SIAM J. COMPUT.
Vol. 16, No. 1, February 1987

ON PROVING LIMITING COMPLETENESS*

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Abstract. We give two proofs of Wadsworth's classic approximation theorem for the pure λ-calculus. One of these illustrates a new method utilising a certain kind of intermediate semantics for proving correspondences between denotational and operational semantics. The other illustrates a direct technique of Milne, employing recursively-specified inclusive relations.

Key words. lambda calculus, semantics

1. Introduction. Suppose that we have both a (standard) denotational semantics and an operational semantics for some programming language. (For examples see [Sto1], [Mil2], [deB].) We would like to prove that they are *equivalent*, in that the output given by the operational semantics, for each program and input, corresponds exactly to the output specified by the denotational semantics. Thus not only is the former to be *consistent* with the latter, but also it is to be *complete*. For diverging computations, completeness is only required "in the limit": the (perhaps partial) outputs given by finite computations of the operational semantics are to converge to the full output specified by the denotational semantics. Following [Wad], we shall refer to this property as *limiting completeness.*

In [Wad], Wadsworth studied the λ-calculus, considering arbitrary λ-terms as programs (including their "input"). The "approximate normal forms" of terms, obtained by finite sequences of β-reductions (followed by the replacement of remaining β-redexes by a special constant symbol Ω that denoted $\perp$), were regarded as outputs. The classic Approximation Theorem [Wad, Thm. 5.2] shows that the denotations of the approximate normal forms of a term do indeed converge to the denotation of the term itself. Thus the theory of β-conversion, although too weak for proving all equations between terms whose denotations are equal, is complete in this limiting sense, and adequate for the evaluation of λ-terms.

In general, it is quite easy to prove consistency, using structural induction. See [Sto1], [deB] for some examples. We shall not consider consistency proofs any further in this paper. Sometimes, it is also possible to prove limiting completeness quite directly using structural induction with subsidiary appeal to fixpoint induction for recursive constructs such as loops (where structural induction fails). However, when the programming languages concerned allow self-application—either explicitly as with Algol 60 procedures, or implicitly as with dynamic bindings in LISP—then the direct method seems to be precluded, for structural induction fails but there is no obvious recursive construct in the language where fixed-point methods could be applied. In such cases, the domains of denotations are defined reflexively (that is, recursively), and what one wants is to use induction on the level of the projective approximations of the domains.

Wadsworth solved this problem in his study of the λ-calculus [Wad] by labelling expressions M (and their subexpressions) with integers n, so that $M^{(n)}$ denoted the nth projection of the denotation of M. Having introduced some extra syntax to make

* Received by the editors December 26, 1984, and in revised form June 3, 1985. This work was supported in part by the Science and Engineering Research Council.

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the levels visible, he then studied the operational properties of the $M^{(n)}$ induced by this semantics and also their relation to the operational properties of the original M. Thus one parameter—the labelling—was used both for inductions relating to the denotational semantics (the projection levels) and for inductions relating to the operational semantics. It seems fair to say that as a result a somewhat heavy apparatus was obtained.

We present two other approaches to the proof of (the difficult part of) the Approximation Theorem. In one the two inductions are separated and the $M^{(n)}$ avoided by means of an intermediate semantics. In the other, using ideas of Milne [Mil1], [Mil2], [Sto1], [Sto2], the second induction is avoided entirely. In both approaches recursively-specified inclusive relations are used [Mil1]; these play only a technical role in the first one but lie at the heart of the second.

Our intermediate semantics is defined just like the standard denotational semantics, except that intermediate denotations take an argument k in $\mathbb{1}^{\infty}$ (the chain cpo of the extended natural numbers $0 \sqsubseteq 1 \sqsubseteq \cdots \sqsubseteq k \sqsubseteq \cdots \sqsubseteq \infty$). The operational idea is that at finite values of k, an intermediate denotation is to correspond to (perhaps partial) output produced by the operational semantics after at most k steps. In particular, at $k=0$ it is the least element $\perp$. At $k=\infty$ it is by continuity the limit of the intermediate denotations.

The denotational idea is (roughly) that the intermediate denotation at ∞ of any phrase is just a function of the intermediate denotations at ∞ of its sub-phrases. This makes it possible to see the relation between the intermediate and standard denotations. To relate them formally, one defines a recursively-specified inclusive relation between the standard and the intermediate domains. The existence and properties of this relation are established by induction on the projective levels, and in fact the techniques are, for the most part, well known (see [Sto1], [Mul] for example). Then a simple structural induction establishes the relation between the standard and intermediate semantics.

The idea of the second approach is to strengthen the hypothesis so that structural induction succeeds. To do this, one defines (for every environment) a relation between semantic values and terms of the language which holds when the value is less than the limiting value of the term given via its operational semantics and moreover when in all suitable contexts the relation still holds. As explained by Stoy [Sto2] one can think of the relation as being that the value approximates to the term. The suitable contexts are given by other related pairs of values and terms and are determined by examining where the original proof by Structural Induction failed. As usual, the existence of such recursively specified relations is determined by induction on the projective levels and the desired result is shown by establishing by structural induction that the denotation of a term is related to the term itself.

Comparing the two approaches, we see that the second is the more direct while the first provides more information, via the intermediate semantics, on the details of the operational semantics. In that connection we should also remark that Wadsworth's labelled expressions have been very useful in the study of the λ-calculus [Bar]. It may be that the intermediate semantics with its feeling of explaining resource-bounded denotations will also be of use. Technically, as the reader will see, the proofs are less straightforward with the second approach, at least in the present case. The references demonstrate the wide applicability of the second approach; we expect this also for the intermediate semantics method although we have no precise general recipe available for defining such semantics.

We assume the general framework of denotational semantics [Ten], [Sto1], [Mil2], [Gor]. We take domains to be ω-complete partial orders (which are just partial orders

$\mathbf{D}=(D,\sqsubseteq)$ with a least element $\perp_D$, and lubs $\bigsqcup_n x_n$ of increasing sequences). The important functions between domains are those $f:\mathbf{D}\to\mathbf{E}$ which are *continuous*, meaning that if x_n is an increasing sequence in $\mathbf{D}$ then $f(x_n)$ is in $\mathbf{E}$ and then, too, $f(\bigsqcup_n x_n)=\bigsqcup_n f(x_n)$; we write $\mathbf{D}\to\mathbf{E}$ for the cpo of all continuous functions from $\mathbf{D}$ to $\mathbf{E}$, with the pointwise ordering $f\sqsubseteq g$ iff for all in $\mathbf{D}$, $f(x)\sqsubseteq g(x)$. See [SP] for further information on this topic, especially as regards constructions of domains including the solution of recursive domain equations. Our notation is essentially that of [Wad], to facilitate comparison of results, and that of [SP] to handle cpos.

2. The intermediate semantics approach. We shall now prove the completeness of an operational semantics for the λ-calculus relative to its usual denotational semantics, by introducing an appropriate intermediate semantics.

First let us recall the syntax of the λ-calculus. We assume a denumerable set **Var** of variables. Basically, the set **Exp** of *λ-terms* is taken to be the least such that:

- if x is in **Var** then x is in **Exp**;
- if x is in **Var** and M is in **Exp** then $(\lambda x.M)$ is in **Exp**;
- if M and N are in **Exp** then (MN) is in **Exp**.

However, for convenience, we extend **Exp** to include *partial terms* by means of a clause for the special symbol Ω:

- Ω is in **Exp**.

We may now regard **Exp** as a poset, taking the least partial order $\sqsubseteq$ on **Exp** such that:

- $\Omega\sqsubseteq M$, for all M in **Exp**;
- if $M\sqsubseteq M'$ then $(\lambda x.M)\sqsubseteq(\lambda x.M')$ for all M and M' in **Exp** and x in **Var**;
- if $M\sqsubseteq M'$ and $N\sqsubseteq N'$ then $(MN)\sqsubseteq(M'N')$ for all M, M', N, N' in **Exp**.

Note that $(\mathbf{Exp},\sqsubseteq)$ is *not* a cpo. However, we shall ensure that all functions that we define on **Exp** are monotonic.

The standard denotational semantics of the λ-calculus is given by the (monotonic) function $\mathcal{V}:\mathbf{Exp}\to(\mathbf{Env}\to\mathbf{D})$ specified in Table 1. Here $\mathbf{D}$ is taken to be the (initial) solution of $\mathbf{D}\cong\mathbf{D}\to\mathbf{D}$ that includes some arbitrary nontrivial cpo $\mathbf{D}_0$ ([SP] (see also our Appendix)). (This is a slight generalisation of Scott's original $\mathbf{D}_\infty$-model [Sco], [Wad], where only complete lattices were considered.) We omit the isomorphism $\mathbf{D}\cong\mathbf{D}\to\mathbf{D}$ from formulae, when there is no danger of confusion.

TABLE 1
Standard semantics.

