \mathcal{O}(P) \to \overline{\mathbb{R}}_+$, with modularity meaning that for all open subsets U, V of P:

$$\mu(U) + \mu(V) = \mu(U \cup V) + \mu(U \cap V)$$

The ordering of $\mathcal{V}(P)$ is defined pointwise as are its addition and scalar multiplication. We refer to (Tix, Keimel, Plotkin 2005) for details of this construction and its properties but recall the main points here. There is a bilinear continuous integration functional $\int : \mathcal{L}(P) \times \mathcal{V}(P) \longrightarrow \overline{\mathbb{R}}_+$; this yields the Riesz-type isomorphism $\mathcal{V}(P) \cong \mathcal{L}(P)^*$ mentioned in the introduction, sending μ to $f \mapsto \int f d\mu$; the inverse of this isomorphism sends φ to $U \mapsto \varphi(\chi_U)$ where χ_U is the characteristic function of U.

In case P is a domain $\mathcal{V}(P)$ is too, and it is the free d-cone over P, see (Kirch 1993; Gierz *et al.* 2003). The unit map $\eta : P \to \mathcal{V}(P)$ is given by $\eta(x) = U \mapsto \chi(U)(x)$. The extension of a continuous map $P \to \overline{\mathbb{R}}_+$ to a linear continuous map $\mathcal{V}(P) \to \overline{\mathbb{R}}_+$ is given by integration and yields an isomorphism of d-cones $\mathcal{L}(P) \cong \mathcal{V}(P)^*$.

Putting all this together we obtain that, for any domain P, $\mathcal{L}(P)$ and $\mathcal{V}(P)$ are both reflexive continuous d-cones. Finally, let us remark that, again in the case where P is a domain, the weak Scott topology on $\mathcal{V}(P)$ coincides with its Scott topology (Kirch 1993; Tix 1995) and if P is also coherent then $\mathcal{V}P$ is too (Tix, Keimel, Plotkin 2005, 2.10). Thus, for every coherent domain P, the extended probabilistic powerdomain $\mathcal{V}(P)$ is a *convenient d-cone* in the following sense: it is continuous, reflexive and its weak Scott topology coincides with its Scott topology and, furthermore, its dual C^* is continuous and has an additive way-below relation. This rather strong notion is useful for the formulation of our results in Section 6.

Powercones have extra structure, a continuous semilattice operation $\cup$, i.e., an associative, commutative and idempotent binary operation, satisfying properties which we now briefly consider abstractly. A *d-cone semilattice* is a d-cone together with a continuous semilattice 'union' operation $\cup$ over which addition and scalar multiplication distribute, the latter meaning that the equations $x+(y\cup z) = (x+y)\cup(y+z)$ and $r(x\cup y) = rx\cup ry$ both hold. We write ConeSL for the category of all d-cone semilattices and $\cup$-preserving linear continuous functions. But in general, $\cup$ need not at all be the join or the meet with respect to the d-cone ordering $\leq$.

The partial order associated with the semilattice operation is written as $\subseteq$ and $x \subseteq y$ holds if and only if $x \cup y = y$; it is closed under directed sups, scalar multiplication and addition. In all the powercones it turns out that $\subseteq$ is the ordinary subset relation, but $\cup$ is not the ordinary union operation, but rather ordinary union followed by the application of a suitable closure operation. The cone $\overline{\mathbb{R}}_+$ can be viewed as a d-cone semilattice in precisely two ways: either as a d-cone join-semilattice, meaning that $\cup = \vee$, or as a d-cone meet-semilattice, meaning that $\cup = \wedge$.

If S is a d-cone semilattice then so is S^C when equipped with the pointwise union. It is important to note that this is not true of $[C, S]$ as the pointwise union of two additive functions need not be additive. For example, taking $S = \overline{\mathbb{R}}_+$ and $C = \mathcal{L}(P)$ we have $C^* \cong \mathcal{V}(P)$, but the latter need have neither binary sups nor binary meets.

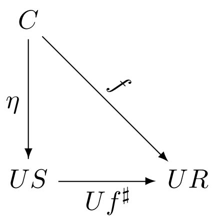
However we can at least define a pointwise partial order $\subseteq$ on $[C, S]$, and that is closed under all the d-cone operations. If C is also a d-cone semilattice then the $\subseteq$-*monotonic* functions in $[C, S]$ (those preserving $\subseteq$) form a sub-d-cone, as is straightforwardly verified using the closure properties of $\subseteq$ on S. The pointwise union $h = f \cup g$ of two maps $f, g \in [C, S]$ is $\subseteq$-*sublinear*, that is, it is homogeneous and $\subseteq$-*additive*, where the latter means that $f(x+y) \subseteq f(x) + f(y)$ for all $x, y \in C$.

3. Powercone and powerdomain constructions

Chapter 4 of (Tix, Keimel, Plotkin 2005) presents three convex powercone and corresponding powerdomain constructions. We begin by developing some of their common properties in an abstract setting. We suppose we have full subcategories K and L of Cone

and ConeSL, respectively, and we write $U : \mathsf{L} \to \mathsf{Cone}$ for the evident forgetful functor. For every d-cone semilattice S, the graph of the order $\Delta_S = \{(y, z) \in S \times S \mid y \leq z\}$ is a sub-d-cone semilattice of $S \times S$; it is continuous if S is, and, further, it is then coherent if S is. We suppose that Δ_S is an object of L for every object S in L.

For a d-cone C say that a linear continuous map $\eta : C \to US$, with S in L, is *universal* if for any other such map $f : C \to UR$ there is a unique $\cup$-preserving linear continuous map $f^\sharp : S \to R$ such that the following diagram commutes:



In other words, S is the free L-d-cone semilattice over C, with unit map η.

We now suppose that for any C in K there is a free L-d-cone semilattice FC in L with UFC also in K and with universal map $\eta_C : C \to UFC$. This allows to define a monad T on K, setting $TC =_{def} UFC$ and $Tf =_{def} (\eta_D \circ f)^\sharp$, for any linear continuous $f : C \to D$, and with unit η and multiplication $\mu_C =_{def} (\mathrm{id}_{TC})^\sharp$.

We now consider properties of extension $f \mapsto f^\sharp$ considered as a function from $[C, S]$ to $[TC, S]$ for a given choice of C in K and S in L.

Proposition 1 *For objects C in K and S in L, extension $f \mapsto f^\sharp$, considered as a map $[C, S] \to [TC, S]$, is continuous and homogeneous.*

Proof. We begin by proving it is monotonic. Suppose that $f \leq g$ for f, g in $[C, S]$. The set $\Delta_S = \{(y, z) \in S^2 \mid y \leq z\}$ is a sub-d-cone semilattice of S^2 and, by presupposition, belongs to L. Since $f \leq g$, we can define a linear continuous map $h : C \to \Delta_S$ by putting $h(x) = (fx, gx)$. We have $\pi_0 h^\sharp \eta_C = \pi_0 h = f$, where π_0 is the restriction to Δ_S of the first projection on S. It follows, by universality, that $f^\sharp = \pi_0 h^\sharp$ and, similarly, that $g^\sharp = \pi_1 h^\sharp$, with π_1 the corresponding restriction of the second projection. But $\pi_0 \leq \pi_1$ and so $f^\sharp \leq g^\sharp$, as required.

To finish the proof of continuity, let $f_\lambda : C \to S$ be a directed family. Then we have: $(\bigvee^{\uparrow} f_\lambda^\sharp)\eta_C = \bigvee^{\uparrow} f_\lambda^\sharp \eta_C = \bigvee^{\uparrow} f_\lambda$, and so, by universality, $(\bigvee^{\uparrow} f_\lambda)^\sharp = \bigvee^{\uparrow} f_\lambda^\sharp$. A similar argument shows that extension is homogeneous. $\square$

Note that it follows that each action $T : [C, D] \to [TC, TD]$ is continuous and homogeneous.

We now recall the three powercone and powerdomain constructions, but make a slight change of terminology with respect to (Tix, Keimel, Plotkin 2005): in the case of cones we drop the word *convex*, saying *lower powercone* instead of *lower convex powercone*, and so on. However, we use the word in the case of powerdomains, speaking of the *lower* or *upper convex powerdomains* or the *biconvex powerdomain* (rather than the clumsy 'order-convex convex powerdomain') to distinguish these powerdomains from the standard powerdomains for (non-probabilistic) nondeterminism. We also present some additional material giving explicit formulas for extensions, particularly Kleisli extensions, which are extensions of maps with codomain of the form TD, and also for monad multiplications.

3.1. *The lower powercone and lower convex powerdomain*

Let C be a d-cone. Then its *lower powercone* $\mathcal{H}C$ is formed from the set of all its nonempty closed convex subsets. The lower powercone is ordered by inclusion, with directed sups being given by the closure of the union; addition and scalar multiplication are defined by $X +_H Y =_{def} \overline{X + Y}$, the closure of $X + Y$, and $r \cdot_H X =_{def} rX$. If C is continuous then so is $\mathcal{H}(C)$. This defines the object part of a functor on Cone that cuts down to a functor on CCone; its action on morphisms is given by: $\mathcal{H}(f)(X) =_{def} \overline{f(X)}$.

We further have that $\mathcal{H}C$ is a join-semilattice, with $X \vee Y =_{def} \overline{\mathrm{conv}(X \cup Y)}$, and, indeed, it is characterised by a universal property (Tix, Keimel, Plotkin 2005, Theorem 4.10): via the map $\eta_C : C \to \mathcal{H}C$, where $\eta_C(c) = {\downarrow}\{c\}$, it is the free d-cone join-semilattice over C, more precisely:

For every continuous linear map f from a d-cone C to a d-cone join-semilattice S there is a unique join-preserving continuous linear map $f^\sharp : \mathcal{H}C \to S$ such that $f^\sharp \circ \eta_C = f$. The extension $f^\sharp$ is defined by:

$$f^\sharp(X) = \bigvee_{x \in X} f(x)$$

The above framework therefore applies, taking K to be either Cone or CCone and L to be the full subcategory of ConeSL of all d-cone join-semilattices. We need some additional information on the Kleisli extension and the monad multiplication for $\mathcal{S}$. For this, we prove a lemma:

Lemma 1 *If $\mathcal{X}$ is a closed subset of $\mathcal{H}C$, then $A =_{def} \bigcup_{X \in \mathcal{X}} X$ is a closed subset of C.*

Proof. Let $y \leq x \in A$. There is an $X \in \mathcal{X}$ such that $x \in X$. As X is closed, we have $y \in X$, too, whence $y \in A$.

Let (x_i) be directed in A. First note that ${\downarrow}x_i \in \mathcal{X}$, as x_i is contained in some member X_i of $\mathcal{X}$, whence ${\downarrow}x_i \subseteq X_i$, and this implies ${\downarrow}x_i \in \mathcal{X}$ as $\mathcal{X}$ is a lower set. It then follows that $\bigvee^{\uparrow} x_i \in \overline{\bigcup {\downarrow}x_i} = \bigvee^{\uparrow}_{\mathcal{H}C} {\downarrow}x_i \in \mathcal{X}$, as $\mathcal{X}$ is closed. □

Proposition 2 *Let C and D be d-cones.*

(a) *The Kleisli extension* $f^\sharp : \mathcal{H}C \to \mathcal{H}D$ *of a linear continuous map* $f : C \to \mathcal{H}D$ *is given by:* $f^\sharp(X) = \overline{\bigcup_{x \in X} f(x)}$.
(b) *The monad multiplication is given by:* $\mu_C(\mathcal{X}) = \bigcup_{X \in \mathcal{X}} X$.

Proof. (a) $\bigvee_{x \in X} f(x) = \overline{\mathrm{conv}(\bigcup_{x \in X} f(x))}$ by the characterisation of arbitrary sups in $\mathcal{H}D$ given in (Tix, Keimel, Plotkin 2005). However $\bigcup_{x \in X} f(x)$ is convex. For if c is a convex combination $ra + (1-r)b$ of elements a, b then there are $x, y \in X$ such that $a \in f(x)$ and $b \in f(y)$, and it follows that $c \in f(rx + (1-r)y)$. We therefore have $f^\sharp(X) = \overline{\bigcup_{x \in X} f(x)}$, as required.

(b) Since $\mu_C = (\mathrm{id}_{\mathcal{H}C})^\sharp$, we have $\mu_C(\mathcal{X}) = \overline{(\bigcup_{X \in \mathcal{X}} X)} = \bigcup_{X \in \mathcal{X}} X$ with the last equality holding because of Lemma 1. □

Combining the extended probabilistic powerdomain functor $\mathcal{V}$ and $\mathcal{H}$ we obtain the *lower convex powerdomain* $\mathcal{H}\mathcal{V}(P)$ of a dcpo P. This is a domain if P is, and it is then the free d-cone join-semilattice over P.

3.2. *The upper powercone and upper convex powerdomain*

Let C be a continuous d-cone. Then its *upper powercone* $\mathcal{S}C$ is formed from the set of all its nonempty compact saturated convex subsets. The upper powercone is ordered by reverse inclusion, with directed sups being given by intersection; addition and scalar multiplication are defined by $X \underset{S}{+} Y =_{def} \uparrow(X + Y)$ and $r \underset{S}{\cdot} X =_{def} \uparrow(rX)$. The d-cone $\mathcal{S}C$ is itself continuous, and we have that $X \ll Y$ iff Y is contained in the interior of X. We now have the object part of a functor on CCone, the category of continuous d-cones. Its action on morphisms is given by: $\mathcal{S}(f)(X) =_{def} \uparrow f(X)$.

We further have that $\mathcal{S}C$ has continuous binary meets, with $X \wedge Y =_{def} \uparrow \mathrm{conv}(X \cup Y)$. Indeed, via the map $\eta_C : C \to \mathcal{S}C$, where $\eta_C(c) = \uparrow\{c\}$, the d-cone $\mathcal{S}C$ is the free continuous d-cone meet-semilattice over C, see (Tix, Keimel, Plotkin 2005, Theorem 4.23):

For every continuous linear map f from a continuous d-cone C to a continuous d-cone meet-semilattice S, there is a unique meet-preserving continuous linear map $f^\sharp : \mathcal{S}C \to S$ such that $f^\sharp \circ \eta_C = f$. The extension $f^\sharp$ is given by:

$$f^\sharp(X) = \bigwedge f(X)$$

The above framework therefore applies, taking K to be CCone and L to be the full subcategory of continuous d-cone meet-semilattices in ConeSL.

