Quantitative Algebraic Reasoning

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Abstract

We develop a quantitative analogue of equational reasoning which we call quantitative algebra. We define an equality relation indexed by rationals: $a =_\varepsilon b$ which we think of as saying that "a is approximately equal to b up to an error of ε". We have 4 interesting examples where we have a quantitative equational theory whose free algebras correspond to well known structures. In each case we have finitary and continuous versions. The four cases are: Hausdorff metrics from quantitive semilattices; p-Wasserstein metrics (hence also the Kantorovich metric) from barycentric algebras and also from pointed barycentric algebras and the total variation metric from a variant of barycentric algebras.

1. Introduction

One of the exciting themes in research in programming language theory is the algebraic study of computational phenomena initiated by Moggi (Moggi 1988, 1991) where he showed how one can view notions of computation as monads. This allowed the incorporation of computational effects into a functional core in a compositional way. This became enormously influential and even led to monads being directly incorporated into programming languages like Haskell. It was a decade later that Plotkin and Power (Plotkin and Power 2001, 2002) began the study of computational effects from the point of view of equations and operations. From a categorical perspective one is moving from monads to Lawvere theories; see the excellent historical survey by Hyland and Power for more details (Hyland and Power 2007).

One aspect of computational effects that has attracted significant attention is probabilistic computation (Saheb-Djahromi 1978, 1980; Kozen 1981, 1985; Jones and Plotkin 1989). This is, in fact, growing significantly with recent work spurred by interest from the machine learning community; see for example (Borgström et al. 2011; Gordon et al. 2014) among many other research efforts on the theory and practice of probabilistic programming as it applies to machine learning applications and (Foster et al. 2015) for a recently developed probabilistic programming language for network applications. Early work on lambda-calculi for probabilistic programming is due to Saheb-Djahromi (Saheb-Djahromi 1980). Claire Jones (Jones 1990) developed a probabilistic λ-calculus in her thesis, gave an operational semantics and proved adequacy results. The fundamental work on probability monads is due to Lawvere (Lawvere 1964) (before monads were invented!) and Gıry (Giry 1981). One can develop a probabilistic λ-calculus using this monad (Ramsey and Pfeffer 2002).

In the present paper we develop an equational approach to reasoning about quantitative phenomena. The key new idea is to introduce equations annotated with rational numbers written $=_\varepsilon$ to capture the notion of *approximate* equality. One should think of $s =_\varepsilon t$ as saying that s and t are "within ε of each other." Essentially we are working with enriched Lawvere theories; see (Robinson 2002) for an expository account of this subject. We do not emphasize the category-theoretic underpinnings here; instead we concentrate on presenting the notion of quantitative equations as concretely as possible. The bulk of the paper is spent on some very pleasing examples and on the general notions developed in the spirit of traditional universal algebra. In later work we will carefully spell out the categorical picture.

The examples are all of the following form: we give a simple set of equations and define the algebras of the resulting theory. We then induce metrics on the free algebra and identify them with commonly defined metrics. Thus, for example, we show that the Hausdorff metric arises from a quantitative version of semilattices. We show that the total variation metric arises from an axiomatization of convexity in terms of barycentric axioms. We show that the famous Kantorovich[1] metric (Villani 2008; van Breugel and Worrell 2001; Panangaden 2009) arises from a variation of the same axioms. In fact, already the p-Wasserstein metric, which is a generalization of the Kantorovich metric arises from a variation of the same axioms. These metrics (especially Kantorovich) play a fundamental role in the study of probabilistic bisimulation (Panangaden 2009) and transport theory (Villani 2008). We present both finitary and infinitary versions of these constructions.

Metric ideas have been important in denotational semantics from the beginning especially in Jaco de Bakker's school; see (van Breugel 2001) for a survey. It may seem that for probabilistic reasoning one needs to work with measure theory. This is, of course, true but measure theory works best when there is metric structure; as witnessed, for example, by the ubiquity of Polish spaces in discussions of measure theory. The algebraic approach to effects (Plotkin and Power 2001, 2002, 2003, 2004; Hyland et al. 2006, 2007) has not, until now, been considered in a metric context. Owing to the increasing importance of probability in computer science it seems worthwhile to investigate this now. The first order of business then is to see how some familiar and important monads fit into this approach. In this paper, we only consider monads related to probabilistic and nondeterministic systems. However the well-known basic examples (exceptions, states, I/O) also fit into the framework of this paper, albeit with some inessential limita-

[1] This metric goes by many names: Hutchinson, Wasserstein (with numerous variations in spelling) and Kantorovich-Rubinstein. Perhaps the most commonly used name is Wasserstein.