Domains	Denotations
$\mathbf{D}\cong\mathbf{D}\to\mathbf{D}$ (see text) $\mathbf{Env}=\mathbf{D}^{\mathbf{Var}}$	$\mathcal{V}:\mathbf{Exp}\to\mathbf{Env}\to\mathbf{D}$ $\mathcal{V}[\![x]\!]\rho=\rho(x)$ $\mathcal{V}[\![\lambda x.M]\!]\rho=\lambda d\in\mathbf{D}.\mathcal{V}[\![M]\!](\rho[d/x])$ $\mathcal{V}[\![MN]\!]\rho=(\mathcal{V}[\![M]\!]\rho)(\mathcal{V}[\![N]\!]\rho)$ $\mathcal{V}[\![\Omega]\!]\rho=\perp_{\mathbf{D}}$

Our operational semantics for the λ-calculus is given by the ω-indexed family of (monotonic) functions $\mathcal{R}^n:\mathbf{Exp}\to\mathbf{Exp}$ defined in Table 2 for any n and any M in **Exp**; $\mathcal{R}^n(M)$ may be regarded as the *partial* normal form of M determined by n steps of a (parallel) reduction algorithm. That is, $\mathcal{R}^n(M)$ can be obtained by making some β-reductions on M, followed by replacing all remaining β-redexes by Ω. Thus each $\mathcal{R}^n(M)$ is an approximate normal form of M in the terminology of [Wad]. Moreover,

TABLE 2
Operational semantics.

$$\begin{array}{l}
\mathscr{R}^n\colon \mathbf{Exp}\to\mathbf{Exp}\ (\text{each } n\in\omega)\\
\quad\mathscr{R}^0(M)=\Omega\\
\quad\mathscr{R}^{n+1}(x)=x\\
\quad\mathscr{R}^{n+1}(\lambda x\,.\,M)=\lambda x\,.\,M' \text{ where } M'=\mathscr{R}^{n+1}(M)\\
\quad\mathscr{R}^{n+1}(MN)=\begin{cases}\Omega \text{ if } M'=\Omega,\\ \mathscr{R}^n([N'/x]M''), \quad \text{if } M'=\lambda x\,.\,M'',\\ (M'N') \quad \text{otherwise,}\\ \text{where } M'=\mathscr{R}^{n+1}(M),\ N'=\mathscr{R}^{n+1}(N)\end{cases}\\
\quad\mathscr{R}^{n+1}(\Omega)=\Omega\\
\quad([-/x]-\colon \mathbf{Exp}\times\mathbf{Exp}\to\mathbf{Exp} \text{ is the usual substitution operator})
\end{array}$$

$\mathscr{R}^n(M)$ is monotonic in n: $\mathscr{R}^n(M)\sqsubseteq\mathscr{R}^{n+1}(M)$, for all n, as can be shown by a simple induction on (n, M) (lexicographically ordered).

The completeness of this operational semantics (relative to the given denotation semantics) is just that for any term M, its denotation $\mathcal{V}\llbracket M\rrbracket$ is included in the limit of the denotations $\mathcal{V}\llbracket\mathscr{R}^n(M)\rrbracket$, as n goes to ∞—this limit exists since $\mathcal{V}$ is monotonic (and $\mathbf{Env}\to\mathbf{D}$ is a cpo).

THEOREM 1. *For M in* $\mathbf{Exp}$

$$\mathcal{V}\llbracket M\rrbracket\sqsubseteq\bigsqcup_{n\geqq 0}\mathcal{V}\llbracket\mathscr{R}^n(M)\rrbracket.$$

An immediate corollary of this theorem is that for all M,

$$\mathcal{V}\llbracket M\rrbracket\sqsubseteq\bigsqcup\{\mathcal{V}\llbracket A\rrbracket \mid A \text{ is an approximate normal form of } M\},$$

which is the hard part of Wadworth's classic Approximation Theorem for the λ-calculus [12]. (The reverse directions of both these theorems follow easily from the consistency of β-reduction and the minimality of Ω, with respect to $\mathcal{V}$. We shall not consider consistency any further in this paper.)

To prove our theorem, we introduce an intermediate semantics $\mathcal{V}'\colon\mathbf{Exp}\to\mathbb{1}^\infty\to\mathbf{Env}'\to\mathbf{D}'$, as given in Table 3 ($\mathcal{V}'\colon\mathbf{Exp}\to\mathbf{Env}'\to\mathbb{1}^\infty\to\mathbf{D}'$ might be considered more natural, but leads to clumsier statements of theorems). Recall that $\mathbb{1}^\infty$ is the chain cpo of the extended natural numbers. There is an evident embedding of ω in $\mathbb{1}^\infty$, such that the usual operations of successor, predecessor and minimum on ω have unique continuous extensions to $\mathbb{1}^\infty$—we get $\infty=\infty+1=\infty-1$ and $k \min \infty = k$. The domain

TABLE 3
Intermediate semantics.

Domains	Denotations
$\mathbf{D}'=(\mathbb{1}^\infty\to\mathbf{D}')\to\mathbf{D}'$ (see text) $\mathbf{Env}'=(\mathbb{1}^\infty\to\mathbf{D}')^{\mathbf{Var}}$	$\mathcal{V}'\colon\mathbf{Exp}\to\mathbb{1}^\infty\to\mathbf{Env}'\to\mathbf{D}'$ $\mathcal{V}'\llbracket x\rrbracket k\rho'=\begin{cases}\bot_{\mathbf{D}'} & \text{if } k=0,\\ \rho'(x)k & \text{otherwise,}\end{cases}$ $\mathcal{V}'\llbracket\lambda x\,.\,M\rrbracket\in k\rho'=\begin{cases}\bot_{\mathbf{D}} & \text{if } k=0,\\ \lambda c'\mathbb{1}^\infty\to\mathbf{D}'\,.\,\mathcal{V}'\llbracket M\rrbracket k(\rho'[c'/x]) & \text{otherwise}\end{cases}$ $\mathcal{V}'\llbracket MN\rrbracket k\rho'=\begin{cases}\bot_{\mathbf{D}'} \quad \text{if } k=0,\\ (\mathcal{V}'\llbracket M\rrbracket(k-1)\rho')(\lambda k'\in\mathbb{1}^\infty\,.\,\mathcal{V}'\llbracket N\rrbracket(k' \min k)\rho')\\ \quad\text{otherwise}\end{cases}$ $\mathcal{V}'\llbracket\Omega\rrbracket k\rho'=\bot_{\mathbf{D}'}$

$\mathbf{D}'$ is taken to be the (initial) solution of $\mathbf{D}' \simeq (\mathbb{1}^{\infty} \to \mathbf{D}') \to \mathbf{D}'$ that includes the same nontrivial domain $\mathbf{D}_0$ that we included earlier in $\mathbf{D}$. Again we often omit the isomorphism. As it happens, $\mathcal{V}'$ does not respect β-reduction: in general only $\mathcal{V}'[\![(\lambda^x M)N]\!] \sqsubseteq \mathcal{V}'[\![[N/x]M]\!]$ holds—but that is not important here.

Our first lemma relates $\mathcal{R}^n(M)$ to $\mathcal{V}'$ at n.

LEMMA 1. *For all $n < \infty$ and M in* **Exp**,

$$\mathcal{V}'[\![M]\!]n \sqsubseteq \mathcal{V}'[\![\mathcal{R}^n(M)]\!]n.$$

We shall prove this lemma by induction on (n, M), later in this section. By the continuity of $\mathcal{V}'[\![M]\!]n$ in n, we then get

$$(1) \qquad \mathcal{V}'[\![M]\!]\infty \sqsubseteq \bigsqcup_{n \geqq 0} \mathcal{V}'[\![\mathcal{R}^n(M)]\!]\infty.$$

Now putting ∞ for k in Table 3, it seems quite obvious that $\mathcal{V}[\![M]\!]$ is equivalent to $\mathcal{V}'[\![M]\!]\infty$, and that we should be able to infer the required result, namely,

$$(2) \qquad \mathcal{V}[\![M]\!] \sqsubseteq \bigsqcup_{n \geqq 0} \mathcal{V}[\![\mathcal{R}^n(M)]\!].$$

However, to prove this "obvious" equivalence between $\mathcal{V}[\![M]\!]$ and $\mathcal{V}'[\![M]\!]\infty$, we need to relate values of $\mathbf{D}$ and $\mathbf{D}'$. As we have already remarked, the techniques for this are quite standard but they appeal to the construction of solutions to reflexive domain equations, and we relegate the details to Appendix I. It would be routine to relate any reflexively-defined domain to an analogous one involving $\mathbb{1}^{\infty}$, in essentially the same way. (Thus the proofs of the remaining lemmas except perhaps that of Lemma 4 should not really be considered when assessing the complexity of our approach.)