We will need an explicit formula for the Kleisli extension and for the multiplication of the monad $\mathcal{S}$. For this we prove a lemma:

Lemma 2 *Let C be a continuous d-cone and let $\mathcal{X}$ be a compact convex subset of $\mathcal{S}C$. Then $A =_{def} \bigcup_{X\in\mathcal{X}} X$ is a compact saturated convex subset of C.*

Proof. First A is convex, the argument being the same as that of Part (a) of Proposition 2; further, A is saturated, as all members of $\mathcal{X}$ are saturated. It remains to prove that A is compact. For this let U_i be a directed family of open sets covering A. Then, for every $X \in \mathcal{X}$, there is an index i_X such that $X \subseteq U_{i_X}$. By (Tix, Keimel, Plotkin 2005) U_{i_X} contains some compact convex saturated set Y_X which is a neighborhood of X (so $Y_X \ll_{\mathcal{S}C} X$). Thus $\Uparrow_{\mathcal{S}C} Y_X$ is a neighborhood of X in $\mathcal{S}C$. As $\mathcal{X}$ is a compact subset of $\mathcal{S}C$, there are finitely many $X_1, \ldots, X_n \in \mathcal{X}$ such that $\mathcal{X} \subseteq \Uparrow_{\mathcal{S}C} Y_{X_1} \cup \cdots \cup \Uparrow_{\mathcal{S}C} Y_{X_n}$. Thus, for all $X \in \mathcal{X}$, there is an index j such that $Y_{X_j} \ll_{\mathcal{S}C} X$. We conclude that X is in the interior of Y_{X_j} and, a fortiori, $X \subseteq U_{X_j}$. We conclude that $A \subseteq U_{X_1} \cup \cdots \cup U_{X_n}$. □

Proposition 3 *Let C and D be continuous d-cones.*

(a) *The Kleisli extension $f^\sharp : \mathcal{S}C \to \mathcal{S}D$ of a continuous linear map $f : C \to \mathcal{S}D$ is given by: $f^\sharp(X) = \bigwedge_{x\in X} f(x) = \bigcup_{x\in X} f(x)$.*

(b) *The monad multiplication is given by $\mu_C(\mathcal{X}) = \bigcup_{X\in\mathcal{X}} X$.*

Proof. (a) Let $X \in \mathcal{S}C$. We have seen before that $f^\sharp(X) = \bigwedge_{x\in X} f(x)$. Since X is compact and convex and f is continuous and linear, $\mathcal{X} = \{f(x) \mid x \in X\}$ is compact and convex. It follows from Lemma 2 that $\bigcup_{x\in X} f(x)$ is compact and convex. As it is saturated, it is a member of $\mathcal{S}D$. As this d-cone is ordered by reverse inclusion, we must therefore have $\bigwedge_{x\in X} f(x) = \bigcup_{x\in X} f(x)$ and the conclusion follows.

(b) This follows immediately from the proof of (a), using the fact that $\mu_C = (\mathrm{id}_{\mathcal{S}C})^{\#}$. □

Combining the extended probabilistic powerdomain functor $\mathcal{V}$ and $\mathcal{S}$ we obtain the *upper convex powerdomain* $\mathcal{S}\mathcal{V}(P)$ of a domain P. This is also a domain, and it is the free continuous d-cone meet-semilattice over P. We remark that $\mathcal{S}\mathcal{V}(P)$ is even the free d-cone meet-semilattice over P; the proof will appear elsewhere.

3.3. *The convex powercone and the biconvex powerdomain*

Let C be a coherent continuous d-cone. Then the *convex powercone* $\mathcal{P}C$ is formed from its *convex lenses* which, by definition, are those subsets which are nonempty intersections of a closed convex set with a compact saturated convex set. The convex powercone is ordered by the Egli-Milner ordering $X \sqsubseteq_{EM} Y$ iff $X \subseteq {\downarrow} Y$ and $\uparrow X \supseteq Y$, and addition and scalar multiplication are defined by $X +_P Y =_{def} (X+Y)^\ell$, where $Z^\ell =_{def} \overline{Z} \cap \uparrow Z$, and $r \cdot_P X =_{def} rX$. The d-cone $\mathcal{P}C$ is itself coherent and continuous and this defines the object part of a functor on CCone^c, the category of coherent continuous d-cones. Its action on morphisms is given by: $\mathcal{P}(f)(X) =_{def} f(X)^{\ell}$. We further have that $\mathcal{P}C$ has a continuous semilattice operation, given by: $X \uplus Y =_{def} (\mathrm{conv}(X \cup Y))^{\ell}$. Indeed, via the map $\eta_C : C \to \mathcal{P}C$, where $\eta_C(c) = \{c\}$, the d-cone $\mathcal{P}C$ is the free coherent continuous d-cone semilattice over C, see (Tix, Keimel, Plotkin 2005, Theorem 4.37):

For any continuous linear map f from a coherent continuous d-cone C to a coherent continuous d-cone semilattice S, there is a unique $\cup$-preserving continuous linear map $f^{\sharp} : \mathcal{P}C \to S$ such that $f^{\sharp} \circ \eta_C = f$. The above framework therefore applies, taking K to be CCone^c and L to be the full subcategory of ConeSL of all coherent continuous d-cone semilattices.

In order to find Kleisli extension formulas, we first look at the relationship between the convex powercone and the other two, beginning with the lower one. Some of this material appears already in (Tix, Keimel, Plotkin 2005) following the statement of Theorem 4.24, in particular in Lemmas 4.25 and 4.26.

By the universal property of $\mathcal{P}$, for every coherent continuous d-cone C there is a unique $\cup$-preserving, linear, continuous map $\downarrow_C : \mathcal{P}C \to \mathcal{H}C$ extending the unit $\eta_C : C \to \mathcal{H}C$, and one then has that $\downarrow_{\_}$ is a map of monads; one can show that $\downarrow_C(X) = \downarrow X$.

Lemma 3 *Let C, D be coherent continuous d-cones and let $f : C \to \mathcal{P}D$ be a linear continuous map. Then the following diagram commutes:*

$$\begin{array}{ccc} \mathcal{P}C & \xrightarrow{f^{\#}} & \mathcal{P}D \\ \downarrow_C \Big\downarrow & & \Big\downarrow \downarrow_D \\ \mathcal{H}C & \xrightarrow[(\downarrow_D \circ f)^{\#}]{} & \mathcal{H}D \end{array}$$

Proof. One shows that $\downarrow_D f^{\#}$ and $(\downarrow_D f)^{\#} \downarrow_C$ are both $\cup$-preserving linear continuous maps extending $\downarrow_D f$ along the unit, and then applies the universal property of $\mathcal{P}C$. $\square$

There is a $\cup$-preserving linear continuous map $l_C : \mathcal{H}C \to \mathcal{P}C$ in the other direction, where $l_C(X) = X$ (but $l_{\_}$ is not a natural transformation, let alone a map of monads); the proof that this map is monotonic relies on the fact that every d-cone has a least element. Note that l_C is right-inverse to $\downarrow_C$, and also that $\mathrm{id}_{\mathcal{P}C} \geq l_C \downarrow_C \supseteq \mathrm{id}_{\mathcal{P}C}$.

Turning to the relationship with the upper powercone, by the universal property of $\mathcal{P}$, for every coherent continuous d-cone C there is a unique $\cup$-preserving linear continuous map $\uparrow_C : \mathcal{P}C \to \mathcal{S}C$ extending the unit $\eta_C : C \to \mathcal{S}C$ (and one then has that $\uparrow_{\_}$ is a map of monads); one can show that $\uparrow_D(X) = \uparrow X$. We then have the following proposition whose proof is analogous to that of the preceding one.

Lemma 4 *Let C, D be coherent continuous d-cones and let $f : C \to \mathcal{P}D$ be a linear continuous map. Then the following diagram commutes:*

$$\begin{array}{ccc} \mathcal{P}C & \xrightarrow{f^{\#}} & \mathcal{P}D \\ \downarrow{\scriptstyle \uparrow_C} & & \downarrow{\scriptstyle \uparrow_D} \\ \mathcal{S}C & \xrightarrow[(\uparrow_D \circ f)^{\#}]{} & \mathcal{S}D \end{array}$$

There is a $\cup$-preserving linear continuous map $u_C : \mathcal{S}C \to \mathcal{P}C$ in the other direction, where $u_C(X) = X$ (but u_- is not a natural transformation, let alone a map of monads); the proof that this map is monotonic relies on the fact that every d-cone has a greatest element. Note that u_C is right-inverse to $\uparrow_C$, and also that $\mathrm{id}_{\mathcal{P}C} \leq u_C \uparrow_C \supseteq \mathrm{id}_{\mathcal{P}C}$.

Proposition 4 *Let C and D be coherent continuous d-cones.*

(a) *The Kleisli extension $f^\sharp : \mathcal{P}C \to \mathcal{P}D$ of a linear continuous map $f : C \to \mathcal{P}D$ is given by: $f^\sharp(X) = \biguplus_{x \in X} f(x) = (\bigcup_{x\in X} f(x))^\ell$.*

(b) *The monad multiplication is given by: $\mu_C(\mathcal{X}) = \mathrm{conv}_{\leq}(\bigcup_{X \in \mathcal{X}} X)$.*

Proof. (a) The proof consists of two calculations, relating to the lower and upper powercones respectively. First, we have:

$$\begin{array}{rcll} \downarrow f^{\#}(X) & = & \downarrow_D f^{\#}(X) & \\ & = & (\downarrow_D f)^{\#}(\downarrow_C(X)) & \text{(by Lemma 3)} \\ & = & \overline{\bigcup_{x \in \downarrow X} \downarrow f(x)} & \text{(by Proposition 2)} \\ & = & \overline{\bigcup_{x \in X} f(x)} & \end{array}$$

Second, we have:

$$\begin{array}{rcll} \uparrow f^{\#}(X) & = & \uparrow_D f^{\#}(X) & \\ & = & (\uparrow_D f)^{\#}(\uparrow_C(X)) & \text{(by Lemma 4)} \\ & = & \bigcup_{x \in \uparrow X} \uparrow f(x) & \text{(by Proposition 3)} \\ & = & \uparrow(\bigcup_{x \in X} f(x)) & \end{array}$$

Putting these together we have:

$$f^{\#}(X) = \downarrow f^{\#}(X) \cap \uparrow f^{\#}(X) = (\bigcup_{x \in X} f(x))^\ell$$

So, $(\bigcup_{x\in X} f(x))^\ell$ is the smallest convex lens containing $f(x)$ for all $x \in X$, whence it equals $\biguplus_{x \in X} f(x)$.

(b) As $\mu_C = (\mathrm{id}_{\mathcal{P}C})^{\#}$, we have:

$$\begin{array}{rcll} \downarrow\mu_C(\mathcal{X}) & = & \overline{\bigcup_{X\in\downarrow\mathcal{X}}\downarrow X} & \text{(following the proof of part (a))} \\ & = & \bigcup_{X\in\downarrow\mathcal{X}}\downarrow X & \text{(by Lemma 1)} \\ & = & \downarrow(\bigcup_{X\in\mathcal{X}} X) & \end{array}$$

As we also have $\uparrow\mu_C(\mathcal{X}) = \uparrow(\bigcup_{X\in\mathcal{X}} X)$ following the proof of part (a), the conclusion follows. □

Combining the extended probabilistic powerdomain functor $\mathcal{V}$ and $\mathcal{P}$ we obtain the *biconvex powerdomain* $\mathcal{PV}(P)$ of a coherent domain P. This is also a coherent domain, and it is the free coherent continuous d-cone semilattice over P. We remark that $\mathcal{PV}(P)$ is even the free d-cone semilattice over P; the proof will appear elsewhere.

4. Functional representations

We begin with some generalities on functional representations and then consider the three powercones: lower, upper and convex. To this end, we return to the framework of Section 3. Let C and D be d-cones in K. We can define a map:

$$\Lambda : TC \longrightarrow TD^{[C,TD]}$$

by:

$$\Lambda_\gamma = f \mapsto f^{\#}(\gamma)$$

The continuity and homogeneity of each Λ_γ is assured by Proposition 1. Further, Λ itself is a morphism of d-cone semilattices as every $f^\sharp$ is. We regard Λ_γ as the *functional representation map*, relative to the choice of D. Assuming that $\overline{\mathbb{R}}_+$ is an object of K, $\overline{\mathbb{R}}_+$ is the natural *standard* choice for D.

We assume that, with respect to the order $\subseteq$ on TD, the set $\{f(x) \mid \eta_C(x) \subseteq \gamma\}$ always has a least upper bound $\bigcup_{\eta_C(x)\subseteq\gamma} f(x)$ and that:

$$\text{(U)} \qquad \Lambda_\gamma(f) = f^{\#}(\gamma) = \bigcup_{\eta_C(x)\subseteq\gamma} f(x)$$

This condition implies that Λ_γ is $\subseteq$-monotonic, and Propositions 2, 3, and 4 assure us that it is satisfied in our three special cases. The formula looks even simpler in these cases as then the elements γ of TD are subsets of D and $\eta_C(x) \subseteq \gamma$ iff $x \in \gamma$.

For every x, the evaluation map $f \mapsto f(x) \colon [C, TD] \to TD$ is linear and $\subseteq$-monotonic. (Recall that the d-cone $[C, TD]$ carries the pointwise defined partial order $\subseteq$.) Thus, formula (U) above shows:

Proposition 5 *Under the above hypotheses, Λ_γ is the pointwise $\bigcup$ of the continuous $\subseteq$-monotonic linear maps $f \mapsto f(x)$, $\eta_C(x) \subseteq \gamma$.*

Corollary 1 *Each Λ_γ is continuous, $\subseteq$-monotonic, and $\subseteq$-sublinear.*

Proof. We already know that Λ_γ is continuous, $\subseteq$-monotonic and homogeneous. We now show $\subseteq$-subadditivity. As, for every x, we have:

$$(f+g)(x) = f(x) + g(x) \subseteq \bigcup_{\eta_C(x)\subseteq\gamma} f(x) + \bigcup_{\eta_C(x)\subseteq\gamma} g(x) = \Lambda_\gamma(f) + \Lambda_\gamma(g)$$

we conclude, using Proposition 5, that:

$$\Lambda_\gamma(f+g) = \bigcup_{\eta_C(x)\subseteq\gamma} (f+g)(x) \subseteq \Lambda_\gamma(f) + \Lambda_\gamma(g)$$

□

4.1. *The lower powercone*

Here the monad T is:

$$\mathcal{H}\colon \mathsf{Cone} \to \mathsf{Cone}$$

and we can simplify the standard representation a little. The free d-cone join-semilattice over $\overline{\mathbb{R}}_+$ is $\overline{\mathbb{R}}_+$ itself with the usual supremum as semilattice operation. So, using $\overline{\mathbb{R}}_+$ in place of the standard choice $\mathcal{H}\overline{\mathbb{R}}_+$, we obtain an equivalent functional representation $\Lambda\colon \mathcal{H}C \longrightarrow \overline{\mathbb{R}}_+^{C^*}$ where, by Proposition 2:

$$\Lambda_X(f) = \sup_{x\in X} f(x)$$

We see that each Λ_X is the pointwise supremum of continuous linear functionals, hence continuous and sublinear. Since $\subseteq$ and $\leq$ coincide in the case of d-cone join-semilattices, this can be viewed as a special case of Proposition 5 and its Corollary.