LICS 16 Month July, 2016, New York, NY, USA

tions arising from our working with operations with finite discrete arities.

2. Quantitative Equational Theories

An algebraic similarity type consists of a finite set of function symbols each with fixed *finite* arity. Consider an algebraic similarity type Ω and algebras of this type.

Given a countable set X of variables, let $\mathbb{T}X$ be the set of terms constructed over Ω from X, this is the *term algebra* of Ω over X.

A *substitution* is a function $\sigma : X \longrightarrow \mathbb{T}X$. It can be canonically extended to terms $\sigma\colon \mathbb{T}X \longrightarrow \mathbb{T}X$ by:

- for any $f : n \in \Omega$, $\sigma(f(t_1, ..t_n)) = f(\sigma(t_1), ..\sigma(t_n))$.

In what follows a substitution is just a function $\sigma : \mathbb{T}X \longrightarrow \mathbb{T}X$ satisfying the conditions stated above and $\Sigma(X)$ denotes the set of substitutions on $\mathbb{T}X$.

If $\Gamma \subseteq \mathbb{T}X$ and $\sigma \in \Sigma(X)$, let $\sigma(\Gamma) = \{\sigma(t) \mid t \in \Gamma\}$.

Let $\mathcal{V}(X)$ denote the set of *indexed equalities* of the form $x =_\epsilon y$ for $x, y \in X$ and $\epsilon \in \mathbb{Q}_+$; similarly, let $\mathcal{V}(\mathbb{T}X)$ denote the set of indexed equalities of the form $t =_\epsilon s$ for $t, s \in \mathbb{T}X$, $\epsilon \in \mathbb{Q}_+$. We call them *quantitative equations*.

Definition 2.1 (Deducibility Relation). *Given an algebraic similarity type Ω and a set X of variables, a* deducibility relation of type Ω over X *is a relation $\vdash \subseteq 2^{\mathcal{V}(\mathbb{T}X)} \times \mathcal{V}(\mathbb{T}X)$ closed under the following rules stated for arbitrary $t, s, u, t_1, \cdots t_n \in \mathbb{T}X$, $\epsilon, \epsilon' \in \mathbb{Q}_+, \Gamma, \Gamma' \subseteq \mathcal{V}(\mathbb{T}X)$ and $\phi, \psi \in \mathcal{V}(\mathbb{T}X)$; where $(\Gamma, \phi) \in \vdash$ is written as $\Gamma \vdash \phi$:*

(Refl) $\emptyset \vdash t =_0 t$

(Symm) $\{t =_\epsilon s\} \vdash s =_\epsilon t$.

(Triang) $\{t =_\epsilon s, s =_{\epsilon'} u\} \vdash t =_{\epsilon+\epsilon'} u$.

(Max) *For* $\epsilon' > 0$, $\{t =_\epsilon s\} \vdash t =_{\epsilon+\epsilon'} s$.

(Arch) *For* $\epsilon \geq 0$, $\{t =_{\epsilon'} s \mid \epsilon' > \epsilon\} \vdash t =_\epsilon s$.

(NExp) $\{t_1 =_\epsilon s_1, \dots t_n =_\epsilon s_n\} \vdash f(t_1, ..t_n) =_\epsilon f(s_1, ..s_n)$, *for any* $f : n \in \Omega$,

(Subst) *If* $\sigma \in \Sigma(X)$, $\Gamma \vdash t =_\epsilon s$ *implies* $\sigma(\Gamma) \vdash \sigma(t) =_\epsilon \sigma(s)$.

(Cut) *If* $\Gamma \vdash \phi$ *for all* $\phi \in \Gamma'$ *and* $\Gamma' \vdash \psi$, *then* $\Gamma \vdash \psi$.

(Assumpt) *If* $\phi \in \Gamma$, *then* $\Gamma \vdash \phi$.