LEMMA 2. *There exists a relation* $\sim \subseteq \mathbf{D} \times \mathbf{D}'$ *such that*

(i) *for all d in $\mathbf{D}_0$, $d \sim d$ under the inclusion of $\mathbf{D}_0$ in $\mathbf{D}$ and $\mathbf{D}'$*;

(ii) *for all d in $\mathbf{D}$ and d' in $\mathbf{D}'$, $d \sim d'$ iff for all e in $\mathbf{D}$ and c' in $\mathbb{1}^{\infty} \to \mathbf{D}'$, $e \sim c'(\infty)$ implies $d(e) \sim d'(c')$.*

The proof of Lemma 2 may be found in the Appendix.

The next lemma states that $\mathcal{V}[\![M]\!]$ and $\mathcal{V}'[\![M]\!]\infty$ are related by the relation provided by Lemma 2.

LEMMA 3. *For all M in* **Exp** *and ρ in* **Env** *and ρ' in* **Env**$'$, *if for all z in* **Var**, *$\rho(z) \sim \rho'(z)\infty$ then $\mathcal{V}[\![M]\!]\rho \sim \mathcal{V}'[\![M]\!]\infty\rho'$.*

We shall prove this lemma by induction on M, later in this section.

According to the next lemma, if $d \sim d'$ then d is (continuously) determined by d'. This will allow us to infer (2) from (1) above.

LEMMA 4. *There exist continuous functions $s: \mathbf{D} \to \mathbf{D}'$ and $r: \mathbf{D}' \to \mathbf{D}$ such that*

(i) *for all d in $\mathbf{D}$, $d \sim s(d)$*;

(ii) *for all d in $\mathbf{D}$ and d' in $\mathbf{D}'$, if $d \sim d'$ then $d = r(d')$.*

The proof of Lemma 4 may be found in the Appendix.

We now show how our theorem follows from Lemmas 1, 3 and 4.

Proof of Theorem 1. Let M in **Exp** and ρ in **Env** be arbitrary. The required result will follow if we show that

$$\mathcal{V}[\![M]\!]\rho \sqsubseteq \bigsqcup_{n \geqq 0} \mathcal{V}[\![\mathcal{R}^n(M)]\!]\rho.$$

Define ρ' in **Env**$'$ by

$$\rho'(x) = \lambda k \in \mathbb{1}^{\infty} . \, s(\rho(x))$$

where $s: \mathbf{D} \to \mathbf{D}'$ is as in Lemma 4(i), so that for all z in **Var**,

$$\rho(z) \sim s(\rho(z)) = \rho'(z)\infty.$$

Now by Lemma 1, for all n in $\omega(\subseteq \mathbb{1}^\infty)$ we have

$$\mathcal{V}'[\![M]\!]n\rho' \sqsubseteq \mathcal{V}'[\![\mathcal{R}^n(M)]\!]n\rho'$$

which by continuity in n and $\bigsqcup_{n\geqq 0} n = \infty$ (in $\mathbb{1}^\infty$) gives

$$\mathcal{V}'[\![M]\!]\infty\rho' \sqsubseteq \bigsqcup_{n\geqq 0} \mathcal{V}'[\![\mathcal{R}^n(M)]\!]\infty\rho'. \tag{3}$$

Lemma 3 gives

$$\mathcal{V}[\![M]\!]\rho \sim \mathcal{V}'[\![M]\!]\infty\rho' \tag{4}$$

and that for all n,

$$\mathcal{V}[\![\mathcal{R}^n(M)]\!]\rho \sim \mathcal{V}'[\![\mathcal{R}^n(M)]\!]\infty\rho'. \tag{5}$$

Let $r:\mathbf{D}'\to\mathbf{D}$ be as in Lemma 4(ii). Then we have

$$\begin{aligned}\mathcal{V}[\![M]\!]\rho &= r(\mathcal{V}'[\![M]\!]\infty\rho') \quad \text{from (4)}\\ &\sqsubseteq r\Big(\bigsqcup_{n\geqq 0} \mathcal{V}'[\![\mathcal{R}^n(M)]\!]\infty\rho'\Big) \quad \text{by (3)}\\ &= \bigsqcup_{n\geqq 0} r(\mathcal{V}'[\![\mathcal{R}^n(M)]\!]\infty\rho') \quad \text{by continuity}\\ &= \bigsqcup_{n\geqq 0} \mathcal{V}[\![\mathcal{R}^n(M)]\!]\rho \quad \text{from (5),}\end{aligned}$$

as required.

We now return to our lemmata. The proof of Lemma 1 will make use of the following standard lemma about substitution, which confirms that the operator $[-/x]-$ has been defined correctly so as to respect the static determination of bindings.

Substitution Lemma. *For all M and N in* **Exp** *and ρ' in* **Env**′ *and k in $\mathbb{1}^\infty$,*

$$\mathcal{V}'[\![[N/x]M]\!]k\rho' = \mathcal{V}'[\![M]\!]k(\rho'[(\lambda k' \in \mathbb{1}^\infty.\, \mathcal{V}'[\![N]\!]k'\rho')/x]).$$

We shall omit the straightforward but tedious proof by induction (on M), which is entirely analogous to the usual substitution lemma for the standard semantics, $\mathcal{V}$.

Proof of Lemma 1. We are to show that for all $n<\infty$ and M in **Exp**,

$$\mathcal{V}'[\![M]\!]n \sqsubseteq \mathcal{V}'[\![\mathcal{R}^n(M)]\!]n.$$

We use induction on (n, M). For $n=0$ and any M, we have $\mathcal{V}'[\![M]\!]0 = \bot \sqsubseteq \mathcal{V}'[\![\mathcal{R}^0(M)]\!]0$, as required.

For $n+1$ and any M_0, let v and w abbreviate $\mathcal{V}'[\![M_0]\!](n+1)$ and $\mathcal{V}'[\![\mathcal{R}^{n+1}(M_0)]\!](n+1)$, respectively. We shall show that $v\sqsubseteq w$, as required, by cases on M_0.

For Ω, we have $v=\bot$, hence $v\sqsubseteq w$.

For x, we have $\mathcal{R}^{n+1}(x)=x$, giving $v=w$.

For $(\lambda x.\, M)$, we have $\mathcal{R}^{n+1}(\lambda x.\, M) = (\lambda x.\, M')$, where $M'=\mathcal{R}^{n+1}(M)$. By the induction hypothesis for $(n+1, M)$ we have $\mathcal{V}'[\![M]\!](n+1) \sqsubseteq \mathcal{V}'[\![M']\!](n+1)$, hence $v\sqsubseteq w$ (by monotonicity).

Finally, for (MN), take ρ' in **Env**′. Then

$$v\rho' = (\mathcal{V}'[\![M]\!]n\rho')(\lambda k' \in \mathbb{1}^\infty.\, \mathcal{V}'[\![N]\!](k' \min n+1)\rho').$$

Let $M'=\mathcal{R}^{n+1}(M)$ and $N'=\mathcal{R}^{n+1}(N)$. By the induction hypothesis, we have

$$\begin{aligned}\mathcal{V}'[\![M]\!]n &\sqsubseteq \mathcal{V}'[\![\mathcal{R}^n(M)]\!]n\\ &\sqsubseteq \mathcal{V}'[\![M']\!]n \quad \text{(by monotonicity);}\end{aligned}$$

also for any $k' \in \mathbb{1}^\infty$, we have $k' \min n+1 \leqq n+1$ so that

$$\begin{aligned}\mathcal{V}'\llbracket N\rrbracket(k' \min n+1) &\sqsubseteq \mathcal{V}'\llbracket \mathscr{R}^{k' \min n+1}(N)\rrbracket(k' \min n+1)\\ &\sqsubseteq \mathcal{V}'\llbracket N'\rrbracket(k' \min n+1) \quad \text{(by monotonicity).}\end{aligned}$$

Putting these calculations together, we get

$$(6) \qquad v\rho' \sqsubseteq (\mathcal{V}'\llbracket M'\rrbracket n\rho')(\lambda k' \in \mathbb{1}^\infty .\ \mathcal{V}'\llbracket N'\rrbracket(k' \min n+1)\rho').$$

We shall now consider the cases for M', but note first that when $n=0$, we have $\mathcal{V}'\llbracket M'\rrbracket 0\rho' = \bot$, so that (6) gives $v\rho' \sqsubseteq \bot \sqsubseteq w\rho'$, and hence $v \sqsubseteq w$ as required. We may now suppose $n > 0$.

For $M' = \Omega$, we have $\mathcal{V}'\llbracket \Omega\rrbracket n\rho' = \bot$, and then (6) again gives $v\rho' = \bot$ and $v \sqsubseteq w$ as required.