4.2. *The upper powercone*

Here the monad T is:

$$\mathcal{S}\colon \mathsf{CCone} \to \mathsf{CCone}$$

and we can again simplify the standard representation a little. The free d-cone meet-semilattice over $\overline{\mathbb{R}}_+$ is $\overline{\mathbb{R}}_+$ itself with the usual infimum as semilattice operation. So, we obtain a functional representation $\Lambda\colon \mathcal{S}C \longrightarrow \overline{\mathbb{R}}_+^{C^*}$ equivalent to the standard one where by Proposition 3:

$$\Lambda_X(f) = \inf_{x\in X} f(x)$$

We see that each Λ_X is the pointwise infimum of continuous linear functionals, hence continuous and superlinear. Since $\subseteq$ and $\geq$ coincide in the case of d-cone meet-semilattices, this can also be viewed as a special case of Proposition 5 and its Corollary.

4.3. *The convex powercone*

Here the monad T is:

$$\mathcal{P}\colon \mathsf{CCone}^c \longrightarrow \mathsf{CCone}^c$$

and with $S = \mathcal{P}\overline{\mathbb{R}}_+$ we have the standard representation:

$$\Lambda\colon \mathcal{P}C \longrightarrow \mathcal{P}\overline{\mathbb{R}}_+^{[C,\mathcal{P}\overline{\mathbb{R}}_+]}$$

where, by Proposition 4:

$$\Lambda_X(f) = \biguplus_{x\in X} f(x) = (\bigcup_{x\in X} f(x))^\ell$$

From Proposition 5 and its Corollary we know that each Λ_X is the pointwise $\biguplus$ of continuous $\subseteq$-monotonic linear maps, hence continuous, $\subseteq$-monotonic and $\subseteq$-sublinear.

To be more specific we recall that $\mathcal{P}\overline{\mathbb{R}}_+$ is the collection of all closed intervals $a = [\,\underline{a}, \overline{a}\,]$, $\underline{a} \leq \overline{a}$, in $\overline{\mathbb{R}}_+$. The cone operations on $\mathcal{P}\overline{\mathbb{R}}_+$ are:

$$\begin{aligned} [\,\underline{a}, \overline{a}\,] + [\,\underline{b}, \overline{b}\,] &= [\,\underline{a} + \underline{b}, \overline{a} + \overline{b}\,] \\ r[\,\underline{a}, \overline{a}\,] &= [r\underline{a}, r\overline{a}\,] \end{aligned}$$

and the Egli-Milner order is given by:

$$[\,\underline{a}, \overline{a}\,] \sqsubseteq_{EM} [\,\underline{b}, \overline{b}\,] \iff \underline{a} \leq \underline{b},\ \overline{a} \leq \overline{b}$$

The semilattice operation $\uplus$ gives the convex hull of two intervals and the associated order is subset inclusion:

$$\begin{aligned} [\,\underline{a}, \overline{a}\,] \uplus [\,\underline{b}, \overline{b}\,] &= [\,\underline{a} \wedge \underline{b}, \overline{a} \vee \overline{b}\,] \\ [\,\underline{a}, \overline{a}\,] \subseteq [\,\underline{b}, \overline{b}\,] &\iff \underline{b} \leq \underline{a} \leq \overline{a} \leq \overline{b} \end{aligned}$$

We need a few notations and facts about maps into $\mathcal{P}\overline{\mathbb{R}}_+$. Let D be a d-cone the elements of which will be denoted by f, f', etc. For a function $F\colon D \to \mathcal{P}\overline{\mathbb{R}}_+$, the image of any $f \in D$ is an interval $F(f) = [\,\underline{F(f)},\ \overline{F(f)}\,]$; picking the endpoints of these intervals, we obtain a pair of functions $\underline{F}, \overline{F}\colon D \to \overline{\mathbb{R}}_+$ such that $F(f) = [\,\underline{F}(f), \overline{F}(f)]$. Thus, the functions $F\colon D \to \mathcal{P}\overline{\mathbb{R}}_+$ correspond, in a one-to-one way, to pairs of functions $\underline{F}, \overline{F}\colon D \to \mathcal{P}\overline{\mathbb{R}}_+$ with $\underline{F} \leq \overline{F}$. We employ the notation $F = [\underline{F}, \overline{F}]$ and observe:

Remark 1 (1) *The map F is continuous and linear, respectively, if and only if both $\underline{F}$ and $\overline{F}$ are. Thus, we have the following d-cone isomorphisms and inclusions:*

$$\begin{aligned} (\mathcal{P}\overline{\mathbb{R}}_+)^D &\cong \{[\underline{F}, \overline{F}] \mid \underline{F} \leq \overline{F}\} \subseteq \mathcal{L}(D) \times \mathcal{L}(D) \\ [D, \mathcal{P}\overline{\mathbb{R}}_+] &\cong \{[\underline{F}, \overline{F}] \mid \underline{F} \leq \overline{F}\} \subseteq D^* \times D^* \end{aligned}$$

(2) *F is $\subseteq$-sublinear if and only if $\underline{F}$ is superlinear and $\overline{F}$ is sublinear.*

(3) *If F is the pointwise $\biguplus$ of linear maps $F_i = [\underline{F}_i, \overline{F}_i]\colon D \to \mathcal{P}\overline{\mathbb{R}}_+$, then:*

$$\underline{F}(f) = \inf_i \underline{F}_i(f) \text{ and } \overline{F}(f) = \sup_i \overline{F}_i(f)$$

and the following holds:

$$(*) \qquad \underline{F}(f+f') \le \underline{F}(f) + \overline{F}(f') \le \overline{F}(f+f')$$

Proof. These assertions are all straightforward except for condition (*) which we now verify. The linearity of $\underline{F}_i$ yields the first inequality in condition (*):

$$\begin{aligned}
\underline{F}(f+f') &= \inf_i \underline{F}_i(f+f') \\
&= \inf_i \left(\underline{F}_i(f) + \underline{F}_i(f')\right) \\
&\le \inf_i \left(\underline{F}_i(f) + \sup_i \overline{F}_i(f')\right) \\
&= \underline{F}(f) + \overline{F}(f')
\end{aligned}$$

The second inequality is proved similarly. □

We will say that a $\subseteq$-sublinear map $F\colon D \to \mathcal{P}\overline{\mathbb{R}}_+$ or, equivalently, a pair $\underline{F}, \overline{F}\colon D \to \overline{\mathbb{R}}_+$ of superlinear and sublinear maps, respectively, is *canonical*, if it satisfies condition (*). Note that the second inequality in condition (*) implies that $\underline{F} \le \overline{F}$ (consider the case $f' = 0$).

We apply these considerations to the case where $D = [C, \mathcal{P}\overline{\mathbb{R}}_+]$ for a coherent continuous d-cone C and $F = \Lambda_X$. As above, the functions $f : C \to \mathcal{P}\overline{\mathbb{R}}_+$ correspond in a one-to-one way to the pairs of functions $\underline{f}, \overline{f} : C \to \overline{\mathbb{R}}_+$ with $\underline{f} \le \overline{f}$, the correspondence being given by $f(x) = [\underline{f}(x), \overline{f}(x)]$ and, as before, we use the notation $f = [\underline{f}, \overline{f}]$. If $f \in [C, \mathcal{P}\overline{\mathbb{R}}_+]$, that is, if f is continuous and linear, $\underline{f}$ and $\overline{f}$ are too, that is, $\underline{f}, \overline{f} \in C^*$. In our general considerations we have seen that, for every $X \in \mathcal{P}C$, the map $\Lambda_X\colon [C, \mathcal{P}\overline{\mathbb{R}}_+] \to \mathcal{P}\overline{\mathbb{R}}_+$ is continuous, $\subseteq$-monotonic, and pointwise the $\biguplus$ of the linear maps $f \mapsto f(x), x \in X$, hence $\subseteq$-sublinear. Together with the previous remark this yields the following:

Proposition 6 *For every $X \in \mathcal{P}C$, the functional $\Lambda_X\colon [C, \mathcal{P}\overline{\mathbb{R}}_+] \to \mathcal{P}\overline{\mathbb{R}}_+$ representing X is continuous, $\subseteq$-monotonic, and canonically $\subseteq$-sublinear; equivalently, the functionals $\underline{\Lambda}_X, \overline{\Lambda}_X\colon [C, \mathcal{P}\overline{\mathbb{R}}_+] \to \overline{\mathbb{R}}_+$ are, respectively, superlinear and sublinear and satisfy condition (*) for all $f, f' \in [C, \mathcal{P}\overline{\mathbb{R}}_+]$. Moreover:*

$$\underline{\Lambda}_X(f) = \inf_{x \in X} \underline{f}(x) \text{ and } \overline{\Lambda}_X(f) = \sup_{x \in X} \overline{f}(x)$$

4.4. *The diagonal representation*

We now exhibit another functional representation $\Lambda'\colon \mathcal{P}C \to \mathcal{P}\overline{\mathbb{R}}_+^{C^*}$, where C is a continuous coherent d-cone, and C^* is its dual. As in the previous subsection, the functions

$f : D \to \mathcal{P}\overline{\mathbb{R}}_+$ correspond in a one-to-one way to the pairs of functions $\underline{f}$, $\overline{f} : D \to \overline{\mathbb{R}}_+$ with $\underline{f} \leq \overline{f}$, the correspondence being given by $f(x) = [\underline{f}(x), \overline{f}(x)]$ and, as before, we use the notation $f = [\underline{f}, \overline{f}]$.

We restrict every continuous map $F : [C, \mathcal{P}\overline{\mathbb{R}}_+] \to \mathcal{P}\overline{\mathbb{R}}_+$ to the 'diagonal' of the f in $[C, \mathcal{P}\overline{\mathbb{R}}_+]$ with $\underline{f} = \overline{f}$. Better: we compose every F with the linear continuous map $\Delta_C : \big(g \mapsto [g, g]\big) : C^* \to [C, \mathcal{P}\overline{\mathbb{R}}_+]$, thereby obtaining a d-cone semilattice morphism $R_C : \mathcal{P}\overline{\mathbb{R}}_+^{[C,\mathcal{P}\overline{\mathbb{R}}_+]} \to \mathcal{P}\overline{\mathbb{R}}_+^{C^*}$ where:

$$R_C(F) = F \circ \Delta_C$$

which assigns to $F = [\,\underline{F}\,, \overline{F}\,]$ the pair $F' = [\underline{F'}, \overline{F'}]$ defined by $\underline{F'}(g) = \underline{F}[g, g]$ and $\overline{F'}(g) = \overline{F}[g, g]$. It is crucial that, if F is $\subseteq$-monotonic, then it is already completely determined by F', even more:

Lemma 5 (1) *Let $F : [C, \mathcal{P}\overline{\mathbb{R}}_+] \to \mathcal{P}\overline{\mathbb{R}}_+$ be continuous and $\subseteq$-monotonic. Then:*

$$F(f) = F[\underline{f}, \overline{f}] = [\underline{F}[\underline{f}, \underline{f}], \overline{F}[\overline{f}, \overline{f}]] = [\underline{F'}(\underline{f}), \overline{F'}(\overline{f})]$$

(2) *The map R_C restricts to a d-cone semilattice isomorphism between the sub-d-cone semilattice of the $\subseteq$-monotonic functionals in $\mathcal{P}\overline{\mathbb{R}}_+^{[C,\mathcal{P}\overline{\mathbb{R}}_+]}$ and $\mathcal{P}\overline{\mathbb{R}}_+^{C^*}$. Its inverse is given by: $R_C^{-1}(F')(f) = \big[\underline{F'}(\underline{f}), \overline{F'}(\overline{f})\big]$.*

Proof. (1) Let $g = \underline{f}$ and $h = \overline{f}$. As $g \leq h$, we have $[g, g] \leq [g, h] \leq [h, h]$, so as F is monotonic it follows that:

$$\underline{F}[g, g] \leq \underline{F}[g, h] \;\text{ and }\; \overline{F}[g, h] \leq \overline{F}[h, h]$$

We also have $[g, g] \subseteq [g, h]$ and $[h, h] \subseteq [g, h]$, so as F is $\subseteq$-monotonic, it follows that:

$$\underline{F}[g, g] \geq \underline{F}[g, h] \;\text{ and }\; \overline{F}[h, h] \leq \overline{F}[g, h]$$

and the conclusion follows.