Let $\mathcal{E}(\mathbb{T}X) = \mathcal{P}_f(\mathcal{V}(\mathbb{T}X)) \times \mathcal{V}(\mathbb{T}X)$, where $\mathcal{P}_f(A)$ is the finite powerset of A; we call its elements *quantitative inferences* on $\mathbb{T}X$. If $(V, \phi) \in \mathcal{E}(\mathbb{T}X)$, we refer to the elements of V as the *hypotheses* of the inference. An *unconditional quantitative inference* is a quantitative inference with an empty set of hypotheses.

Of particular interest for us is the subclass $\mathcal{E}(X) = \mathcal{P}_f(\mathcal{V}(X)) \times \mathcal{V}(\mathbb{T}X)$ of quantitative inferences, hereafter called *basic quantitative inferences*, where the hypotheses are finite sets of quantitative equations between variables. The axioms for theories will be basic quantitative inferences.

Notation: Hereafter in the paper we fix a countable set X of variables that we use to define *quantitative equational theories* over various algebraic similarity types.

Definition 2.2 (Quantitative Equational Theory). *Given a set $S \subseteq \mathcal{E}(X)$ of basic quantitative inferences on $\mathbb{T}X$, denote by $\vdash_S$ the smallest deducibility relation that contains S. The* quantitative equational theory induced by S *is the set*

$$\mathcal{U} \stackrel{\text{def}}{=} (\vdash_S) \cap \mathcal{E}(\mathbb{T}X).$$

The elements of S are the axioms *of the theory $\mathcal{U}$.*

Note that in our current setting a quantitative equational theory does not contain any conditional equation with infinitely many hypotheses, nor indeed does the set S. However, in constructing $\mathcal{U}$ from S, we can use the infinitary archimedean rule in derivations. This setting can be extended to include inferences with a countable set of hypothesis; and the basic theory developed in what follows can be easily adapted.

If $\mathcal{U}$ is a quantitative equational theory and $\emptyset \vdash s =_e t \in \mathcal{U}$, we will abuse notation and also write $\mathcal{U} \vdash s =_e t$.

Definition 2.3 (Consistent theories). *A quantitative equational theory $\mathcal{U}$ over $\mathbb{T}X$ is* inconsistent *if $\mathcal{U} \vdash x =_0 y$, where $x, y \in X$ are two distinct variables. $\mathcal{U}$ is* consistent *if it is not inconsistent.*

3. Quantitative Algebras

Definition 3.1 (Quantitative Algebra). *A* quantitative algebra *is a tuple $\mathcal{A} = (A, \Omega^{\mathcal{A}}, d^{\mathcal{A}})$, where $(A, \Omega^{\mathcal{A}})$ is an algebra of type Ω and $d^{\mathcal{A}} : A \times A \longrightarrow \mathbb{R}_+ \cup \{\infty\}$ is a metric on A (possibly taking infinite values) such that all the operators in the signature are non-expansive. i.e., for any $f : n \in \Omega^{\mathcal{A}}$, any $a_i, b_i \in A$, $i = 1, ..n$ and any $\epsilon \geq 0$, $d^{\mathcal{A}}(a_i, b_i) \leq \epsilon$ for all $i = 1, ..n$ implies $d^{\mathcal{A}}(f(a_1, .., a_n), f(b_1, .., b_n)) \leq \epsilon$.*

A quantitative algebra is *degenerate* if its support is empty or it is a singleton.

As expected, a homomorphism of quantitative algebras of signature Ω is just a non-expansive homomorphism of Ω-universal algebras. The quantitative algebras of type Ω and their homomorphisms form a category, denoted Ω-$\mathbf{QA}$.

The quantitative algebra $\mathcal{B} = (B, \Omega, d^{\mathcal{B}})$ is a subalgebra of the quantitative algebra $\mathcal{A} = (A, \Omega, d^{\mathcal{A}})$, denoted by $\mathcal{B} \leq \mathcal{A}$, if $\mathcal{B}$ is a subalgebra of $\mathcal{A}$ as universal algebra and, in addition, for any $a, b \in B$, $d^{\mathcal{B}}(a, b) = d^{\mathcal{A}}(a, b)$.