For $M' = \lambda x.\, M''$, we have

$$\begin{aligned} v\rho' &\sqsubseteq (\mathcal{V}'\llbracket M'\rrbracket n\rho')(\lambda k' \in \mathbb{1}^\infty .\ \mathcal{V}'\llbracket N'\rrbracket k'\rho') \quad \text{by } k' \min n+1 \sqsubseteq k'\\ &= \mathcal{V}'\llbracket M''\rrbracket n(\rho'[(\lambda k' \in \mathbb{1}^\infty .\ \mathcal{V}'\llbracket N'\rrbracket k'\rho')/x])\\ &= \mathcal{V}'\llbracket [N'/x]M''\rrbracket n\rho' \quad \text{by the substitution lemma}\\ &\sqsubseteq \mathcal{V}'\llbracket \mathscr{R}^n([N'/x]M'')\rrbracket n\rho' \quad \text{by the induction hypothesis}\\ &= \mathcal{V}'\llbracket \mathscr{R}^{n+1}(MN)\rrbracket n\rho'\\ &\sqsubseteq \mathcal{V}'\llbracket \mathscr{R}^{n+1}(MN)\rrbracket(n+1)\rho' = w\rho' \end{aligned}$$

giving $v \sqsubseteq w$, as required.

In the remaining cases for M' (viz. x and $(M''N'')$) we have $\mathscr{R}^{n+1}(MN) = (M'N')$. Hence

$$w\rho' = (\mathcal{V}'\llbracket M'\rrbracket n\rho')(\lambda k' \in \mathbb{1}^\infty .\ \mathcal{V}'\llbracket N'\rrbracket(k' \min n+1)\rho')$$

so $v\rho' \sqsubseteq w\rho'$ by (6), giving $v \sqsubseteq w$ as required.

This exhausts the cases for M', thus completing the final case (MN) for M_0. Having now completed the induction step to $n+1$, we may infer the required result. □

We finish this section with a proof of Lemma 3. Its simplicity justifies our earlier remarks about the obviousness of the equivalence of $\mathcal{V}\llbracket M\rrbracket$ and $\mathcal{V}'\llbracket M\rrbracket\infty$.

Proof of Lemma 3. We use (structural) induction on M. Our induction hypothesis is that for all components M of M_0 in **Exp**,

(7) for all ρ in **Env** and ρ' in **Env**$'$, if for all z in **Var**, $\rho(z) \sim \rho'(z)\infty$
then $\mathcal{V}\llbracket M\rrbracket\rho \sim \mathcal{V}'\llbracket M\rrbracket\infty\rho'$.

We shall show that (7) then holds also for M_0, by cases on M_0.

For Ω, we have $\mathcal{V}\llbracket\Omega\rrbracket\rho = \bot_{\mathbf{D}} = \bot_{\mathbf{D}_0} = \bot_{\mathbf{D}'} = \mathcal{V}'\llbracket\Omega\rrbracket\infty\rho'$, for any ρ and ρ' (considering the inclusion of $\mathbf{D}_0$ in $\mathbf{D}$ and $\mathbf{D}'$). But by Lemma 2(i), $\bot_{\mathbf{D}} \sim \bot_{\mathbf{D}'}$. Hence (7) holds for Ω.

For x, we have $\mathcal{V}\llbracket x\rrbracket\rho = \rho(x)$ and $\mathcal{V}'\llbracket x\rrbracket\infty\rho' = \rho'(x)\infty$, for any ρ and ρ'. So (7) holds for x.

For $(\lambda x.\, M)$, take ρ and ρ' such that for all z in **Var**, $\rho(z) \sim \rho'(z)\infty$. Also take e in $\mathbf{D}$ and c' in $\mathbb{1}^\infty \to \mathbf{D}'$ such that $e \sim c'(\infty)$. We have

$$\begin{aligned}(\mathcal{V}\llbracket \lambda x.\, M\rrbracket\rho)(e) &= \mathcal{V}\llbracket M\rrbracket(\rho[e/x])\\ &\sim \mathcal{V}'\llbracket M\rrbracket\infty(\rho'[c'/x]) \quad \text{by the induction hypothesis}\\ &= (\mathcal{V}'\llbracket \lambda x.\, M\rrbracket\infty\rho')(c')\end{aligned}$$

so by Lemma 2(ii) we get $\mathcal{V}\llbracket \lambda x.\, M\rrbracket\rho \sim \mathcal{V}'\llbracket \lambda x.\, M\rrbracket\infty\rho'$, showing that (7) holds for $(\lambda x.\, M)$.

Finally, for (MN), take ρ and ρ' such that for all z, $\rho(z) \sim \rho'(z)\infty$. Now by the induction hypothesis,

$$\begin{aligned} \mathcal{V}[\![M]\!]\rho &\sim \mathcal{V}'[\![M]\!]\infty\rho' \quad \text{and} \\ \mathcal{V}[\![N]\!]\rho &\sim \mathcal{V}'[\![N]\!]\infty\rho' \\ &= (\lambda k' \in \mathbb{1}^{\infty} .\, \mathcal{V}'[\![N]\!](k' \min \infty)\rho')(\infty) \end{aligned}$$

so that

$$\begin{aligned} \mathcal{V}[\![MN]\!]\rho &= (\mathcal{V}[\![M]\!]\rho)(\mathcal{V}[\![N]\!]\rho) \\ &\sim (\mathcal{V}'[\![M]\!]\infty\rho')(\lambda k' \in \mathbb{1}^{\infty} .\, \mathcal{V}'[\![N]\!](k' \min \infty)\rho') \\ &= \mathcal{V}'[\![MN]\!]\infty\rho' \end{aligned}$$

so (7) holds in this case also.

There are no more cases, so we may infer the required result. □

3. The relational approach. Here we use a different operational semantics $\mathcal{S}^n : \mathbf{Exp} \to \mathbf{Exp}$, (given in Table 4) which is "outside-in" whereas the previous one was "inside-out" (and this seems to be needed to make the relational approach work). Again each $\mathcal{S}^n(M)$ is an approximate form of M, monotonic in n. And we will show:

THEOREM 2. *For all M in* **Exp**

$$\mathcal{V}[\![M]\!] \sqsubseteq \bigsqcup_{n \geqq 0} \mathcal{V}[\![\mathcal{S}^n(M)]\!]$$

thereby providing our second proof of Wadsworth's theorem.

If one tries to prove this directly by structural induction on M, the proof breaks down in the case where M is an application. Consequently, following the idea explained above for every ρ in **Env** we wish to define a relation $\sim_\rho$ between **D** and **Exp** so that:

$$d \sim_\rho M \quad \text{iff} \quad d \sqsubseteq \bigsqcup_{n \geqq 0} \mathcal{V}[\![\mathcal{S}^n(M)]\!]\rho \quad \text{and}$$

$$\forall e \in \mathbf{D},\ N \in \mathbf{Exp} .\ e \sim_\rho N \supset d(e) \sim_\rho (MN).$$

These relations are constructed in Appendix II where we also demonstrate the very useful:

LEMMA 5. *Suppose d in* **D** *and M in* **Exp** *are such that whenever $e_i \sim_\rho N_i$ for $i = 1$, n' it is the case that*

$$d(e_1) \cdots (e_{n'}) \sqsubseteq \bigsqcup_{n} \mathcal{V}[\![\mathcal{S}^n(MN_1 \cdots N_{n'})]\!]\rho.$$

Then it follows that $d \sim_\rho M$.

Note that it follows that $\perp \sim_\rho M$ always holds.

Now to demonstrate Theorem 2, it is clearly enough to show that $\mathcal{V}[\![M]\!](\rho) \sim_\rho M$ always holds. This is now done by proving a stronger statement (to handle free variables) by structural induction on M.

TABLE 4
Another operational semantics.

$$\begin{aligned} &\mathcal{S}^n : \mathbf{Exp} \to \mathbf{Exp} \quad (\text{each } n \in \omega) \\ &\mathcal{S}^0(M) = \Omega \\ &\mathcal{S}^{n+1}(\Omega N_1 \cdots N_{n'}) = \Omega \\ &\mathcal{S}^{n+1}(xN_1 \cdots N_{n'}) = x\mathcal{S}^n(N_1) \cdots \mathcal{S}^n(N_{n'}) \\ &\mathcal{S}^{n+1}((\lambda x .\, M)N_1 \cdots N_{n'}) = \begin{cases} \lambda x \mathcal{S}^n(m) & \text{if } n' = 0, \\ \mathcal{S}^n(([N_1/x]M)N_2 \cdots N_{n'}) & \text{if } n' \neq 0 \end{cases} \end{aligned}$$

LEMMA 6. *Let M be an expression and ρ be an environment. Suppose that $a_j \sim_\rho A_j$ for $j = 1, m$. Then*

$$\mathcal{V}\llbracket M \rrbracket \rho[a_1/x_1, \cdots, a_m/x_m] \sim_\rho [A_1/x_1, \cdots, A_m/x_m]M$$

holds, where the x_i are distinct variables.

Proof. By structural induction on M. We will write $\tilde{\rho}$ for $\rho[a_1/x_1, \cdots, a_m/x_m]$ and $\tilde{K}$ for $[A_1/x_1, \cdots, A_m/x_m]K$ (for any term K), where the a_j, A_j, x_j will be understood from the context.

Case I. M is an application, (M_1M_2).

By induction hypothesis we have $\mathcal{V}\llbracket M_i \rrbracket(\tilde{\rho}) \sim_\rho \tilde{M}_i$ $(i = 1, 2)$.