(2) As $R_C^{-1}[G, H]$ is, evidently, $\subseteq$-monotonic and R_C^{-1} is continuous it is only necessary to prove R_C and R_C^{-1} are inverses. To show that R_C is the right inverse of R_C^{-1}, we calculate:

$$\begin{array}{rcll} R_C^{-1}(R_C(F))(f) & = & R_C^{-1}\big(g \mapsto [\underline{F}[g, g], \overline{F}[g, g]]\big)(f) & \\ & = & [\underline{F}[\underline{f}, \underline{f}], \overline{F}[\overline{f}, \overline{f}]] & \\ & = & F(f) & \text{(by Part 1)} \end{array}$$

The proof that it is the left inverse is similar but does not require the use of Part 1. □

We apply the above to our functional representation Λ and we obtain the *diagonal* representation $\Lambda' : \mathcal{P}C \to \mathcal{P}\overline{\mathbb{R}}_+^{C^*}$ given by:

$$\Lambda'_X(g) = \big[\underline{\Lambda}_X[g, g], \overline{\Lambda}_X[g, g]\big]$$

for all $g \in C^*$. Via Lemma 5, Λ' inherits from Λ the properties of being continuous, linear and $\uplus$-preserving. From Proposition 6 we obtain:

Proposition 7 *For every $X \in \mathcal{P}C$, the functional $\Lambda'_X : C^* \to \mathcal{P}\overline{\mathbb{R}}_+$ representing X is continuous and canonically $\subseteq$-sublinear, that is, the functionals $\underline{\Lambda'}_X$, $\overline{\Lambda'}_X : C^* \to \overline{\mathbb{R}}_+$ are superlinear and sublinear, respectively, and they satisfy condition* (*). *Moreover:*

$$\underline{\Lambda'}_X(g) = \inf_{x \in X} g(x) \text{ and } \overline{\Lambda'}_X(g) = \sup_{x \in X} g(x)$$

As Λ_X is $\subseteq$-monotonic, Lemma 5 allows us to recover Λ from Λ', as follows:

$$\Lambda_X(f) = \left[\underline{\Lambda'}_X(\underline{f}), \overline{\Lambda'}_X(\overline{f})\right]$$

Thus the diagonal representation can be considered to be equivalent to the standard one, Λ. The last part of Proposition 7 shows that it combines the functional representations of the lower and the upper powercones $\mathcal{H}C$ and $\mathcal{S}C$. For $g \in C^*$ one has indeed:

$$\begin{aligned} \Lambda'_X(g) &= [\inf_{x \in X} g(x), \sup_{x \in X} g(x)] \\ &= [\inf_{x \in \uparrow X} g(x), \sup_{x \in \downarrow X} g(x)] \\ &= [\Lambda_{\uparrow X}(g), \Lambda_{\downarrow X}(g)] \end{aligned}$$

How should we view condition (*)? Note first that it implies that $G \leq H$ (choose $g = 0$). Suppose you have a concave function G and a convex function H of one real variable with $G \leq H$. The above condition expresses that from every point on one the two curves you can see every point of the other curve; in other words: if we draw the line segment from a point on the lower curve to a point on the upper curve, then this line segment lies between the two curves. Our interest in condition (*) stems from the fact that a $\subseteq$-sublinear map from a continuous d-cone with an additive way-below relation to $\mathcal{P}\overline{\mathbb{R}}_+$ is pointwise the $\biguplus$ of continuous linear maps if and only if it is canonical as we shall see at the end of the next section.

5. Continuous Sublinear and Superlinear Functionals

In order to characterise the functionals representing the objects constituting our three powercones we need the following information about sublinear and superlinear functionals:

Main Lemma 1 *Let $G, H \colon D \to \overline{\mathbb{R}}_+$ be continuous superlinear and sublinear functionals, respectively, on a continuous d-cone D. Let A and B be the sets of all linear continuous functionals f on D with $f \leq H$ and $G \leq f$, respectively. We then have:*

(1) $$H(x) = \sup\{f(x) | f \in A\} \text{ for all } x \in D$$

(2) $$G(x) = \inf\{f(x) | f \in B\} \text{ for all } x \in D$$

Suppose in addition that the way-below relation is additive on D and that the following condition is satisfied:

(∗) $$G(u + v) \leq G(u) + H(v) \leq H(u + v) \text{ for all } u, v \in D$$

Then, with $L = A \cap B$, we have:

(3) $$H(x) = \sup\{f(x) | f \in L\} \text{ and } G(x) = \inf\{f(x) | f \in L\} \text{ for all } x \in D$$

For the proof of the Theorem we need some lemmas.

Lemma 6 *Let b be an element in an arbitrary d-cone D. Then $P\colon D \to \overline{\mathbb{R}}_+$ defined by:*

$$P(x) = \inf\{r | rb \geq x\}$$

is a sublinear continuous functional with $P(b) \leq 1$.

Proof. Clearly $P(rx) = rP(x)$. For all $r > P(x)$ and all $s > P(y)$ one has $rb \geq x$ and $sb \geq y$, whence $(r + s)b \geq x + y$, and consequently $P(x) + P(y) \geq P(x + y)$. Thus, P is sublinear. Clearly, P is monotonic. For continuity, let $x = \sup_i x_i$ for a directed family (x_i). Choose any $r > \sup_i P(x_i)$. Then $rb \geq x_i$ for all i, whence $rb \geq \sup x_i = x$ and we conclude that $r \geq P(x)$. We conclude that $\sup_i P(x_i) \geq P(x)$. As the converse inequality follows from monotonicity, continuity is proved. □

Lemma 7 *Let P and H be sublinear functionals on a d-cone D. By defining $J\colon D \to \overline{\mathbb{R}}_+$ by:*

$$J(x) = \inf\{P(y) + H(z) | x \leq y + z\}$$

one obtains the greatest monotonic sublinear functional minorizing P and H.

Proof. Clearly J is monotonic and $J(rx) = rJ(x)$. To prove $J(x + x') \leq J(x) + J(x')$, choose arbitrary $r > J(x)$ and $r' > J(x')$. Then there are $y, z \in D$ such that $y + z \geq x$ and $r \geq P(y) + H(z)$ and there are $y', z' \in D$ such that $y' + z' \geq x'$ and $r' \geq P(y') + H(z')$. We conclude that $y + y' + z + z' \geq x + x'$ and, using the sublinearity of H and P, $r + r' \geq P(y) + P(y') + H(y) + H(y') \geq P(y + y') + H(y + y')$. We conclude that $J(x) + J(x') \geq J(x + x')$.

Clearly J is below H and P. Now, let E be any monotonic sublinear functional minorizing H and P. For all y, z such that $y + z \geq x$, we then have:

$$P(y) + H(z) \geq E(y) + E(z) \geq E(y + z) \geq E(x)$$

We conclude that $J(x) \geq E(x)$. □

Lemma 8 *Let a, b be elements of a continuous d-cone C with $a \ll b$. Then there is a continuous superlinear functional $Q\colon C \to \mathbb{R}_+$ such that $Q(b) \geq 1$ and $Q(x)a \leq x$ for all $x \in C$.*

Proof. By local convexity there is a convex open neighbourhood V of b contained in $\Uparrow a$. We look at the Minkowski functional of V:

$$Q(x) = \sup\{r > 0 \mid x \in rV\}$$

Clearly $Q(b) \geq 1$ as $b \in 1 \cdot V$. Consider any x. Whenever $0 < r < Q(x)$ we have $x \in rV \subseteq r\Uparrow a$, whence $ra \ll x$. So:

$$Q(x)a = \sup\{r \mid 0 < r < Q(x)\} \cdot a = \sup\{ra \mid 0 < r < Q(x)\} \leq x$$

Thus, the two inequalities are established. It is straightforward that Q is homogeneous. For the proof of superlinearity we use that the equality $rV + sV = (r+s)V$ holds for convex sets V:

$$\begin{aligned}
Q(x) + Q(y) &= \sup\{r > 0 \mid x \in rV\} + \sup\{s > 0 \mid x \in sV\} \\
&= \sup\{r + s \mid r, s > 0, x \in rV, y \in sV\} \\
&\leq \sup\{r + s \mid r, s > 0, x + y \in rV + sV = (r+s)V\} \\
&= \sup\{t > 0 \mid x + y \in tV\} \\
&= Q(x + y)
\end{aligned}$$

In order to prove the continuity of Q in x, choose an r such that $0 < r < Q(x)$. Then x is in the open set rV. Thus, there is a $y \in rV$ such that $y \ll x$ which implies $r \leq Q(y)$. □

Lemma 9 *Let G and Q be monotonic superlinear functionals on a d-cone D. Then $E\colon D \to \overline{\mathbb{R}}_+$ defined by:*

$$E(x) = \sup\{G(y) + Q(z) \mid y + z \leq x\}$$

is the least monotonic superlinear functional majorizing G and Q. If D is a continuous d-cone with an additive way-below relation and if G and Q are continuous, then E is continuous, too.

Proof. Clearly E majorizes G and Q. Let E' be any monotonic superlinear functional majorizing G and Q. Let $x \in D$. For arbitrary y, z with $y + z \leq x$, we then have $E'(x) \geq E'(y + z) \geq E'(y) + E'(z) \geq G(y) + Q(z)$. It follows that $E'(x) \geq E(x)$.

It is clear that $E(rx) = rE(x)$. For superadditivity, chose any $r < E(x)$ and any $r' < E(x')$. Then there are elements y, z, y', z' such that $y + z \leq x, y' + z' \leq x'$ and $r < G(y) + Q(z), r' < G(y') + Q(z')$. We conclude that $y + y' + z + z' \leq x + x'$ and that

$r + r' < G(y) + G(y') + Q(z) + Q(z') \le G(y + y') + Q(z + z')$, whence $r + r' \le E(x + x')$. We therefore have that $E(x) + E(x') \le E(x + x')$.

Clearly, E is monotonic. Suppose now that C is a continuous d-cone with an additive way-below relation. For continuity of E, suppose that x is the supremum of a directed family (x_i). Consider any $r < E(x)$. Then there are y, z such that $y + z \le x$ and $r < G(y) + Q(z)$. If G and Q are continuous, we may find $y' \ll y$ and $z' \ll z$ such that $r \le G(y') + Q(z')$. By additivity of the way-below relation, $y' + z' \ll y + z \le x = \sup_i x_i$. Thus $y' + z' \le x_i$ for some i. We conclude that $r \le E(x_i)$ for some i. As this holds for every $r < E(x)$, we conclude that $E(x) \le \sup_i E(x_i)$. As the converse inequality follows from the monotonicity of E, continuity of E is proved. □

For the proof of the Main Lemma 1 we consider continuous superlinear and sublinear functionals G and H satisfying condition (*), which implies $G \le H$, on a continuous d-cone D. As in the Theorem, denote by L the set of all linear continuous functionals f on D such that $G \le f \le H$. Choose any $b \in D$. If $G(b) = H(b)$, then there is a linear functional $f \in L$ with $G(b) = f(b) = H(b)$ by the Sandwich Theorem of (Tix, Keimel, Plotkin 2005, 3.2). So suppose henceforward that $G(b) < H(b)$. Let r be any real number such that $G(b) < r < H(b)$. Claim (3) of the Theorem is a direct consequence of the following two lemmas, claim (2) follows from the first one, with H the functional having value $+\infty$ except at 0, and claim (1) from the second,with G being the zero functional.

Lemma 10 *There is a linear continuous functional $f \in L$ such that $r \le f(b)$ provided that the way-below relation is additive on D. The latter hypothesis is superfluous, if G is the zero functional (and $A = L$).*

Proof. Without loss of generality we may suppose $r = 1$. By the continuity of H and the continuity of D, there is an $a \ll b$ such that that $1 < H(a)$. For these elements a and b form the continuous superlinear functional Q as in Lemma 8 which satisfies $Q(b) \ge 1$ and $Q(z)a \le z$ for all z. For G and this Q, we now form the continuous superlinear functional E as in Lemma 9. We clearly have then $G \le E$ and $1 \le E(b)$. We further prove that $E \le H$. For every $z \in D$, we have indeed $Q(z)a \le z$. We deduce $Q(z)H(a) = H(Q(z)a) \le H(z)$. At the other hand, $Q(z) \le Q(z)H(a)$, as $1 \le H(a)$. Thus $Q(z) \le H(z)$ for every z. For arbitrary elements y, z with $y + z \le x$ we now have:

$$G(y) + Q(z) \le G(y) + H(z) \le H(y + z) \le H(x)$$

where we have used hypothesis (*) for the inequality in the middle. We conclude that $E(x) \le H(x)$.

We now can apply the Sandwich Theorem of (Tix, Keimel, Plotkin 2005, 3.2) to E and H and we find a linear continuous functional f in between. It has the desired properties.

The additivity of the way-below relation is only needed in this proof, when we use Lemma 9. If G is the zero functional, we do not need this lemma, as we may choose $E = Q$. □

Lemma 11 *There is a linear continuous functional $f \in L$ such that $f(b) \leq r$.*

Proof. Without loss of generality we may suppose again $r = 1$. For the given b we first form the continuous sublinear functional P as in Lemma 6. We first show that $G(x) \leq P(x)$ for all x. For every $r > P(x)$ we have indeed $rb \geq x$, hence $G(rb) \geq G(x)$; as $G(b) \leq 1$ we conclude that $r \geq rG(b) = G(rb) \geq G(x)$.

We then form the sublinear functional J as in Lemma 7(a). We have $J \leq H$ and $J \leq P$, whence $J(b) \leq P(b) \leq 1$. For all x and all y, z such that $x \leq y + z$ we have:

$$G(x) \leq G(y + z) \leq G(y) + H(z) \leq P(y) + H(z)$$

where we have used hypothesis (*) for the inequality in the middle. We conclude that $G(x) \leq J(x)$ by the definition of J.

We now can apply the Sandwich Theorem, (Tix, Keimel, Plotkin 2005, 3.2), to G and J and we find a linear continuous functional f in between them; it has the desired properties. □

The following Theorem is crucial. In particular, part (3) helps clarify the the significance of the strange-looking condition (*) used to define the canonically $\subseteq$-sublinear maps.

Theorem 1 *Let D be a continuous d-cone, whose way-below relation is additionally assumed to be additive for (2) and (3) below.*

(1) *A functional $H : D \to \overline{\mathbb{R}}_+$ is continuous and sublinear if and only if it is pointwise the supremum of continuous linear functionals, i.e., $H(x) = \sup_{f \in A} f(x)$ for some subset $A \subseteq D^*$. We may choose $A = \{f \in D^* \mid f \leq H\}$ which is convex and weak* Scott-closed in D^*.*

(2) *A functional $G : D \to \overline{\mathbb{R}}_+$ is continuous and superlinear if and only if it is pointwise the infimum of a weak* Scott-compact set of continuous linear functionals, i.e., $G(x) = \inf_{f \in B} f(x)$ for some weak* Scott-compact subset $B \subseteq D^*$. We may choose $B = \{f \in D^* \mid G \leq f\}$ which is convex, saturated, and weak* Scott-compact.*

(3) *A map $F : D \to \mathcal{P}\overline{\mathbb{R}}_+$ is continuous and canonically $\subseteq$-sublinear if and only if it is pointwise the $\biguplus$ of a weak* Scott-compact set of continuous linear maps $f : D \to \mathcal{P}\overline{\mathbb{R}}_+$, equivalently, $F(x) = \biguplus_{f \in L} [f(x), f(x)]$ for some weak* Scott-compact subset $L \subseteq D^*$. The set L may be chosen to be the intersection of the weak* Scott-closed convex subset $A = \{f \in D^* \mid f \leq \overline{F}\}$ and the weak* Scott-compact convex saturated subset $B = \{f \in D^* \mid \underline{F} \leq f\}$.*

Proof. (1) The pointwise supremum $H(x) = \sup_{f \in A} f(x)$ of any set A of continuous functions $f : D \to \overline{\mathbb{R}}_+$ is continuous. If all $f \in A$ are linear, then the pointwise supremum is sublinear. Conversely, if H is continuous and sublinear, then it is the pointwise supremum of the set A of those $f \in D^*$ with $f \leq H$ by part 1 of the Main Lemma. Clearly A is weak* Scott closed and convex in D^*.