Definition 3.2 (Universal mapping property). *Let $\mathbb{K}$ be a subcategory of quantitative algebras of type Ω, $\mathbb{C}$ an arbitrary category, $G\colon \mathbb{K} \longrightarrow \mathbb{C}$ a functor and C an object in $\mathbb{C}$. A* universal morphism from C to G *is a pair $(\mathcal{A}, \alpha)$ consisting of a quantitative algebra $\mathcal{A} \in \mathbb{K}$ and a morphism $\alpha\colon C \longrightarrow G\mathcal{A}$ in $\mathbb{C}$, such that for every pair $(\mathcal{B}, \beta)$ with $\mathcal{B} \in \mathbb{K}$ and $\beta\colon C \longrightarrow G\mathcal{B}$ a morphism in $\mathbb{C}$, there exists a unique homomorphism of quantitative algebras $h\colon \mathcal{A} \longrightarrow \mathcal{B}$ such that $Gh \circ \alpha = \beta$. Diagrammatically*

$$\begin{tikzcd}
 & \textit{in } \mathbb{C} & \textit{in } \mathbb{K} \\
C \arrow[r, "\alpha"] \arrow[dr, "\beta"'] & G\mathcal{A} \arrow[d, "Gh"] & \mathcal{A} \arrow[d, dashed, "h"] \\
 & G\mathcal{B} & \mathcal{B}
\end{tikzcd}$$

A quantitative algebra $\mathcal{A}$ has the universal mapping property for C to G *if there exists a universal morphism $(\mathcal{A}, \alpha)$ from C to G.*

4. Algebraic Semantics

Given a quantitative algebra $\mathcal{A} = (A, \Omega^{\mathcal{A}}, d^{\mathcal{A}})$ of type Ω and a set X of variables, an *assignment* on $\mathcal{A}$ is a function $\iota : X \longrightarrow A$ that is canonically extended to $\iota : \mathbb{T}X \longrightarrow A$ over Ω-terms by

- for any $f : n \in \Omega$, $\iota(f(t_1, \ldots, t_n)) = f^{\mathcal{A}}(\iota(t_1), \ldots \iota(t_n))$.

We denote by $\mathbb{T}(X|\mathcal{A})$ the set of assignments on $\mathcal{A}$.

DEFINITION 4.1 (Satisfiability). *Consider a quantitative algebra $\mathcal{A} = (A, \Omega^{\mathcal{A}}, d^{\mathcal{A}})$ and a set X of variables. $\mathcal{A}$* satisfies *a quantitative inference $\Gamma \vdash s =_\epsilon t \in \mathcal{E}(\mathbb{T}X)$ over $\mathbb{T}X$, written $\Gamma \models_{\mathcal{A}} s =_\epsilon t$, if for all assignments $\iota \in \mathbb{T}(X|\mathcal{A})$ it is the case that $d^{\mathcal{A}}(\iota(t'), \iota(s')) \leq \epsilon'$ for all $s' =_{\epsilon'} t' \in \Gamma$ implies $d^{\mathcal{A}}(\iota(s), \iota(t)) \leq \epsilon$.*

In these cases we say that $\mathcal{A}$ is a *model* of the inference. Similarly, for a set of quantitative inferences Γ, we say that $\mathcal{A}$ is a model of Γ if $\mathcal{A}$ satisfies each element of Γ. A quantitative inference (a quantitative equational theory) is *satisfiable* if it has a model.

Instead of $\emptyset \models_{\mathcal{A}} s =_\epsilon t$ we also write $\mathcal{A} \models s =_\epsilon t$.

DEFINITION 4.2 (Equational Class of Quantitative Algebras). *For a quantitative equational theory $\mathcal{U}$ over the Ω-terms $\mathbb{T}X$, the* equational class induced by *$\mathcal{U}$ is the class of quantitative algebras of signature Ω satisfying $\mathcal{U}$.*

We denote this class as well as the full subcategory of Ω-quantitative algebras satisfying $\mathcal{U}$ by $\mathbb{K}(\Omega, \mathcal{U})$. It is closed under isomorphic images, subalgebras and small products.

5. Completeness for Quantitative Algebras

Fix a signature Ω and a quantitative equational theory $\mathcal{U}$ over Ω-terms in $\mathbb{T}X$ with variables in the countable set X. We consider a set M of generators and we construct a quantitative algebra $\mathbb{T}[M]$ with support a quotient of $\mathbb{T}M$ w.r.t. 0-provability induced by $\mathcal{U}$.