So $\mathcal{V}\llbracket M_1 \rrbracket(\tilde{\rho})(\mathcal{V}\llbracket M_2 \rrbracket(\tilde{\rho})) \sim_\rho (\tilde{M}_1\tilde{M}_2)$. But as the left-hand side is $\mathcal{V}\llbracket M_1M_2 \rrbracket(\tilde{\rho})$ and the right-hand side $\widetilde{(M_1M_2)}$ so we are finished.

Case II. M is Ω. This is by the remark after Lemma 5.

Case III. M is a variable, x:

Subcase 1. x is some x_j.

Then $\mathcal{V}\llbracket x \rrbracket \tilde{\rho} = a_j \sim_\rho A_j = \tilde{x}$.

Subcase 2. x is no x_j. Then we must show that $\mathcal{V}\llbracket x \rrbracket \rho \sim_\rho x$ and we employ Lemma 5, taking $e_i \sim_\rho N_i$ $(i = 1, n')$ and calculating:

$$\begin{aligned}
&\mathcal{V}\llbracket x \rrbracket(\rho)(e_1) \cdots (e_{n'}) \\
&\quad \sqsubseteq \mathcal{V}\llbracket x \rrbracket(\rho)\Big(\bigsqcup_n \mathcal{V}\llbracket \mathcal{S}^n N_1 \rrbracket(\rho)\Big) \cdots \Big(\bigsqcup_n \mathcal{V}\llbracket \mathcal{S}^n N_{n'} \rrbracket(\rho)\Big) \qquad (\text{since } e_i \sim_\rho N_i) \\
&\quad = \bigsqcup_n \mathcal{V}\llbracket x \rrbracket(\rho)(\mathcal{V}\llbracket \mathcal{S}^n N_1 \rrbracket(\rho)) \cdots (\mathcal{V}\llbracket \mathcal{S}^n N_{n'} \rrbracket(\rho)) \\
&\quad = \bigsqcup_n \mathcal{V}\llbracket x(\mathcal{S}^n N_1) \cdots (\mathcal{S}^n N_{n'}) \rrbracket(\rho) \\
&\quad = \bigsqcup_n \mathcal{V}\llbracket \mathcal{S}^{n+1}(xN_1 \cdots N_{n'}) \rrbracket(\rho)
\end{aligned}$$

as required.

Case IV. M is an abstraction, $\lambda x . M'$. We may assume without loss of generality that x is no x_j and x does not occur free in any of the A_j.

We apply Lemma 5 and take $e_i \sim_\rho N_i$ $(i = 1, n')$.

Subcase 1. $n' = 0$. We calculate

$$\begin{aligned}
\mathcal{V}\llbracket \lambda x . M' \rrbracket \tilde{\rho} &= \lambda d \in \mathbf{D} . \mathcal{V}\llbracket M' \rrbracket \widetilde{\rho[d/x]} \qquad (\text{as } x \text{ is no } x_j) \\
&\sqsubseteq \lambda d \in \mathbf{D} . \bigsqcup_n \mathcal{V}\llbracket \mathcal{S}^n \tilde{M}' \rrbracket \rho[d/x] \\
&\qquad\qquad (\text{by the induction hypothesis applied to } M') \\
&= \bigsqcup_n \lambda d \in \mathbf{D} . \mathcal{V}\llbracket \mathcal{S}^n \tilde{M}' \rrbracket(\rho[d/x]) \\
&= \bigsqcup_n \mathcal{V}\llbracket \lambda x . \mathcal{S}^n \tilde{M}' \rrbracket(\rho) \\
&= \bigsqcup_n \mathcal{V}\llbracket \mathcal{S}^{n+1}(\lambda x . \tilde{M}') \rrbracket(\rho) \\
&= \bigsqcup_n \mathcal{V}\llbracket \mathcal{S}^{n+1}(\widetilde{\lambda x . M'}) \rrbracket(\rho)
\end{aligned}$$

as required.

Subcase 2. $n'>0$. First calculate that:

$$\begin{aligned}
&\mathcal{V}[\![\lambda x\,.\,M']\!](\tilde{\rho})(e_1)\\
&\quad=\mathcal{V}[\![M']\!]\rho[e_1/x, a_1/x_1,\cdots,a_m/x_m] \quad (\text{as } x \text{ is no } x_j)\\
&\quad\sim_\rho [N_1/x, A_1/x_1,\cdots,A_m/x_m]M'\\
&\qquad\qquad\qquad (\text{as } e_1\sim_\rho N_1 \text{ and by induction hypothesis applied to } M')\\
&\quad=[N_1/x]\tilde{M}' \ (\text{since } x \text{ is no } x_j \text{ and does not occur free in any } A_j).
\end{aligned}$$

So as $e_2\sim_\rho N_2,\cdots,e_{n'}\sim_\rho N_{n'}$ we have that

$$\mathcal{V}[\![\lambda x\,.\,M']\!](\tilde{\rho})e_1\cdots e_{n'}\sim_\rho([N_1/x]\tilde{M}')N_2\cdots N_{n'}$$

and so we see that

$$\begin{aligned}
\mathcal{V}[\![\lambda x\,.\,M']\!](\tilde{\rho})e_1\cdots e_{n'}&\sqsubseteq\bigsqcup_n \mathcal{V}[\![\mathcal{S}^n(([N_1/x]\tilde{M}')N_2\cdots N_{n'})]\!](\rho)\\
&=\bigsqcup_n \mathcal{V}[\![\mathcal{S}^{n+1}((\lambda x\,.\,\tilde{M}')N_1N_2\cdots N_{n'})]\!](\rho)\\
&=\bigsqcup_n \mathcal{V}[\![\mathcal{S}^{n+1}((\widetilde{\lambda x\,.\,M'})N_1\cdots N_{n'})]\!](\rho)
\end{aligned}$$

as required, concluding the proof. □

Appendix I.

Construction of **D** *and* **D**′. This proceeds as described in [SP]. Let $\mathbf{D}_0$ be a fixed (nontrivial) cpo. For the construction of **D** satisfying

$$\mathbf{D}\cong(\mathbf{D}\to\mathbf{D})$$

first define an ω-chain $\Delta=\langle\mathbf{D}_n,f_n\rangle$ of cpos $\mathbf{D}_n$ and embeddings $f_n:\mathbf{D}_n\to\mathbf{D}_{n+1}$ by

$$\mathbf{D}_{n+1}=(\mathbf{D}_n\to\mathbf{D}_n)$$

and

$$\begin{aligned}
f_0(d)&=\lambda e\in\mathbf{D}_0\,.\,d, & f_0^R(g)&=g(\perp_{\mathbf{D}_0}),\\
f_{n+1}(g)&=f_n\circ g\circ f_n^R, & f_{n+1}^R(h)&=f_n^R\circ h\circ f_n.
\end{aligned}$$

Next **D** is the colimit of Δ, being the sub-cpo of $\prod_n \mathbf{D}_n$ of all sequences d with, for every n, $d_n=f_n^R(d_{n+1})$. We have a cone $\mu:\Delta\to D$ of embeddings where

$$(\mu_n(d_n))_m=\begin{cases} f_{nm}(d_n) & (m\geqq n),\\ f_{mn}^R(d_n) & (m<n),\end{cases}$$

and $\mu_n^R(d)=d_n$, for all n (and where $f_{nm}=f_{m-1}\circ\cdots\circ f_n$, $f_{nm}^R=f_n^R\circ\cdots\circ f_{m-1}^R$). Note that μ_0 is the inclusion of $\mathbf{D}_0$ in **D**. Now we have a cone $\nu:\Delta^-\to(\mathbf{D}\to\mathbf{D})$ where $\Delta^-=\langle\mathbf{D}_{n+1},f_{n+1}\rangle$ is Δ less its first element and $\nu_n(g)=\mu_n\circ g\circ\mu_n^R$ (and $\nu_n^R(h)=\mu_n^R\circ h\circ\mu_n$).

Then the isomorphism pair

$$\mathbf{D}\underset{\Psi}{\overset{\Phi}{\rightleftarrows}}(\mathbf{D}\to\mathbf{D})$$

is given by the formulae:

$$\Phi=\bigsqcup_{n\geqq 0}\nu_n\circ\mu_{n+1}^R,\qquad \Psi=\bigsqcup_{n\geqq 0}\mu_{n+1}\circ\nu_n^R.$$

We need to examine the first formula in more detail.