(2) The pointwise infimum of a set B of linear functionals is superlinear. In general, when the functionals $f \in B$ are all continuous, the pointwise infimum need not be continuous. But if B is weak* Scott-compact in D^*, then this is true. Indeed, the map $(f, x) \mapsto f(x) : D^* \times D \to \overline{\mathbb{R}}_+$ is separately continuous in each argument, where on D^* we take the weak* Scott topology. As D is continuous, this map is automatically jointly continuous. Corollary (9) in (Keimel and Gierz 1982) tells us that, if X is a T_0-space and Y a locally compact space then, for every continuous map g from $X \times Y$ into a continuous lattice and every compact subset B of X, the pointwise infimum $\inf_{x \in B} g(x, y)$ is continuous on Y. This allows to conclude that, for each weak*-compact subset $B \subseteq D^*$, the pointwise infimum is continuous.

Conversely, when G is a superlinear continuous functional on C, then, by part 2 of the Main Lemma, it is pointwise the infimum of the set B of continuous linear functionals $f \geq G$. Clearly, B is convex and saturated. As we suppose here that the way-below relation is additive on C, Corollary 2 of the Banach-Alaoglu Theorem in (Plotkin 2006) yields that B is weak* Scott-compact.

(3) We first have make it clear, what we mean by the weak* Scott topology on the d-cone of continuous linear maps f from D to $\mathcal{P}\overline{\mathbb{R}}_+$: it is the weakest topology making the evaluations $f \mapsto f(x)$ continuous for all $x \in D$. This is equivalent to requiring that all the maps $f \mapsto \underline{f}(x)$ and $f \mapsto \overline{f}(x)$ are continuous.

Let $F(x) = \biguplus_{f \in L} f(x)$ for some set L of continuous linear maps $f : D \to \mathcal{P}\overline{\mathbb{R}}_+$. By Remark 1, we know that F is canonically $\subseteq$-sublinear and that $\underline{F}(x) = \inf_{f \in L} \underline{f}(x)$ and that $\overline{F}(x) = \sup_{f \in L} \overline{f}(x)$. Then $\overline{F}$ is continuous as the the pointwise supremum of the continuous linear functionals $\overline{f}, f \in L$ (see item (1)). If L is weak* Scott-compact, then $\underline{F}$ is continuous as the pointwise infimum of the weak* Scott-compact set of continuous linear functionals $\underline{f}, f \in L$ (see item (2)). It follows that $F = [\underline{F}, \overline{F}]$ is also continuous.

Conversely, let $F : D \to \mathcal{P}\overline{\mathbb{R}}_+$ be a canonical $\subseteq$-sublinear continuous map. The set A of all $g \in D^*$ with $g \leq \overline{F}$ is weak* Scott-closed and convex in D^* by item (1) and the set B of all $g \in D^*$ with $g \geq \underline{F}$ is weak* Scott-compact, convex and saturated by item (2). By part 3 of the Main Lemma, $\underline{F}(x) = \inf_{g \in L} g(x)$ and $\overline{F}(x) = \sup_{g \in L} f(x)$ and consequently $F(x) = [\underline{F}(x), \overline{F}(x)] = [\inf_{g \in L} f(x), \sup_{g \in L} g(x)]$. Thus F is pointwise the $\biguplus$ of the linear maps $[g, g] : D \to \mathcal{P}\overline{\mathbb{R}}_+$ with $g \in L$. $\square$

6. The Functional Representation Theorems

We are going to characterise the functionals Λ_X for our three types of powercones. In all cases we have to restrict ourselves to reflexive continuous d-cones. This strong hypothesis is satisfied by our main example, the extended probabilistic powercone $\mathcal{V}(X)$ of all continuous valuations on a domain X.

6.1. *The lower powercone*

From Section 4 we know that, for every d-cone C, the representation function:

$$\Lambda\colon \mathcal{H}C \to \overline{\mathbb{R}}_+{}^{C^*}$$

is a morphism of d-cone join-semilattices that transforms every closed convex subset X of C into the continuous sublinear functional $\Lambda_X\colon C^* \to \overline{\mathbb{R}}_+$ defined by:

$$\Lambda_X(f) = \sup_{x\in X} f(x)\,.$$

We want to show that, under appropriate additional hypotheses, the sublinear continuous functionals on C^* form a sub-d-cone join-semilattice of the d-cone join-semilattice of all the continuous functionals on C^* and, further, that Λ is then a d-cone join-semilattice isomorphism of $\mathcal{H}C$ and the sublinear continuous functionals. This will follow from the above remarks if we can show that Λ is an order-embedding and that its range includes all the sublinear continuous functionals.

Proposition 8 *For a continuous d-cone C, the map Λ is an order embedding, that is, for $X, Y \in \mathcal{H}C$, we have $\Lambda_X \leq \Lambda_Y$ if and only if $X \subseteq Y$.*

Proof. As Λ is monotonic by the general considerations in Section 4, it remains to show that, if $\Lambda_X \leq \Lambda_Y$, then $X \subseteq Y$. For this, we suppose $X \not\subseteq Y$. Choose an element $a \in X \setminus Y$. As Y is closed and as a continuous d-cone is locally convex, there is a convex open neighbourhood U of a disjoint from Y. By the Separation Theorem of (Tix, Keimel, Plotkin 2005) there is a linear continuous functional $f\colon C \to \overline{\mathbb{R}}_+$ such that $f(a) > 1$, but $f(y) \leq 1$ for all $y \in Y$. Hence $\Lambda_X(f) = \sup f(X) \geq f(a) > 1$, but $\Lambda_Y(f) = \sup f(Y) \leq 1$ which implies $\Lambda_X \not\leq \Lambda_Y$. □

Suppose that C is a reflexive continuous d-cone whose dual d-cone C^* is also continuous, and let H be a continuous sublinear functional on C^*. We may apply Theorem 1(1) with $D = C^*$ and $D^* = C$, and we find a convex weak Scott-closed subset $X \subseteq C$ such that $H(f) = \sup_{x\in X} f(x) = \Lambda_X(f)$, whence $\Lambda_X = H$. Note that weak Scott-closed sets are closed. Together with the previous proposition this yields:

Theorem 2 *Let C be a reflexive continuous d-cone whose dual C^* is also continuous. Then the sublinear functionals form a sub-d-cone join-semilattice of $\overline{\mathbb{R}}_+^{C^*}$ and Λ cuts down to a d-cone join-semilattice isomorphism between $\mathcal{H}C$ and the continuous sublinear functionals.*

Proof. Since Λ is a morphism of d-cone join-semilattices, its range is a sub-d-cone join-semilattice of $\overline{\mathbb{R}}_+^{C^*}$. As its range consists of the continuous sublinear functionals, the first assertion follows. Finally, as Λ is an order-embedding as well as a morphism of d-cone join-semilattices, it cuts down to an isomorphism of d-cone join-semilattices as asserted. □

Let us now consider d-cones of the form $\mathcal{V}(P)$ for a domain P. Here we have a representation function:

$$\mathcal{H}\mathcal{V}(P) \to \overline{\mathbb{R}}_+^{\mathcal{V}(P)^*} \cong \overline{\mathbb{R}}_+^{\mathcal{L}(P)}$$

which we also call Λ and which is given by:

$$\Lambda_X(f) = \sup_{\mu \in X} \int f d\mu$$

As $\mathcal{V}(P)$ is a reflexive continuous d-cone when P is a domain and as the dual cone $\mathcal{V}(P)^* \cong \mathcal{L}(P)$ is continuous, too, we may apply Theorem 2.

Corollary 2 *Let P be a domain. Then the sublinear functionals form a sub-d-cone join-semilattice of $\overline{\mathbb{R}}_+^{\mathcal{L}(P)}$ and Λ cuts down to a d-cone join-semilattice isomorphism between $\mathcal{H}\mathcal{V}(P)$ and the continuous sublinear functionals.*

6.2. *The upper powercone*

From Section 4.2 we know that, for every continuous d-cone C, the representation function:

$$\Lambda \colon \mathcal{S}C \to \overline{\mathbb{R}}_+^{\,C^*}$$

is a morphism of d-cone meet-semilattices that transforms every compact convex saturated subset X of C into the continuous superlinear functional $\Lambda_X \colon C^* \to \overline{\mathbb{R}}_+$ defined by:

$$\Lambda_X(f) = \inf_{x \in X} f(x)$$

Analogously to the previous case, we want to show that, under appropriate additional hypotheses, Λ is a d-cone meet-semilattice isomorphism between $\mathcal{S}$ and the sub-d-cone meet-semilattice of the superlinear continuous functionals, and to this end we again need only further show it is an order embedding whose range includes the superlinear continuous functionals.

Proposition 9 *For a continuous d-cone C, the map Λ is an order embedding, that is, for $X, Y \in \mathcal{S}C$, we have $\Lambda_X \leq \Lambda_Y$ if and only if $X \supseteq Y$.*

Proof. As Λ is monotonic by the general considerations in Section 4.1, it remains to show that, if $\Lambda_X \leq \Lambda_Y$, then $X \supseteq Y$. For this, we suppose $X \not\supseteq Y$. Choose an element $b \in Y \setminus X$. As X is a compact convex saturated set, by the Strict Separation Theorem of (Tix, Keimel, Plotkin 2005), there is a linear continuous functional $f: C \to \overline{\mathbb{R}}_+$ such that $f(b) \leq 1$, but $f(x) \geq r > 1$ for some $r > 0$ and all $x \in X$. It follows that $\Lambda_X(f) = \inf f(X) \geq r > 1$, but $\Lambda_Y(f) = \inf f(Y) \leq f(b) \leq 1$ which implies $\Lambda_X \not\leq \Lambda_Y$. □

Now let G be a continuous superlinear functional on C^*. Supposing that C is a convenient d-cone – that is, it is continuous, reflexive and its weak Scott topology coincides with its Scott topology and, furthermore, the dual cone C^* is continuous and has an additive way-below relation – we may apply Theorem 1(2) for $D = C^*$ and $D^* = C$, and we find a compact convex saturated subset $X \subseteq C$ such that $G(f) = \sup_{x \in X} f(x) = \Lambda_X(f)$, whence $\Lambda_X = G$. Together with the previous proposition this yields:

Theorem 3 *Let C be a convenient d-cone. Then the superlinear functionals form a sub-d-cone meet-semilattice of $\overline{\mathbb{R}}_+^{C^*}$ and Λ cuts down to a d-cone meet-semilattice isomorphism between $\mathcal{S}C$ and the continuous superlinear functionals.*

Let us now consider d-cones of the form $\mathcal{V}(P)$ for a domain P. Here we have a representation function:

$$\mathcal{S}\mathcal{V}(P) \to \overline{\mathbb{R}}_+^{\mathcal{V}(P)^*} \cong \overline{\mathbb{R}}_+^{\mathcal{L}(P)}$$

which we again also call Λ and which is given by:

$$\Lambda_X(f) = \inf_{\mu \in X} \int f d\mu$$

As, for a coherent domain P, the extended probabilistic powerdomain $\mathcal{V}(P)$ is a convenient d-cone, we may apply Theorem 3.

Corollary 3 *Let P be a coherent domain. Then the superlinear functionals form a sub-d-cone meet-semilattice of $\overline{\mathbb{R}}_+^{\mathcal{L}(P)}$ and Λ cuts down to a d-cone meet-semilattice isomorphism between $\mathcal{S}\mathcal{V}(P)$ and the continuous sublinear functionals.*

6.3. *The convex powercone*

For every coherent continuous d-cone C, we have two representations according to Section 4.3 and 4.4, the standard one:

$$\Lambda: \mathcal{P}C \to \mathcal{P}\overline{\mathbb{R}}_+^{[C, \mathcal{P}\overline{\mathbb{R}}_+]}$$

and the diagonal one:

$$\Lambda' : \mathcal{P}C \to \mathcal{P}\overline{\mathbb{R}}_+^{C^*}$$

Both representations are morphisms of d-cone semilattices. Every convex lens $X \subseteq C$ is represented by a pair of continuous real valued functionals $\Lambda_X = \left[\underline{\Lambda}_X, \overline{\Lambda}_X\right]$ defined on the d-cone $[C, \mathcal{P}\overline{\mathbb{R}}_+]$ in the case of Λ, and by a pair of continuous real valued functionals $\Lambda'_X = \left[\underline{\Lambda'}_X, \overline{\Lambda'}_X\right]$ defined on the dual cone C^* in the case of Λ'. The latter are defined by:

$$\underline{\Lambda'}_X(g) = \inf_{x \in X} g(x)$$

and:

$$\overline{\Lambda'}_X(g) = \sup_{x \in X} g(x)$$

and the two representations are related by the formulas:

$$\Lambda'_X(g) = \Lambda_X[g, g]$$

$$\Lambda_X[g, h] = \left[\underline{\Lambda'}_X(g)\overline{\Lambda'}_X(h)\right]$$

for $g \in C^*$ and $[g, h] \in [C, \mathcal{P}\overline{\mathbb{R}}_+]$. For each $X \in \mathcal{P}\overline{\mathbb{R}}_+$ the functional $\Lambda_X : [C, \mathcal{P}\overline{\mathbb{R}}_+]$ and similarly the functional $\Lambda'_X : C^* \to \overline{\mathbb{R}}_+$ is continuous and canonically $\subseteq$-sublinear.