Define the pseudometric $d_{\mathcal{U}} : \mathbb{T}X \times \mathbb{T}X \longrightarrow \mathbb{R}_+ \cup \{\infty\}$ by

$$d_{\mathcal{U}}(t, s) = \inf\{\epsilon \mid \emptyset \vdash t =_\epsilon s \in \mathcal{U}\}$$

It is not difficult to verify that $d_{\mathcal{U}}$ is also characterized by

$$d_{\mathcal{U}}(t, s) = \inf\{\epsilon \mid \forall V \in \mathcal{P}_f(\mathcal{V}(X)), V \vdash s =_\epsilon t \in \mathcal{U}\}\,.$$

Let $\mathbb{P}$ be the set of all pseudometrics that makes all the assignments in $\mathbb{T}(X|\mathbb{T}M)$ non-expansive. We define the following pseudometric on $\mathbb{T}M$ for arbitrary $p, q \in \mathbb{T}M$:

$$d(p, q) = \sup\{\delta(p, q) \mid \delta \in \mathbb{P}\}\,,$$

This construction is known as the *final pseudometric for a cone of functions*, where in this specific case the cone is $\mathbb{T}(X|\mathbb{T}M)$.

Let $(\mathbb{T}[M], d^{\cong})$ be the metric space induced by the pseudometric d after quotienting $\mathbb{T}M$ w.r.t. the equivalence relation $\cong = \{(p, q) \mid d(p, q) = 0\}$. Let $\mathbb{T}[M]$ be the set of $\cong$-equivalence classes on $\mathbb{T}M$; hence, $d^{\cong}(p^{\cong}, q^{\cong}) = d(p, q)$, for any $p, q \in \mathbb{T}M$.

The fact that the equational theory is axiomatized by basic quantitative inferences allows us to prove that $\cong$ is a congruence w.r.t. the operators in Ω, i.e., for any $f : n \in \Omega$ and $p_i, q_i \in \mathbb{T}M$, $i = 1, \ldots, n$,

$$p_i \cong q_i \qquad \text{implies} \qquad f(p_1, \ldots, p_n) \cong f(q_1, \ldots, q_n)\,.$$

Due to this property, we can endow $\mathbb{T}[M]$ with the structure of an Ω-algebra by interpreting $f : n \in \Omega$ as follows:

$$f(p_1^{\cong}, \ldots, p_n^{\cong}) = (f(p_1, \ldots p_n))^{\cong}\,.$$

Thus we get a quantitative algebra $\mathbb{T}[M] = (\mathbb{T}[M], \Omega, d^{\cong})$.

THEOREM 5.1. $\mathbb{T}[M] = (\mathbb{T}[M], \Omega, d^{\cong}_{\mathcal{U}}) \in \mathbb{K}(\Omega, \mathcal{U})$.

Term Quantitative algebra. In particular, the previous construction can also be done for the case when $M = X$ and we obtain the quantitative algebra $\mathbb{T}[X] = (\mathbb{T}[X], \Omega, d^{\cong})$ of terms modulo 0-provability.

We prove now that when we construct the distance d on $\mathbb{T}X$ what we get is, in fact, exactly $d_{\mathcal{U}}$.

Note that any assignment $\iota \in \mathbb{T}(X|T[X])$ is a substitution on $\mathbb{T}X$ and applying (Subst), for any $\iota \in \mathbb{T}(X|T[X])$,

$$\emptyset \vdash s =_e t \in \mathcal{U} \;\text{ iff }\; \emptyset \vdash \iota(s) =_e \iota(t) \in \mathcal{U}.$$

An immediate consequence of this is that for any $\iota \in \mathbb{T}(X|T[X])$,

$$d_{\mathcal{U}}(s, t) = d_{\mathcal{U}}(\iota(s), \iota(t)).$$

Hence, any $\iota \in \mathbb{T}(X|T[X])$ is non-expansive. This means that $d_{\mathcal{U}} \in \mathbb{P}$, then by definition $d \geq d_{\mathcal{U}}$.

On the other hand, $d_{\mathcal{U}} \geq d$, because d must make all the maps in $\mathbb{T}(X|\mathbb{T}X)$ non-expansive, and in particular, it must make the identity on $\mathbb{T}X$ non-expansive.

Hence, for $M = X$ we get that $d = d_{\mathcal{U}}$.

This equality allows us to further speak about $\mathbb{T}[X] = (\mathbb{T}[X], \Omega, d^{\cong}_{\mathcal{U}})$ as the algebra generated by the set X.