For Φ we calculate that for any d, e in $\mathbf{D}$

$$(\Phi(d)(e)) = \left(\bigsqcup_m \mu_m(d_{m+1}(\mu_m^R(e)))\right) = \bigsqcup_m \mu_m(d_{m+1}(e_m))$$

and then note that for any n, m

$$\begin{aligned}(\Phi(d)(e))_n &= \bigsqcup_{m \geqq n} \mu_n^R(\mu_m(d_{m+1}(e_m)))\\ &= \bigsqcup_{m \geqq n} f_{nm}^R(d_{m+1}(e_m))\end{aligned}$$

(making use of the calculation, $\mu_n^R \circ \mu_m = (\mu_m \circ f_{nm})^R \circ \mu_m = f_{nm}^R \circ \mu_m^R \circ \mu_m = f_{nm}^R$). In the case where $e = \mu_n(e_n)$ with e_n in $\mathbf{D}_n$ we have

$$\begin{aligned}(\Phi(d)(\mu_n e_n))_n &= \bigsqcup_{m \geqq n} f_{nm}^R(d_{m+1}((\mu_n e_n)_m))\\ &= \bigsqcup_{m \geqq n} f_{nm}^R(d_{m+1}(f_{nm}(e_n)))\\ &= \bigsqcup_{m \geqq n} f_{(n+1)(m+1)}^R(d_{m+1})(e_n)\\ &= d_{n+1}(e_n).\end{aligned}$$

Turning to $\mathbf{D}'$, which is to satisfy the isomorphism

$$\mathbf{D}' \cong (\mathbb{1}^\infty \to \mathbf{D}') \to \mathbf{D}',$$

we take the $\boldsymbol{\omega}$-chain $\Delta' = \langle \mathbf{D}'_n, f'_n \rangle$ of cpos and embeddings where

$$\mathbf{D}'_0 = \mathbf{D}_0, \qquad \mathbf{D}'_{n+1} = (\mathbb{1}^\infty \to \mathbf{D}'_n) \to \mathbf{D}'_n$$

and

$$\begin{aligned}&f'_0(d') = \lambda c' \in (\mathbb{1}^\infty \to \mathbf{D}'_0)\,.\, d', \qquad f_0'^R(g') = g'(\bot_{\mathbb{1}^\infty \to \mathbf{D}'}),\\ &f'_{n+1}(g') = f'_n \circ g' \circ (\lambda c' \in \mathbb{1}^\infty \to \mathbf{D}'_{n+1}\,.\, f_n'^R \circ c),\\ &f_{n+1}'^R(h') = f_n'^R \circ h' \circ (\lambda c' \in \mathbb{1}^\infty \to \mathbf{D}'_n\,.\, f'_n \circ c').\end{aligned}$$

Now $\mathbf{D}'$ is the colimit of Δ' being the sub-cpo of $\prod_n \mathbf{D}'_n$ of all sequences d' with, for every n, $d'_n = f_n'^R(d'_{n+1})$. We have a cone $\mu' : \Delta' \to \mathbf{D}'$ of embeddings defined as before (and with $\mu_n'^R(d') = d'_n$, as before). Now we have a cone ν' from Δ'^- (defined as before) to $(\mathbb{1}^\infty \to \mathbf{D}') \to \mathbf{D}'$, where $\nu'_n(g') = \mu'_n \circ g' \circ (\lambda c \in \mathbb{1}^\infty \to \mathbf{D}'\,.\, \mu_n'^R \circ c)$ (and $\nu_n'^R(h') = \mu_n'^R \circ h' \circ (\lambda c \in \mathbb{1}^\infty \to \mathbf{D}'_n\,.\, \mu'_n \circ c)$).

Then the isomorphism pair

$$\begin{tikzcd} \mathbf{D}' & (\mathbb{1}^\infty \to \mathbf{D}') \to \mathbf{D}' \arrow[l, shift right, "\Phi'"'] \arrow[l, shift left, "\Psi'"] \end{tikzcd}$$

is given by the analogous formulae to those for Φ and Ψ. Upon detailed examination these yield for any d' in $\mathbf{D}'$ and $c' \in (\mathbb{1}^\infty \to \mathbf{D}')$ that

$$(\Phi'(d')(c'))_n = \bigsqcup_{m \geqq n} f_{nm}'^R(d'_{m+1}(\mu_m'^R \circ c')).$$

Also for any $d' \in \mathbf{D}'$ and $c'_n \in (\mathbb{1}^\infty \to \mathbf{D}'_n)$ we have that

$$(\Phi'(d')(\mu'_n \circ c'_n))_n = d'_{n+1}(c'_n)$$

The relation between $\mathbf{D}$ *and* $\mathbf{D}'$. First we inductively construct relations $\sim_n$ between $\mathbf{D}_n$ and $\mathbf{D}'_n$ by

$$d \sim_0 d' \quad \text{iff } d = d';$$

$$d \sim_{n+1} d' \quad \text{iff } \forall e \in \mathbf{D}_n, c' \in (\mathbb{1}^\infty \to \mathbf{D}'_n) \,.\, e \sim_n c'(\infty) \supset d(e) \sim_n d'(c').$$

We recall that for cpos $\mathbf{D}$ and $\mathbf{E}$ a relation $R \subseteq \mathbf{D} \times \mathbf{E}$ is *inclusive* (termed ω-complete in [SP]) iff it is closed under lubs of increasing sequences, which is to say that if d_n, e_n are increasing sequences in $\mathbf{D}$ and $\mathbf{E}$ respectively, then if $R(d_n, e_n)$ holds for every n so does $R(\bigsqcup d_n, \bigsqcup e_n)$.

LEMMA 7. *Each* $\sim_n$ *is inclusive.*

Proof. By induction on n. For $n = 0$ this is obvious. For $n + 1$ assume for all $m \geqq 0$ that $d_m \sim_{n+1} d'_m$ where d, d' are increasing sequences. Take e in $\mathbf{D}_n$, c' in $\mathbb{1}^\infty \to \mathbf{D}'_n$ with $e \sim c'(\infty)$. Then $d_m(e) \sim_n d'_m(c')$ and so $\bigsqcup_m (d_m(e)) \sim_n \bigsqcup_m (d'_m(c'))$, by induction hypothesis and so we have $(\bigsqcup_m d)(e) = \bigsqcup_m (d_m(e)) \sim_n \bigsqcup_m (d'_m(c')) = (\bigsqcup_m d')(c')$, concluding the proof.

LEMMA 8. (i) *For all d in $\mathbf{D}_n$ and d' in $\mathbf{D}'_n$, $d \sim_n d' \supset f_n(d) \sim_{n+1} f'_n(d')$.*

(ii) *For all d in $\mathbf{D}_{n+1}$ and d' in $\mathbf{D}'_{n+1}$, $d \sim_{n+1} d' \supset f^R_n(d) \sim_n f'^R_n(d')$.*

Proof. By simultaneous induction on n. For $n = 0$ suppose first that $d \sim_0 d'$. To show $f_0(d) \sim_1 f'_0(d')$ take e in $\mathbf{D}_0$ and c' in $\mathbb{1}^\infty \to \mathbf{D}'_0$ with $e \sim_0 c'(\infty)$. Then $f_0(d)(e) = d$ and $f'_0(d')(c') = d'$, so $f_0(d)(e) \sim_0 f'_0(d')(c')$ as required. Next suppose $d \sim_1 d'$. As $\bot \sim_0 (\lambda k \in \mathbb{1}^\infty \,.\, \bot)(\infty)$ we have $f^R_0(d) = d(\bot) \sim_0 d'(\lambda k \in \mathbb{1}^\infty \,.\, \bot) = f'^R_0(d')$ as required.

For $n + 1$, suppose first that $d \sim_{n+1} d'$ and take e in $\mathbf{D}_{n+1}$ and $c' \in (\mathbb{1}^\infty \to \mathbf{D}'_{n+1})$ such that $e \sim_{n+1} c'(\infty)$. Then $f^R_n(e) \sim_n f'^R_n(c'(\infty))$ (by induction hypothesis); so $d(f^R_n(e)) \sim_n d'(f'^R_n \circ c')$ (by assumption); so $f_n(d(f^R_n(e))) \sim_{n+1} f'_n(d(f'^R_n \circ c'))$ (by induction hypothesis). But this is just $f_{n+1}(d)(e) \sim_{n+1} f'_{n+1}(d')(c')$ as required.

Suppose next that $d \sim_{n+2} d'$ and take e in $\mathbf{D}_n$ and $c' \in (\mathbb{1}^\infty \to \mathbf{D}'_n)$ such that $e \sim_n c'(\infty)$. Then $f_n(e) \sim_{n+1} f'_n(c'(\infty))$ (by induction hypothesis); so $d(f_n(e)) \sim_{n+1} d'(f'_n \circ c')$ (by assumption); so $f^R_n(d(f_n(e))) \sim_n f'^R_n(d'(f'_n \circ c'))$ (by induction hypothesis). But this is just $f^R_{n+1}(d)(e) \sim f'^R_{n+1}(d')(c')$ as required, concluding the proof. □

Now we can define our relation between $\mathbf{D}$ and $\mathbf{D}'$ by

$$d \sim d' \quad \text{if } \forall n \,.\, d_n \sim_n d'_n.$$

By Lemma 7 and the componentwise calculation of lubs of ω-sequences in $\mathbf{D}$ and $\mathbf{D}'$ we get that $\sim$ is inclusive. By Lemma 8 we get that for any d in $\mathbf{D}_n$ and d' in $\mathbf{D}'_n$ if $d \sim_n d'$ then $\mu_n(d) \sim \mu'_n(d')$.