Proposition 10 *For a coherent continuous d-cone C, the maps Λ and Λ' are order embeddings, that is, for $X, Y \in \mathcal{P}C$, we have $\Lambda_X \le \Lambda_Y$ and $\Lambda'_X \le \Lambda'_Y$, respectively, if and only if $X \sqsubseteq_{EM} Y$.*

Proof. By the general considerations in Section 4.3 Λ and Λ' are monotonic. Conversely, if $\Lambda_X \le \Lambda_Y$, then $\Lambda'_X \le \Lambda'_Y$ by the definition of Λ'. From $\Lambda'_X \le \Lambda'_Y$ we firstly deduce $\Lambda_{\downarrow X} \le \Lambda_{\downarrow Y}$ and so $\downarrow X \subseteq \downarrow Y$ by Proposition 8, where now Λ is the lower powercone functional representation, and secondly $\Lambda_{\uparrow X} \le \Lambda_{\uparrow Y}$ and so $\uparrow X \subseteq \uparrow Y$ by Proposition 9, where now Λ is the upper powercone functional representation. So we have $X \sqsubseteq_{EM} Y$, as required. □

Now let C be a convenient d-cone. Then $D =_{def} C^*$ is continuous and has an additive way-below relation and the weak* Scott topology on $D^* \cong C$ is identical to the Scott topology. Theorem 1(3) allows to conclude that for every continuous canonically $\subseteq$-sublinear functional $F : C^* \to \mathcal{P}\overline{\mathbb{R}}_+$, there is a convex lens $X \subseteq D^* \cong C$ such that $\Lambda'_X = F$.

Let us next consider a continuous $\subseteq$-monotonic canonically $\subseteq$-sublinear functional $F : [C, \mathcal{P}\overline{\mathbb{R}}_+] \to \mathcal{P}\overline{\mathbb{R}}_+$. By Lemma 5, the map $F' =_{def} R_C(F) : C^* \to \mathcal{P}\overline{\mathbb{R}}_+$ is continuous and canonically $\subseteq$-sublinear. By the preceding, there is a convex lens $X \subseteq C$ such that $\Lambda'_X = F'$. As, again by Lemma 5, F can be recovered from F' in the same way as Λ_X from Λ'_X by applying R_C^{-1}, we conclude that $F = \Lambda_X$.

Together with the previous proposition, we have:

Theorem 4 *Let C be a convenient coherent d-cone. Then Λ and Λ' cut down to isomorphisms between the continuous coherent d-cone semilattice $\mathcal{P}C$ and sub-d-cone semilattices of $\mathcal{P}\overline{\mathbb{R}}_+^{[C,\mathcal{P}\overline{\mathbb{R}}_+]}$ and $\mathcal{P}\overline{\mathbb{R}}_+^{C^*}$, respectively, consisting of all canonically $\subseteq$-sublinear functionals which, in the first case, are also $\subseteq$-monotonic.*

Let us now specialise to d-cones of the form $C = \mathcal{V}(P)$ for a domain P. We recall that $\mathcal{V}(P)$ is a convenient d-cone and that the dual cone $\mathcal{V}(P)^*$ is naturally isomorphic to $\mathcal{L}(P)$. Similarly, the cone $[\mathcal{V}(P), \mathcal{P}\overline{\mathbb{R}}_+]$ is naturally isomorphic to the cone $\mathcal{P}\overline{\mathbb{R}}_+^P$ of all continuous functions $f : P \to \mathcal{P}\overline{\mathbb{R}}_+$ which can be represented as pairs $[g, h]$ of functions $g, h \in \mathcal{L}(P)$ with $g \leq h$. We therefore have representation functions:

$$\begin{array}{ccccc} \mathcal{P}\mathcal{V}(P) & \longrightarrow & \mathcal{P}\overline{\mathbb{R}}_+^{[\mathcal{V}(P),\mathcal{P}\overline{\mathbb{R}}_+]} & \cong & (\mathcal{P}\overline{\mathbb{R}}_+)^{\mathcal{P}\overline{\mathbb{R}}_+^P} \\ \mathcal{P}\mathcal{V}(P) & \longrightarrow & \mathcal{P}\overline{\mathbb{R}}_+^{\mathcal{V}(P)^*} & \cong & \mathcal{P}\overline{\mathbb{R}}_+^{\mathcal{L}(P)} \end{array}$$

which we again also call Λ and Λ', respectively; Λ' is given by the formulas:

$$\underline{\Lambda'}_X(g) = \inf_{\mu \in X} \int g d\mu$$

and:

$$\overline{\Lambda'}_X(g) = \sup_{\mu \in X} \int g d\mu$$

And Λ can be calculated from Λ' as above:

$$\Lambda_X[g, h] = \left[\underline{\Lambda'}_X(g), \overline{\Lambda'}_X(h)\right]$$

We may now apply Theorem 4.

Corollary 4 *Let P be a coherent domain. Then Λ and Λ' cut down to isomorphisms between the continuous d-cone semilattice $\mathcal{P}\mathcal{V}(P)$ and sub-d-cone semilattices of $(\mathcal{P}\overline{\mathbb{R}}_+)^{\mathcal{P}\overline{\mathbb{R}}_+^P}$ and $(\mathcal{P}\overline{\mathbb{R}}_+)^{\mathcal{L}(P)}$, respectively, consisting of all canonically $\subseteq$-sublinear maps which, in the first case, are also $\subseteq$-monotonic.*

7. Predicate transformers

As in the case of functional representations, a certain amount of the development can be carried out at a general level. We place ourselves in the framework of Sections 3 and 4, assume the category K of d-cones contains $\overline{\mathbb{R}}_+$, and work with the standard representation:

$$\Lambda : TC \longrightarrow T\overline{\mathbb{R}}_+^{[C,T\overline{\mathbb{R}}_+]}$$

given by:

$$\Lambda_\gamma = f \mapsto f^\#(\gamma)$$

We take 'predicates' on a d-cone C to be linear continuous maps from C to $T\overline{\mathbb{R}}_+$ and predicate transformers from one d-cone D to another C to be continuous maps $\Phi : [D, T\overline{\mathbb{R}}_+] \to [C, T\overline{\mathbb{R}}_+]$. The general question is then the relation between such predicate transformers and 'state transformers' from C to D which we take to be linear continuous maps from C to TD.

There is an evident isomorphism of d-cones:

$$\mathrm{t} : [C, T\overline{\mathbb{R}}_+^{[D, T\overline{\mathbb{R}}_+]}] \cong [C, T\overline{\mathbb{R}}_+]^{[D, T\overline{\mathbb{R}}_+]}$$

defined by transposition: $\mathrm{t}(m')(f)(x) =_{def} m'(x)(f)$. Composing with Λ and applying t, one then obtains a linear continuous map:

$$W_{C,D} : [C, TD] \longrightarrow [C, T\overline{\mathbb{R}}_+]^{[D, T\overline{\mathbb{R}}_+]}$$

where $W_{C,D}(m) =_{def} \mathrm{t}(\Lambda \circ m)$. More explicit formulas for this map are:

$$W_{C,D}(m)(f)(x) = \Lambda_{mx}(f) = f^{\#}(mx)$$

Using the last formula, it is easily verified that this defines the morphism part of a locally linear and continuous functor:

$$W : \mathsf{K}_T \longrightarrow \mathsf{PT}^{\mathrm{op}}$$

which acts as the identity on objects. Here K_T is, as usual, the Kleisli category of T, and is our category of state transformers; PT is the category with the same objects as K and with the morphisms from C to D being the predicate transformers from C to D.

It further follows from the second formula for $W_{C,D}$, together with Corollary 1, that every predicate transformer in the range of W is $\subseteq$-monotonic and $\subseteq$-sublinear. The collection of such predicate transformers from a given C to a given D forms a sub-d-cone of the d-cone of all predicate transformers from C to D.

Proposition 11 *If Λ is an order-embedding, so is W (locally).*

Proof. The map $W_{C,D}$ consists, by the assumption, of a composition with an order-embedding, which is itself an order-embedding, followed by an isomorphism. □

The converse also holds, but we don't need it.

We now specialise the discussion to free d-cones on domains. Suppose that J is a full subcategory of the category of domains and that we have an adjunction:

$$\mathcal{V} \dashv G : \mathsf{K} \longrightarrow \mathsf{J}$$

where G is the evident forgetful functor and $\mathcal{V}$ is the appropriate restriction of the valuation functor, and suppose further that the natural transformation:

$$\psi_{P,D} : D^P = \mathsf{J}(P, D) \cong \mathsf{K}(\mathcal{V}P, D) = [\mathcal{V}P, D]$$

is an isomorphism of d-cones. Here, and below, we neglect to write the forgetful functor G and consider $\mathcal{V}$ to be a left adjoint or a monad, as convenient.

With these assumptions we have a monad on J which may be written as $T\mathcal{V}$. We then take state transformers at the level of domains to be continuous functions $P \to T\mathcal{V}Q$ in J, and so take the category of state transformers to be $\mathsf{J}_{T\mathcal{V}}$. We can define a full and faithful functor $\mathcal{V}_T : \mathsf{J}_{T\mathcal{V}} \longrightarrow \mathsf{K}_T$, locally an isomorphism of d-cones, by putting:

$$\mathcal{V}_T(P) = \mathcal{V}(P)$$

on objects, and:

$$\mathcal{V}_T(m) = \psi_{P,T\mathcal{V}Q}(m)$$

on morphisms; functoriality is a straightforward, if tedious, calculation.

We take predicates on a domain P in J to be continuous maps from P to $T\overline{\mathbb{R}}_+$ and predicate transformers from another such Q to P to be continuous maps $\Phi : T\overline{\mathbb{R}}_+^Q \to T\overline{\mathbb{R}}_+^P$, yielding the category PT_d of predicate transformers. We can define a useful functor $\mathcal{V}_p : \mathsf{PT}_d \to \mathsf{PT}$ by putting:

$$\mathcal{V}_p(P) = \mathcal{V}(P)$$

on objects, and:

$$\mathcal{V}_p(\Phi) = (\psi_{P,T\overline{\mathbb{R}}_+}) \circ \Phi \circ (\psi_{Q,T\overline{\mathbb{R}}_+})^{-1}$$

on morphisms.

Next, since $W\mathcal{V}_T(P) = \mathcal{V}(P)$ we can define a functor $W_d : \mathsf{J}_{T\mathcal{V}} \longrightarrow {\mathsf{PT}_d}^{\mathrm{op}}$ which is the identity on objects, and on morphisms $m : P \to T\mathcal{V}Q$ is given by:

$$W_d(m) = (\mathcal{V}_p)^{-1}_{Q,P}(W \circ \mathcal{V}_T) = (\psi_{P,T\overline{\mathbb{R}}_+})^{-1} \circ W(\mathcal{V}_T(m)) \circ \psi_{Q,T\overline{\mathbb{R}}_+}$$

Note that $\mathcal{V}_p \circ W_d = W \circ \mathcal{V}_T$. One then calculates that:

$$\begin{array}{rcl} W_d(m)(\theta)(x) & = & W(\mathcal{V}_T(m))(\psi_{Q,T\overline{\mathbb{R}}_+}(\theta))(\eta x) \\ & = & \Lambda_{\mathcal{V}_T(m)(\eta x)}(\psi_{Q,T\overline{\mathbb{R}}_+}(\theta)) \\ & = & \Lambda_{mx}(\psi_{Q,T\overline{\mathbb{R}}_+}(\theta)) \\ & = & \psi_{Q,T\overline{\mathbb{R}}_+}(\theta)^{\#}(mx) \end{array}$$

Note that W_d is locally the composition of W with isomorphisms of d-cones, viz $\mathcal{V}_T$ and $(\psi_{P,T\overline{\mathbb{R}}_+})^{-1} \circ - \circ \psi_{Q,T\overline{\mathbb{R}}_+}$. So W_d is locally continuous and linear as W is; it also preserves $\cup$ as $\psi_{Q,T\overline{\mathbb{R}}_+}(\theta)^{\#}$ does.

7.1. *The lower powercone and powerdomain*

Here we follow Section 4.1 simplifying from the d-cone join-semilattice $\mathcal{H}\overline{\mathbb{R}}_+$ to the isomorphic one on $\overline{\mathbb{R}}_+$, yielding the functional representation:

$$\Lambda : \mathcal{H}C \longrightarrow \overline{\mathbb{R}}_+^{C^*}$$

where:

$$\Lambda_X(f) = \sup_{x \in X} f(x)$$

Predicates on a d-cone C are now linear continuous maps from C to $\overline{\mathbb{R}}_+$, and the predicate transformers from a d-cone D to another C are continuous maps $\Phi : D^* \to C^*$. These last provide the morphisms of the category PT, which retains the same objects as before. The morphism part of the locally linear and continuous functor $W : \mathsf{Cone}_{\mathcal{H}} \longrightarrow \mathsf{PT}^{\mathrm{op}}$ is given by the calculation:

$$W_{C,D}(m)(f)(x) = f^{\#}(mx) = \sup_{y \in mx} f(y)$$

and, as $\subseteq$ and $\leq$ coincide in the join-semilattice case, all predicate transformers in the range of W are sublinear and, locally, form a sub-d-cone of the d-cone of all predicate transformers.

Let PT_l be the subcategory of PT restricted to the sublinear predicate transformers, the 'healthy' ones. It is easily verified that the sublinear predicate transformers form a sub-d-cone of the d-cone all predicate transformers.

Theorem 5 *The functor W cuts down to a locally linear and continuous order-embedding:*

$$W : \mathsf{CCone}_{\mathcal{H}} \to \mathsf{PT}_l^{\mathrm{op}}$$

It further cuts down to an equivalence of the full subcategories of reflexive continuous d-cones with continuous duals that is locally an isomorphism of d-cones.

Proof. The first part of the theorem follows from Proposition 8 and Proposition 11. For the second part, note that, by Theorem 2, if D is a reflexive continuous d-cone then the maps $m' : C \to \overline{\mathbb{R}}_+^{D^*}$ whose range consists of sublinear functionals are in bijective correspondence, via composition with Λ, with the state transformers $m : C \to \mathcal{H}\overline{\mathbb{R}}_+$. So then W is locally a bijection and the conclusion follows, applying the first part of the theorem. □

Turning to powerdomains, we now take K to be CCone and J to be Dom, recalling that $\mathcal{H}$ preserves continuity; PT then has continuous d-cones as objects; and PT_d has domains as objects and the morphisms from P to Q are the continuous maps $\Phi : \mathcal{L}(P) \to \mathcal{L}(Q)$, simplifying from $\mathcal{H}\overline{\mathbb{R}}_+^P$ to $\overline{\mathbb{R}}_+^P$. The functor $W : \mathsf{CCone}_{\mathcal{H}} \to \mathsf{PT}^{\mathrm{op}}$ is then the restriction of the W considered immediately above. The functor $W_d : \mathsf{Dom}_{\mathcal{HV}} \to \mathsf{PT}_d^{\mathrm{op}}$ is locally linear, continuous and $\vee$-preserving ; it is also an order-embedding, since, by Theorem 5, the same is true of W. Its action on morphisms $m : P \to \mathcal{HV}Q$ is given by the calculation:

$$W_d(m)(f)(x) = (\psi_{Q,\overline{\mathbb{R}}_+} f)^\#(mx) = \sup_{\mu \in mx} (\psi_{Q,\overline{\mathbb{R}}_+} f)(\mu) = \sup_{\mu \in mx} \int f \, d\mu$$

Now let PT_{dl} be the subcategory of PT_d of the sublinear predicate transformers. The following corollary is an immediate consequence of Theorem 5, given the relationship between W and W_d, and the fact that W_d is locally a morphism of d-cone join semilattices:

Corollary 5 *The sublinear predicate transformers form a sub-d-cone join semilattice of the predicate transformers, and the functor W_d cuts down to an equivalence of $\mathsf{Dom}_{\mathcal{HV}}$ and PT_{dl} that is locally an isomorphism of d-cone join-semilattices.*

The functor W_d is, essentially, the greatest pre-expectation function *wlp* defined in the conclusion of (Tix, Keimel, Plotkin 2005); the difference is that in the latter only the action on endomorphisms of a coherent domain is considered. The corollary therefore characterises the greatest liberal pre-expectation function transformers associated to state transformers of the form $P \to \mathcal{HV}P$, for P a coherent domain, and, indeed, more generally.