We are now in a position to supply the proof of Lemma 2.

Proof of Lemma 2. (i) What this claims is that for any d_0 in $\mathbf{D}_0$ it holds that $\mu_0(d_0) \sim \mu'_0(d_0)$, which follows at once by the above remark and the fact that $d_0 \sim_0 d_0$.

(ii) $\Rightarrow$: First suppose $d \sim d'$ (so that $d_{n+1} \sim d'_{n+1}$ for any n). Take e in $\mathbf{D}$ and c' in $\mathbb{1}^\infty \to \mathbf{D}'$ such that $e \sim c'(\infty)$ (to show that $d(e) \sim d'(c')$, meaning that $\Phi(d)(e) \sim \Phi'(d')(c')$). Now since $e \sim c'(\infty)$ we have for any m that $e_m \sim_m (\mu'^R_m \circ c')(\infty)$ and so by assumption, we have $d_{m+1}(e_m) \sim_m d'_{m+1}(\mu'^R_m \circ c')$. Therefore by Lemmas 7 and 8(ii) we have for any n that

$$\bigsqcup_{m \geqq n} f^R_{nm}(d_{m+1}(e_m)) \sim_n \bigsqcup_{m \geqq n} f'^R_{nm}(d'_{m+1}(\mu'^R_m \circ c'))$$

and by the above remarks on Φ and Φ' this is just

$$(\Phi(d)(e))_n \sim_n (\Phi'(d')(c'))_n$$

showing $\Phi(d)(e) \sim \Phi'(d')(c')$ as required.

(iii) $\Leftarrow$: Suppose that whenever e in $\mathbf{D}$ and c' in $\mathbb{1}^{\infty} \to \mathbf{D}'$ satisfy $e \sim c'(\infty)$ then $\Phi(d)(e) \sim \Phi'(d')(c')$. We wish to show that $d \sim d'$ and by Lemma 8(ii) it is enough to show $d_{n+1} \sim_{n+1} d'_{n+1}$ for all n. So take e in $\mathbf{D}_n$ and c' in $\mathbb{1}^{\infty} \to \mathbf{D}'_n$ such that $e \sim_n c'(\infty)$. Then by a previous remark we have $\mu_n(e) \sim (\mu'_n \circ c')\infty$ and so, by our supposition, that $\Phi(d)(\mu_n(e)) \sim \Phi'(d')(\mu'_n \circ c')$.

So we have

$$\begin{aligned} d_{n+1}(e) &= (\Phi(d)(\mu_n(e)))_n \quad \text{(by a calculation given above)} \\ &\sim_n (\Phi'(d')(\mu'_n \circ c'))_n \quad \text{(as just shown)} \\ &= d'_{n+1}(c') \quad \text{(by a calculation given above)} \end{aligned}$$

showing that $d_{n+1} \sim_{n+1} d'_{n+1}$ and thereby concluding the proof. □

The functions s and r. First we inductively define continuous $s_n : \mathbf{D}_n \to \mathbf{D}'_n$ and $r_n : \mathbf{D}'_n \to \mathbf{D}_n$ by

$$s_0 = r_0 = id_{\mathbf{D}_0},$$

$$s_{n+1}(d) = \lambda c' \in \mathbb{1}^{\infty} \to \mathbf{D}'_n \,.\, s_n(d(r_n(c'(\infty)))),$$

$$r_{n+1}(d') = \lambda e \in \mathbf{D}_n \,.\, r_n(d'(\lambda k \in \mathbb{1}^{\infty} \,.\, s_n(e))).$$

LEMMA 9. (i) *For all n,* $f'^R_n \circ s_{n+1} \sqsupseteq s_n \circ f^R_n$.

(ii) *For all n,* $f^R_n \circ r_{n+1} \sqsupseteq r_n \circ f'^R_n$.

Proof. By simultaneous induction on n. For $n = 0$, first we have $f'^R_0(s_1(d)) = s_1(d)(\bot) = s_0(d(r_0(\bot(\infty)))) = d(\bot) = s_0(f^R_0(d))$; and second $f^R_0(r_1(d')) = r_1(d')(\bot) = r_0(d'(\lambda k \in \mathbb{1}^{\infty} \,.\, s_0(\bot))) = d'(\bot) = r_0(f'^R_0(d'))$.

For $n+1$ and part (i) we first calculate that for any d in $\mathbf{D}_{n+2}$ and c' in $\mathbb{1}^{\infty} \to \mathbf{D}'_n$,

$$\begin{aligned} f'^R_{n+1}(s_{n+2}(d))(c') &= f'^R_n(s_{n+2}(d)(f'_n \circ c')) \\ &= f'^R_n(s_{n+1}(d(r_{n+1}(f'_n(c'(\infty)))))) \end{aligned}$$

and then calculate that

$$s_{n+1}(f^R_{n+1}(d))(c') = s_n(f^R_n(d(f_n(r_n(c'(\infty)))))).$$

But $(f'^R_n \circ s_{n+1}) \sqsupseteq (s_n \circ f^R_n)$ (by induction hypothesis, part (i)) and also $(r_{n+1} \circ f'_n) \sqsupseteq f_n \circ f^R_n \circ r_{n+1} \circ f'_n \sqsupseteq f_n \circ r_n \circ f'^R_n \circ f'_n$ (by induction hypothesis, part (ii)) $= (f_n \circ r_n)$ and applying these two facts enables us to complete the above calculation, showing that

$$f'^R_{n+1}(s_{n+2}(d))(c') \sqsupseteq s_{n+1}(f^R_{n+1}(d))(c')$$

as required. For part (ii) the proof is similar. □

Because of part (i) of this lemma, for any d in $\mathbf{D}$ and any n, the sequence $\langle f'^R_{nm}(s_m(d_m)) \rangle_{m \geqq n}$ is increasing and so we can define $s : \mathbf{D} \to \mathbf{D}'$ by:

$$(s(d))_n = \bigsqcup_{m \geqq n} f'^R_{nm}(s_m(d_m))$$

it being simple to verify that $s(d)$ is in $\mathbf{D}'$. Similarly using part (ii), we can define $r : \mathbf{D}' \to \mathbf{D}$ by

$$(r(d'))_n = \bigsqcup_{m \geqq n} f^R_{nm}(r_m(d'_m)).$$

LEMMA 10. (i) *For all d in* $\mathbf{D}_n$, $d \sim_n s_n(d)$.

(ii) *For all d in $\mathbf{D}_n$ and d' in $\mathbf{D}'_n$ if $d \sim_n d'_n$ then $d = r_n(d')$.*

Proof. Simultaneous induction on n, the case $n = 0$ being evident. For $n+1$ and part (i) suppose d is in $\mathbf{D}_{n+1}$. Take e in $\mathbf{D}_n$ and c' in $\mathbb{1}^\infty \rightarrow \mathbf{D}'_n$ such that $e \sim_n c'(\infty)$ (so that $e = r_n(c'(\infty))$, by induction hypothesis part (ii)). Now we just calculate that

$$\begin{aligned} d(e) &\sim_n s_n(d(e)) \quad \text{(by induction hypothesis part (i))} \\ &= s_n(d(r_n(c'(\infty)))) \\ &= s_{n+1}(d)(c') \end{aligned}$$

as required.

For part (ii) suppose d is in $\mathbf{D}_{n+1}$ and d' is in $\mathbf{D}'_{n+1}$ and $d \sim_{n+1} d'$. For any e in $\mathbf{D}_n$ we have

$$\begin{aligned} e &\sim_n s_n(e) \quad \text{(by induction hypothesis part (i))} \\ &= (\lambda k \in \mathbb{1}^\infty \,.\, s_n(e))(\infty) \end{aligned}$$

and so $d(e) \sim_n d'(\lambda k \in \mathbb{1}^\infty \,.\, s_n(e))$, by assumption and so $d(e) = r_n(d'(\lambda k \in \mathbb{1}^\infty \,.\, s_n(e)))$, by induction hypothesis, part (ii). But this just says, since e was chosen arbitrarily, that $d = r_{n+1}(d')$.

We can now prove Lemma 4.

Proof of Lemma 4. (i) Take d in $\mathbf{D}$. Then for any m we have $d_m \sim_m s_m(d_m)$, by Lemma 10(i), and so for any n and any $m \geqq n$ we have by Lemma 8(ii), $d_n \sim_n f'^R_{nm}(s_m(d_m))$ and so, by Lemma 7, $d_n \sim_n (s(d))_n$, showing $d \sim s(d)$ as required.