7.2. *The upper powercone and powerdomain*

Here we follow Section 4.2 simplifying from d-cone meet-semilattice $\mathcal{S}\overline{\mathbb{R}}_+$ to the isomorphic one on $\overline{\mathbb{R}}_+$, yielding the functional representation:

$$\Lambda : \mathcal{S}C \longrightarrow \overline{\mathbb{R}}_+^{C^*}$$

where:

$$\Lambda_X(f) = \inf_{x \in X} f(x)$$

Predicates, predicate transformers, and the category PT are then as in the previous case, that of the lower powercone, but restricted to continuous d-cones. The morphism part of the locally linear and continuous functor $W : \mathsf{CCone}_{\mathcal{S}} \longrightarrow \mathsf{PT}^{\mathrm{op}}$ is given by the calculation:

$$W_{C,D}(m)(f)(x) = f^\#(mx) = \inf_{y \in mx} f(y)$$

Using Propositions 9 and 11, we see that W is a local order-embedding. As $\subseteq$ coincides with $\geq$ in the meet-semilattice case, we further see that all predicate transformers in the range of W are superlinear, and that, locally, they form a sub-d-cone of the d-cone of all predicate transformers.

Let PT_u be the subcategory of PT restricted to the superlinear predicate transformers, the 'healthy' ones. It is easily verified that the superlinear predicate transformers form a sub-d-cone of the d-cone all predicate transformers.

Theorem 6 *The functor W cuts down to a locally continuous linear order-embedding:*

$$W : \mathsf{CCone}_{\mathcal{S}} \to \mathsf{PT}_u^{\mathrm{op}}$$

It further cuts down to an equivalence of the full subcategories of convenient d-cones that is locally an isomorphism of d-cones.

Proof. The proof is much as in the previous case, that of the lower powercone, but now using the remark above on the matter of its being an order-embedding and Theorem 3 for the local order isomorphism. □

Turning to powerdomains, we leave K as CCone and again take J to be Dom; PT and PT_d are also as in the lower case. The functor $W_d : \mathsf{Dom}_{\mathcal{SV}} \to \mathsf{PT}_d^{\mathrm{op}}$ is locally linear, continuous and $\wedge$-preserving ; it is also an order-embedding, since, by the above remarks, the same is true of W. Its action on morphisms $m : P \to \mathcal{SV}Q$ is given by the calculation:

$$W_d(m)(f)(x) = (\psi_{Q,\overline{\mathbb{R}}_+} f)^{\#}(mx) = \inf_{\mu \in mx} (\psi_{Q,\overline{\mathbb{R}}_+} f)(\mu) = \inf_{\mu \in mx} \int f d\mu$$

Now let PT_{du} be the subcategory of PT_d of the superlinear predicate transformers. The following corollary is an immediate consequence of the the fact that W_d is locally a morphism of d-cone meet semilattices and Theorem 6:

Corollary 6 *The superlinear predicate transformers form a sub-d-cone meet semilattice of the predicate transformers, and the functor W_d cuts downs to an equivalence of the full subcategories of $\mathsf{Dom}_{\mathcal{SV}}$ and PT_{du} of coherent domains that is locally an isomorphism of d-cone meet-semilattices.*

The functor W_d is, essentially, the weakest pre-expectation function *wp* defined in the conclusion of (Tix, Keimel, Plotkin 2005). The corollary therefore characterises the weakest pre-expectation functions transformers associated to state transformers of the form $P \to \mathcal{SV}P$, for P a coherent domain, and, indeed, more generally.

7.3. *The convex powercone and the biconvex powerdomain*

We follow Section 4.3 and employ the standard representation:

$$\Lambda : \mathcal{P}C \longrightarrow \mathcal{P}\overline{\mathbb{R}}_+^{[C, \mathcal{P}\overline{\mathbb{R}}_+]}$$

where:

$$\Lambda_X(f) = [\inf_{x \in X} \underline{f}(x), \sup_{x \in X} \overline{f}(x)]$$

Predicates and PT are then as in the general approach, i.e., predicates on C are elements of $[C, \mathcal{P}\overline{\mathbb{R}}_+]$, PT has objects the coherent continuous cones and the predicate transformers from C to D are continuous maps $\Phi : [C, \mathcal{P}\overline{\mathbb{R}}_+] \to [D, \mathcal{P}\overline{\mathbb{R}}_+]$.

The morphism part of the locally linear and continuous functor $W : \mathsf{CCone}^c{}_{\mathcal{P}} \longrightarrow \mathsf{PT}^{\mathrm{op}}$ is given by the calculation:

$$W_{C,D}(m)(f)(x) = \Lambda_{mx}(f) = [\inf_{y \in mx} \underline{f}(y)\, ,\, \sup_{y \in mx} \overline{f}(y)]$$

and we know that, locally, the predicate transformers in the range of W are $\subseteq$-monotonic and $\subseteq$-sublinear. Using Propositions 10 and 11 we also see that W is a local order-embedding.

Following Section 4.4 it is natural to also consider a different kind of predicate transformer, viz maps $\Phi' : C^* \to [D, \mathcal{P}\overline{\mathbb{R}}_+]$, with C and D coherent continuous d-cones as above. We call these 'double' predicate transformers, for reasons which will become clear shortly. First we introduce under- and over-lining conventions for functions of the form $f : C \to [D, \mathcal{P}\overline{\mathbb{R}}_+]$. We write $\underline{f}$ for the function $x \mapsto \underline{f(x)} : C \to D^*$; $\overline{f}$ is defined similarly. This gives a bijection between functions $f : C \to [D, \mathcal{P}\overline{\mathbb{R}}_+]$ and pairs of functions $g, h : C \to D^*$ with $g \leq h$; the inverse of the bijection sends g, h to $[g, h] =_{def} x \mapsto [gx, hx]$.

So, in particular, double predicate transformers $\Phi' : C^* \to [D, \mathcal{P}\overline{\mathbb{R}}_+]$ correspond to pairs of predicate transformers $\underline{\Phi'}, \overline{\Phi'} : C^* \to D^*$ of the type considered in the lower and upper cases in the previous two subsections. With the aid of the bijection, it is further easy to see that they form a category, PT': the identity on an object C is $[\mathrm{id}_C, \mathrm{id}_C]$ and composition is defined by setting $[\underline{\Psi'}, \overline{\Psi'}] \circ [\underline{\Phi'}, \overline{\Phi'}] = [\underline{\Psi'} \circ \underline{\Phi'}, \overline{\Psi'} \circ \overline{\Phi'}]$.

Now let PT_m be the subcategory of PT of the $\subseteq$-monotonic predicate transformers.

Lemma 12 (1) *Let* $\Phi : [C, \mathcal{P}\overline{\mathbb{R}}_+] \to [D, \mathcal{P}\overline{\mathbb{R}}_+]$ *be* $\subseteq$*-monotonic. Then we have:*

$$\Phi[g, h] = [\underline{\Phi[g, g]}, \overline{\Phi[h, h]}]$$

(2) *There is an isomorphism of categories* $R : \mathsf{PT}_m \cong \mathsf{PT}'$*, locally an isomorphism of d-cones. It acts as the identity on objects and on predicate transformers* Φ *from* C *to* D*,* $R_{C,D}(\Phi) = \Phi \circ \Delta_C$*. The inverse of* $R_{C,D}$ *is given by:* $R^{-1}_{C,D}(\Phi')[g, h] = [\underline{\Phi'}(g), \overline{\Phi'}(h)]$

Proof. (1) A predicate transformer $\Phi : [C, \mathcal{P}\overline{\mathbb{R}}_+] \to [D, \mathcal{P}\overline{\mathbb{R}}_+]$ is $\subseteq$-monotonic if and only if $t^{-1}(\Phi)(d)$ is for every d in D. Using this observation, we calculate:

$$\begin{array}{rcll} \underline{\Phi[g, h](d)} & = & \underline{t^{-1}(\Phi)(d)[g, h]} & \\ & = & \underline{t^{-1}(\Phi)(d)[g, g]} & \text{(by Lemma 5.1)} \\ & = & \underline{\Phi[g, g](d)} & \end{array}$$

So $\underline{\Phi[g, h]} = \underline{\Phi[g, g]}$ and similarly for $\overline{\Phi[g, h]}$.

(2) We first check that R is a functor. Note that $R(\Phi)(f) = \Phi[f, f]$. So R evidently

preserves the identity. To check it preserves composition suppose that Φ is a predicate transformer from C to D and Ψ is one from D to E. Then we calculate:

$$\begin{array}{rcll} \underline{R(\Psi)\circ R(\Phi)}(f) & = & \underline{R\Psi}(\underline{R\Phi}(f)) & \\ & = & \underline{\Psi[\underline{\Phi[f,f]}, \underline{\Phi[f,f]}} & \\ & = & \underline{\Psi(\Phi[f,f])} & \text{(by Part 1)} \\ & = & \underline{R(\Psi\circ\Phi)}(f) & \end{array}$$

and similarly for $\overline{R(\Psi)\circ R(\Phi)}$. The rest follows from the second part of Lemma 5. $\square$

Composing W with R (all predicate transformers in the range of W are $\subseteq$-monotonic) we obtain a functor $W' : \mathsf{CCone}^c \to \mathsf{PT}'$. This acts as as the identity on objects and acts on morphisms as follows:

$$W'_{C,D}(m)(f)(x) = [\inf_{y \in mx} f(y)\, ,\ \sup_{y \in mx} f(y)]$$

Thus W' combines the upper and lower predicate transformers. Precisely, we have that $\underline{W'_{C,D}(m)} = W_{C,D}(\uparrow_D \circ m)$ and $\overline{W'_{C,D}(m)} = W_{C,D}(\downarrow_D \circ m)$.

Turning to healthiness conditions we consider the property for a $\subseteq$-monotonic predicate transformer Φ from C to D to be canonically $\subseteq$-sublinear which is $\subseteq$-sublinearity of Φ together with the condition:

$$(*) \qquad \underline{\Phi}(f+g) \leq \underline{\Phi}(f) + \overline{\Phi}(g) \leq \overline{\Phi}(f+g)$$

The corresponding healthiness conditions on a double predicate transformer Φ' from C to D are superlinearity and sublinearity of the functionals $\underline{\Phi'}$ and $\overline{\Phi'}$, respectively, together with the condition:

$$(*) \qquad \underline{\Phi'}(f+g) \leq \underline{\Phi'}(f) + \overline{\Phi'}(g) \leq \overline{\Phi'}(f+g)$$

Using Lemma 12 it is straightforward to show that Φ satisfies this healthiness conditions, if and only if $\Phi' = R(\Phi)$ satisfies the corresponding properties. Further, the healthy (double) predicate transformers, of either kind, form a subcategory of PT, respectively PT'. It is evident that the identity predicate transformer is healthy, and therefore so too, by the above remarks, is the identity double predicate transformer. Similarly, for composition it suffices to check the case of the double predicate transformer. So let Ψ', Φ' be double predicate transformers from D to E and from C to D respectively, and calculate:

$$\begin{array}{rcll} \underline{\Psi'\circ\Phi'}(f+g) & = & \underline{\Psi'}\circ\underline{\Phi'}(f+g) & \\ & \leq & \underline{\Psi'}(\underline{\Phi'}(f) + \overline{\Phi'}(g)) & \text{(as } \Phi' \text{ is healthy)} \\ & \leq & \underline{\Psi'}(\underline{\Phi'}(f)) + \overline{\Psi'}(\overline{\Phi'}((\text{as } \Psi' \text{ is healthy}) & \\ & = & \underline{\Psi'\circ\Phi'}(f) + \overline{\Psi'\circ\Phi'}(g) & \end{array}$$

and similarly for the second inequality.

It is straightforward to verify that the healthy predicate transformers from C to D form a sub-d-cone of all the predicate transformers from C to D and that the same holds of the healthy double predicate transformers.

Theorem 7 *The functor* $W : \mathsf{CCone}^c{}_{\mathcal{P}} \longrightarrow \mathsf{PT}^{\mathrm{op}}$ *(respectively,* $W' : \mathsf{CCone}^c{}_{\mathcal{P}} \longrightarrow \mathsf{PT'}^{\mathrm{op}}$*) cuts down to a locally linear and continuous equivalence of categories of the full subcategory of* $\mathsf{CCone}^c{}_{\mathcal{P}}$ *of the convenient d-cones and the subcategory of* PT *(respectively* $\mathsf{PT'}$*) of the convenient d-cones and the healthy predicate transformers (respectively, the healthy double predicate transformers).*

Proof. First, by Theorem 4, Λ cuts down to a d-cone isomorphism between $\mathcal{P}D$ and those functionals in $[D, \mathcal{P}\overline{\mathbb{R}}_+]^{\mathcal{P}\overline{\mathbb{R}}_+}$ which are canonically $\subseteq$-sublinear. Second, for any m' in $[C, [D, \mathcal{P}\overline{\mathbb{R}}_+]^{\mathcal{P}\overline{\mathbb{R}}_+}]$, $m'(x)$ is canonically $\subseteq$-sublinear for every x in C if, and only if, $t(m')$ is canonically $\subseteq$-sublinear on predicate transformers, and so t cuts down to a d-cone isomorphism of the sub-d-cone of such m' and the sub-d-cone of the healthy predicate transformers. The first assertion is then an immediate consequence.

The second assertion follows from the first as, by Lemma 12, R is locally an isomorphism of d-cones. □

Turning to powerdomains we keep K, and so PT as they are, and take J to be Dom^c. Then PT_d has coherent domains as objects and the morphisms from P to Q are the continuous maps $\Phi : \mathcal{P}\overline{\mathbb{R}}_+^P \to \mathcal{P}\overline{\mathbb{R}}_+^Q$. The functor $W_d : \mathsf{Dom}^c_{\mathcal{PV}} \to \mathsf{PT}_d^{op}$ is locally linear, continuous and $\uplus$-preserving ; it is also a local order-embedding, as W is.

We introduce under- and over-lining notation for functions of the form $f : P \to \mathcal{P}\overline{\mathbb{R}}_+^Q$ in the evident way, as well as the notation $[g, h]$ for pairs of functions $g, h : P \to \mathcal{L}(Q)$ with $g \leq h$, and the usual properties carry over. Using the naturality of ψ one shows $\underline{\psi_{P,\mathcal{P}\overline{\mathbb{R}}_+}(f)} = \psi_{P,\overline{\mathbb{R}}_+}(\underline{f})$ and $\overline{\psi_{P,\mathcal{P}\overline{\mathbb{R}}_+}(f)} = \psi_{P,\overline{\mathbb{R}}_+}(\overline{f})$ for any f in $\mathcal{P}\overline{\mathbb{R}}_+^P$ and also that $\psi_{P,\mathcal{P}\overline{\mathbb{R}}_+}([g, h]) = [\psi_{P,\overline{\mathbb{R}}_+}(g), \psi_{P,\overline{\mathbb{R}}_+}(h)]$ for any g, h in $\mathcal{L}(P)$ with $g \leq h$.

With that, one can calculate the action of W_d on morphisms:

$$\begin{array}{rcl} W_d(m)(f)(x) & = & \Lambda_{mx}(\psi_{Q,\mathcal{P}\overline{\mathbb{R}}_+} f) \\ & = & \left[\inf_{\mu \in mx} \underline{\psi_{Q,\mathcal{P}\overline{\mathbb{R}}_+} f}(\mu)\, , \, \sup_{\mu \in mx} \overline{\psi_{Q,\mathcal{P}\overline{\mathbb{R}}_+} f}(\mu)\right] \\ & = & \left[\inf_{\mu \in mx} (\psi_{Q,\overline{\mathbb{R}}_+} \underline{f})(\mu)\, , \, \sup_{\mu \in mx} (\psi_{Q,\overline{\mathbb{R}}_+} \overline{f})(\mu)\right] \\ & = & \left[\inf_{\mu \in mx} \int \underline{f} d\mu\, , \, \sup_{\mu \in mx} \int \overline{f} d\mu\right] \end{array}$$

Double predicate transformers have the form $\Phi' : \mathcal{L}(P) \to \mathcal{P}\overline{\mathbb{R}}_+^Q$ and they form a category PT'_d much as before: the identity on P is $[\mathrm{id}_P, \mathrm{id}_P]$ and composition is defined by setting $[\underline{\Psi'}, \overline{\Psi'}] \circ [\underline{\Phi'}, \overline{\Phi'}] = [\underline{\Psi'} \circ \underline{\Phi'}, \overline{\Psi'} \circ \overline{\Phi'}]$. We can define a useful functor, locally an isomorphism of d-cones, $\mathcal{V}'_p : \mathsf{PT}'_d \to \mathsf{PT}'$ by putting:

$$\mathcal{V}_p(P) = \mathcal{V}(P)$$

on objects, and:

$$\begin{array}{rcl} \mathcal{V}_p(\Phi') & = & \psi_{Q,\mathcal{P}\overline{\mathbb{R}}_+} \circ \Phi' \circ \psi^{-1}_{P,\overline{\mathbb{R}}_+} \\ & (= & [\psi_{Q,\overline{\mathbb{R}}_+} \circ \underline{\Phi'} \circ (\psi_{P,\overline{\mathbb{R}}_+})^{-1}, \psi_{Q,\overline{\mathbb{R}}_+} \circ \overline{\Phi'} \circ (\psi_{P,\overline{\mathbb{R}}_+})^{-1}]) \end{array}$$

on morphisms, with the last equation making it clear why $\mathcal{V}'_p$ is a functor.

We write PT_{dm} to be the subcategory of PT_d of the monotonic predicate transformers. Note that $\mathcal{V}_p(\Phi)$ is monotonic if, and only if, Φ is, so $\mathcal{V}_p$ cuts down to a functor from PT_{dm} to PT_m.

Lemma 13 *There is an isomorphism of categories $R_d : \mathsf{PT}_{md} \cong \mathsf{PT}'_d$, locally an isomorphism d-cone semilattices. It acts as the identity on objects and on predicate transformers Φ from P to Q, $R_d(\Phi) = \big(f \mapsto \Phi[f, f]\big)$.*

Proof. We define the action of R_d on morphisms $\Phi : \mathcal{P}\overline{\mathbb{R}}_+^P \to \mathcal{P}\overline{\mathbb{R}}_+^Q$ by:

$$R_d(\Phi) = (\mathcal{V}'_p)^{-1}(R \circ \mathcal{V}_p(\Phi))$$

As $\mathcal{V}_p$, R and $\mathcal{V}'_p$ are locally d-cone isomorphisms, using Lemma 12, so is R_d.

We now calculate:

$$\begin{array}{rcl} R_d(\Phi)(f) & = & \psi^{-1}_{Q,\mathcal{P}\overline{\mathbb{R}}_+}(R(\psi_{Q,\mathcal{P}\overline{\mathbb{R}}_+} \circ \Phi \circ \psi^{-1}_{P,\mathcal{P}\overline{\mathbb{R}}_+})(\psi_{P,\overline{\mathbb{R}}_+}(f))) \\ & = & \psi^{-1}_{Q,\mathcal{P}\overline{\mathbb{R}}_+}(\psi_{Q,\mathcal{P}\overline{\mathbb{R}}_+} \circ \Phi \circ \psi^{-1}_{P,\mathcal{P}\overline{\mathbb{R}}_+}([\psi_{P,\overline{\mathbb{R}}_+}(f), \psi_{P,\overline{\mathbb{R}}_+}(f)])) \\ & = & \Phi \circ \psi^{-1}_{P,\mathcal{P}\overline{\mathbb{R}}_+}(\psi_{P,\mathcal{P}\overline{\mathbb{R}}_+}[f, f]) \\ & = & \Phi[f, f] \end{array}$$

Using this formula for $R_d(\Phi)$, we see that R_d preserves unions. □

Clearly, W_d cuts down to a functor to PT_{dm} and so, composing with R_d, we obtain a functor $W'_d : \mathsf{Dom}^c \to \mathsf{PT}'_d$, locally a morphism of d-cone semilattices. This acts as the identity on objects and on morphisms:

$$W_d(m)(f)(x) = \big[\inf_{\mu \in mx} \int f d\mu\, ,\, \sup_{\mu \in mx} \int f d\mu\big]$$

Healthy predicate transformers and healthy double predicate transformers are defined analogously to before. It is straightforward to calculate that both kinds of predicate transformer are closed under (pointwise) unions. One next checks that a (double) predicate transformer is healthy if, and only if, its image under $\mathcal{V}_p$ (respectively $\mathcal{V}'_p$) is. So both kinds of healthy predicate transformers form subcategories of their ambient categories, and, locally, form sub-d-cone semilattices of the d-cone semilattices of all predicate transformers. We also have that a predicate transformer is healthy if, and only if, its image under R_d is.

Corollary 7 *The functor $W_d : \mathsf{Dom}^c_{\mathcal{PV}} \longrightarrow \mathsf{PT}^{\mathrm{op}}_d$ (respectively, $W'_d : \mathsf{Dom}^c_{\mathcal{PV}} \longrightarrow \mathsf{PT'}^{\mathrm{op}}_d$) cuts down to an equivalence of categories, locally a d-cone semilattice isomorphism, of $\mathsf{Dom}^c_{\mathcal{PV}}$ and the subcategory of PT_d (respectively $\mathsf{PT'}_d$) of the coherent domains and the healthy predicate transformers (respectively, the healthy double predicate transformers).*

Proof. We know that W_d is locally a morphism of d-cone semilattices and an order-embedding. Further, locally, its range consists exactly of the healthy predicate transformers as: $\mathcal{V}_p \circ W_d = W \circ \mathcal{V}_{\mathcal{P}}$; by Theorem 7 the range of W consists exactly of the healthy predicate transformers; and $\mathcal{V}_p$ preserves and reflects healthiness. This proves the first of the assertions. The second follows from Lemma 13 and the fact that R_d also preserves and reflects healthiness. □

The functors W'_d and W_d are, essentially, the two forms of the weakest pre-bi-expectation function *wpb* defined in the conclusion of (Tix, Keimel, Plotkin 2005). The corollary therefore characterises the weakest pre-bi-expectation function transformers associated to state transformers of the form $P \to \mathcal{PV}P$, for P a coherent domain, and, indeed, more generally.

References

Bonnesen, T., and Fenchel, W. (1934) *Theorie der konvexen Körper*, Ergebnisse der Mathematik und ihrer Grenzgebiete **3**, Springer Verlag.

Bonsall, F. F. (1954) Sublinear functionals and ideals in partially ordered vector spaces. *Proc. London Math. Soc.*, Ser. 3 **4** 402–418.

Bonsangue, M. M. (1998) Topological Duality in Semantics. *Electronic Notes in Theoretical Computer Science* **8** 1–274.

Dijkstra, E. W. (1976) *A Discipline of Programming*, Prentice-Hall.

Escardó, M. (2004) Synthetic topology of data types and classical spaces. *Electronic Notes in Theoretical Computer Science* **87** 1–150.

Gierz, G., Hofmann, K. H., Keimel, K., Lawson, J. D., Mislove, M. and Scott, D. S. (2003) *Continuous Lattices and Domains*, Encyclopedia of Mathematics and its Applications **93**, Cambridge University Press.

Heckmann, R. (1993) Power domains and second-order predicates. *Theoretical Computer Science* **111** 59–88.

Heckmann, R. (1994) Probabilistic domains. In Proc. of CAAP '94. *Springer-Verlag Lecture Notes in Computer Science* **136** 21–56.

Hörmander, L. (1955) Sur la fonction d'appui des ensembles convexes dans un espace localement convexe. *Ark. Mat.* **3** 181–186.

Huber, P. J. (1981) *Robust Statistics*, Wiley.

Johnstone, P., T. (1985) Vietoris locales and localic semilattices. In: Hoffmann, R.-E., Hofmann, K. H. (eds.) *Continuous Lattices and Their Applications*, Lecture Notes in Pure and Applied Mathematics **101** 155–180, Marcel Dekker.

Jones, C. (1990) *Probabilistic non-determinism*, Ph.D. Thesis, University of Edinburgh, Report ECS-LFCS-90-105.

Jones, C. and Plotkin, G. D. (1989) A probabilistic powerdomain of evaluations. In *Proc. of LICS '89*. IEEE Press 186–195.

Jung, A. and Tix, R. (1998) The troublesome probabilistic powerdomain. In: Edalat, A., Jung A., Keimel, K. and Kwiatkowska, M. (eds.) Proc. of Comprox III. *Electronic Notes in Theoretical Computer Science* **13** 70–91.

Keimel, K., and Gierz, G. (1982) Halbstetige Funktionen und stetige Verbände. In: R.-E. Hoffmann (ed.), *Continuous Lattices and Related Topics*, Mathematik Arbeitspapiere Nr. 27, Universität Bremen, 59–67.

Kirch, O. (1993) *Bereiche und Bewertungen*. Master's thesis, Technische Hochschule Darmstadt, 77pp. `www.mathematik.tu-darmstadt.de:8080/ags/ag14/papers/kirch/`

Kutateladze, S. S., and Rubinov, A. M. (1972) Minkowski duality and its applications. *Russian Mathematical Surveys* **27** (3) 137–191.

Maaß, S. (2002) Exact functionals and their core. *Statistical Papers* **43** 1 75–93.

McIver, A. and Morgan, C. (2001a) Demonic, angelic and unbounded probabilistic choices in sequential programs. *Acta Informatica* **37** 329–354.

McIver, A. and Morgan, C. (2001b) Partial correctness for probabilstic demonic programs. *Theoretical Computer Science* **266** 513–541.

McIver, A. and Morgan, C. (2005) *Abstraction, Refinement and Proof for Probabilistic Systems*, Monographs in Computer Science, Springer Verlag.

McIver, A., Morgan, C. and Seidel, K. (1996) Probabilistic predicate transformers. *ACM Transactions on Programming Languages and Systems* **18** 325–353.

Minkowski, H. (1903) Volumen und Oberfläche. *Mathematische Annalen* **57** 447–495.

Plotkin, G. D. (1980) Dijkstra's predicate transformers and Smyth's power domains. In D. Bjorner (editor), Abstract Software Specifications. *Springer-Verlag Lecture Notes in Computer Science* **86** 527–553.

Plotkin, G. D. (2006) A Domain-Theoretic Banach-Alaoglu Theorem. *Mathematical Structures in Computer Science* **16** 299–312.

Rockafellar, R. T. (1972) *Convex Analysis*, Princeton University Press.

Smyth, M. B. (1983) Power domains and predicate transformers: a topological view. In J. Díaz (editor) Proc. of 10th ICALP. *Springer-Verlag Lecture Notes in Computer Science* **154** 662–675.

Tix, R. (1995) *Stetige Bewertungen auf topologischen Räumen*. Master's thesis, Technische Hochschule Darmstadt, 51pp.
`www.mathematik.tu-darmstadt.de:8080/ags/ag14/papers/tix/`

Tix, R., Keimel, K. and Plotkin, G. D. (2005) Semantic Domains for Combining Probability and Non-Determinism. *Electronic Notes in Theoretical Computer Science* **129** 1–104.

Tolstogonov, A. A. (1976) Support functions of convex compacta (Russian). *Matematicheskie Zametki* **22** 203–231. English translation in: *Mathematical Notes* **22** 604–612.

Walley, P. (1991) *Statistical Inference with Imprecise Probabilities*, Chapman and Hall.

Ying, M. (2003) Reasoning about probabilistic sequential programs in a probabilistic logic. *Acta Informatica* **39** 315–389.