(ii) Suppose $d \sim d'$. Then for any m we have $d_m \sim d'_m$ and so $d_m = r_m(d'_m)$ by Lemma 10(ii). So for any n and $m \geqq n$, by Lemma 8(ii) we have $d_n = f^R_{nm}(d_m) = f^R_{nm}(r_m(d'_m))$ and so $d_n = (r(d'))_n$ by the definition of $r(d')$ showing $d = r(d')$, as required. □

Appendix II. We show that a relation, $\sim_\rho$, with the required properties exists. First define relations $\sim^n_\rho$ between $\mathbf{D}_n$ and $\mathbf{Exp}$ for every $n \geqq 0$ by:

$$d \sim^0_\rho M \quad \text{iff } \forall n' \geqq 0 \forall N_1, \cdots, N_{n'} \,.\, \mu_0(d) \sqsubseteq \bigsqcup_k \mathcal{V}[\![\mathcal{S}^k(MN_1 \cdots N_{n'})]\!](\rho);$$

$$\begin{aligned} d \sim^{n+1}_\rho M \quad &\text{iff } \mu_{n+1}(d) \sqsubseteq \bigsqcup_k \mathcal{V}[\![\mathcal{S}^k M]\!](\rho) \quad \text{and} \\ &(\forall e \in \mathbf{D}_n, N \in \mathbf{Exp} \,.\, e \sim^n_\rho N \supset d(e) \sim^n_\rho (MN)). \end{aligned}$$

LEMMA 11. (i) *$d \sim^0_\rho M$ iff $\mu_0(d) \sqsubseteq \bigsqcup_k \mathcal{V}[\![\mathcal{S}^k M]\!](\rho)$ and for all N, $d \sim^0_\rho (MN)$.*

(ii) *$\perp \sim^0_\rho M$ always holds.*

Proof. Obvious.

LEMMA 12. *Each $\sim^n_\rho$ is inclusive (in its first argument).*

Proof. Easy induction on n.

LEMMA 13. (i) *If $d \sim^n_\rho M$ then $f_n(d) \sim^{n+1}_\rho M$.*

(ii) *If $d \sim^{n+1}_\rho M$ then $f^R_n(d) \sim^n_\rho M$.*

Proof. By simultaneous induction on n. For $n = 0$, assume for i that $d \sim^0_\rho M$. Then first we calculate that:

$$\mu_1(f_0(d)) = \mu_0(d) \sqsubseteq \bigsqcup_k \mathcal{V}[\![\mathcal{S}^k M]\!](\rho)$$

and second, supposing that $e \sim^0_\rho N$ we see that $f_0(d)(e) = d \sim^0_\rho (MN)$, by Lemma 11(i). So $f_0(d) \sim^1_\rho M$ as required.

Next, for (ii) suppose that $d \sim^1_\rho M$. We show, using Lemma 11 that $f^R_0(d) \sim^0_\rho M$. First,

$$\mu_0(f^R_0(d)) = \mu_1(f_0 \circ f^R_0(d)) \sqsubseteq \mu_1(d) \sqsubseteq \bigsqcup_k \mathcal{V}\llbracket \mathcal{S}^k M \rrbracket(\rho),$$

since $d \sim^1_\rho M$. Next for any N, since $\perp \sim^0_\rho N$, by Lemma 11(ii) we have $f^R_0(d) = d(\perp) \sim^0_\rho (MN)$, as required.

For $n+1$, part (i) suppose $d \sim^{n+1}_\rho M$. Then first we have

$$\begin{aligned}\mu_{n+2}(f_{n+1}(d)) &= \mu_{n+1}(d)\\ &\sqsubseteq \bigsqcup_k \mathcal{V}\llbracket \mathcal{S}^k M \rrbracket(\rho) \quad (\text{as } d \sim^{n+1}_\rho M)\end{aligned}$$

and second, taking e in $\mathbf{D}_{n+1}$ and N in **Exp** with $e \sim^{n+1}_\rho N$, we see that $f^R_n(e) \sim^n_\rho N$ (by the induction hypothesis, part (i)) and so that $d(f^R_n(e)) \sim^n_\rho (MN)$ (as $d \sim^{n+1}_\rho M$) and so that $f_n(d(f^R_n(e))) \sim^{n+1}_\rho (MN)$ (by induction hypothesis, part (ii)). But this last is just $f_{n+1}(d)(e) \sim^{n+1}_\rho (MN)$ and so we have $f_{n+1}(d) \sim^{n+2}_\rho M$ as required. The verification of part (ii) is very similar and we omit it. □

Note that $\perp \sim^n_\rho M$ always holds and that if $d' \sqsubseteq d \sim^n_\rho M$ then $d' \sim^n_\rho M$, that is $\sim^n_\rho$ is downwards closed in its left argument. All in all, $\{d \mid d \sim^n_\rho M\}$ is always a nonempty Scott closed set.

Now we can define the relation $\sim_\rho$ by:

$$d \sim_\rho M \quad \text{iff } \forall n\,.\, d_n \sim^n_\rho M.$$

This is clearly inclusive using Lemma 12, and indeed $\perp \sim_\rho M$ always holds (and actually $\{d \mid d \sim_\rho M\}$ is always nonempty Scott closed). Note that, by Lemma 13, if $d \sim^n_\rho M$ then $\mu_n(d) \sim_\rho M$.

Now we show that the recursive specification for $\sim_\rho$ is satisfied.

LEMMA 14. *$d \sim_\rho M$ iff $d \sqsubseteq \bigsqcup_k \mathcal{V}\llbracket \mathcal{S}^k M \rrbracket \rho$ and*

$$(\forall e \sim_\rho N\,.\, d(e) \sim_\rho (MN)).$$

Proof. This is routine, but perhaps worth writing down.

$\Rightarrow$ Suppose $d \sim_\rho M$. Then $d_n \sim^n_\rho M$, so $\mu_n(d_n) \sqsubseteq \bigsqcup_k \mathcal{V}\llbracket \mathcal{S}^k M \rrbracket(\rho)$ holds for every n and so holds for $d = \bigsqcup_n \mu_n(d_n)$ too. Next, suppose $e \sim_\rho N$. Then $e_n \sim^n_\rho N$ and so, as $d_{n+1} \sim^{n+1}_\rho M$, we have $d_{n+1}(e_n) \sim^n_\rho (MN)$ and so $\mu_n(d_{n+1}(e_n)) \sim_\rho (MN)$ and so, as $d(e) = \bigsqcup \mu_n(d_{n+1}(e_n))$, $d(e) \sim_\rho (MN)$, as required.

$\Leftarrow$ Suppose $d \sqsubseteq \bigsqcup_k \mathcal{V}\llbracket \mathcal{S}^k M \rrbracket(\rho)$ and whenever $e \sim_\rho N$ then $d(e) \sim_\rho (MN)$. We will show that $d_{n+1} \sim^{n+1}_\rho M$, for every n. Clearly $\mu_{n+1}(d_{n+1}) \sqsubseteq \bigsqcup_k \mathcal{V}\llbracket \mathcal{S}^k M \rrbracket(\rho)$, from the supposition. Suppose that $e \sim^n_\rho N$. Then $\mu_n(e) \sim_\rho N$ and so $d(\mu_n e) \sim_\rho N$ and so $d_{n+1}(e) = d(\mu_n e)_n \sim^n_\rho N$, as required. □

Proof of Lemma 5. We show by induction on n that for any d, M if for any $n' \geqq 0$ whenever $e_i \sim_\rho N_i$ (for $i = 1, n'$) then

$$d e_1 \cdots e_{n'} \sqsubseteq \bigsqcup_k \mathcal{V}\llbracket \mathcal{S}^k M N_1 \cdots N_{n'} \rrbracket(\rho)$$

then it follows that $d_n \sim^n_\rho M$.

First suppose $n = 0$. Take $N_1, \cdots, N_{n'}$. Then $\perp \sim_\rho N_i$ and so

$$\mu_0(d_0) = \mu_0(d_0)\perp \cdots \perp \sqsubseteq d\perp \cdots \perp \sqsubseteq \bigsqcup_k \mathcal{V}\llbracket \mathcal{S}^k M N_1 \cdots N_{n'} \rrbracket(\rho)$$

as required.

For $n+1$, first

$$\mu_{n+1}(d_{n+1}) \sqsubseteq d \sqsubseteq \bigsqcup_k \mathcal{V}\llbracket \mathcal{S}^k M \rrbracket(\rho)$$

and second, suppose that $e \sim_\rho^n N$. Then $\mu_n(e) \sim_\rho N$. Now suppose that, in order to apply the induction hypothesis to $d(\mu_n e)$, (MN), that $e_i \sim_\rho N_i$ for $i = 1, n'$. Then

$$d(\mu_n e)e_1 \cdots e_{n'} \sqsubseteq \bigsqcup_k \mathcal{V}[\![\mathcal{S}^k(MN)N_1 \cdots N_{n'}]\!](\rho)$$

by assumption on d, M and so $(d(\mu_n e))_n \sim_\rho^n (MN)$. But as $d(\mu_n e)_n = d_{n+1}(e)$, it follows that $d_{n+1}(e) \sim_\rho^n (MN)$ and so, finally, that $d_{n+1} \sim_\rho^{n+1} M$, as required. □

Acknowledgments. We wish to acknowledge the helpful suggestions of the referees.

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