Completeness. These results allow us now to prove the following strong completeness theorem.

THEOREM 5.2 (Completeness). *Given a quantitative equational theory $\mathcal{U}$ over the set X of variables and signature Ω,*

$$[\Gamma \models_{\mathcal{A}} \phi \textit{ for any } \mathcal{A} \in \mathbb{K}(\Omega, \mathcal{U})] \quad \textit{iff}\; \Gamma \vdash \phi \in \mathcal{U}.$$

Proof. The right-to-left implication (soundness) is a direct consequence of the definition of $\mathbb{K}(\Omega, \mathcal{U})$.

It remains for us to prove the left-to-right implication:

$$[\Gamma \models_{\mathcal{A}} \phi \text{ for any } \mathcal{A} \in \mathbb{K}(\Omega, \mathcal{U})] \quad \text{implies}\; \Gamma \vdash \phi \in \mathcal{U}.$$

Suppose that the left-hand side is satisfied. Assume that ϕ is the quantitative equation $s =_e t$.

Let $\overline{\mathcal{U} \cup \Gamma}$ be the quantitative equational theory induced by $\mathcal{U} \cup \{\emptyset \vdash \psi \mid \psi \in \Gamma\}$. Obviously, $\overline{\mathcal{U} \cup \Gamma}$ is a theory over $\mathbb{T}X$. Applying Theorem 5.1, we obtain that $(\mathbb{T}[X], \Omega, d^{\cong}_{\overline{\mathcal{U}\cup\Gamma}})$ is a model for $\overline{\mathcal{U} \cup \Gamma}$, hence both for $\mathcal{U}$ and for $\{\emptyset \vdash \psi \mid \psi \in \Gamma\}$.

Because $(\mathbb{T}[X], \Omega, d^{\cong}_{\overline{\mathcal{U}\cup\Gamma}}) \in \mathbb{K}(\Omega, \mathcal{U})$, $(\mathbb{T}[X], \Omega, d^{\cong}_{\overline{\mathcal{U}\cup\Gamma}})$ satisfies $\Gamma \vdash \phi$. And because $(\mathbb{T}[X], \Omega, d^{\cong}_{\overline{\mathcal{U}\cup\Gamma}})$ is a model of Γ, we obtain that $(\mathbb{T}[X], \Omega, d^{\cong}_{\overline{\mathcal{U}\cup\Gamma}})$ is also a model for $s =_e t$. Consequently, $\inf\{\epsilon \mid \overline{\mathcal{U} \cup \Gamma} \vdash s =_\epsilon t\} \leq e$, i.e., $d_{\overline{\mathcal{U}\cup\Gamma}}(s, t) \leq e$.

Suppose now that $\Gamma \vdash s =_e t \notin \mathcal{U}$. If $\emptyset \vdash s =_e t \in \mathcal{U}$, applying (Cut) we get that $\Gamma \vdash s =_e t \in \mathcal{U}$ - contradiction. Hence, $\emptyset \vdash s =_e t \notin \mathcal{U}$. Also, $\emptyset \vdash s =_e t \notin \{\emptyset \vdash \psi \mid \psi \in \Gamma\}$, because otherwise $s =_e t \in \Gamma$ and (Assumpt) proves $\Gamma \vdash s =_e t$ which would imply $\Gamma \vdash s =_e t \in \mathcal{U}$ - contradiction.

Hence, $\emptyset \vdash s =_e t \notin \overline{\Gamma \cup \mathcal{U}}$.

Let $i = \inf\{\epsilon \mid \overline{\Gamma \cup \mathcal{U}} \vdash s =_\epsilon t\} = d_{\overline{\Gamma\cup\mathcal{U}}}(s, t)$.

If $i \in \mathbb{Q}$, then using (Arch) we can prove that $\overline{\Gamma \cup \mathcal{U}} \vdash s =_i t$ and further (Max) guarantees that $i > e$, since $\emptyset \vdash s =_e t \notin \overline{\Gamma \cup \mathcal{U}}$.

If $i \notin \mathbb{Q}$, from $\emptyset \vdash s =_e t \notin \overline{\Gamma \cup \mathcal{U}}$ we derive that $i \geq e$. But since $e \in \mathbb{Q}$, this means that $i > e$.

Hence, $d_{\overline{\Gamma \cup \mathcal{U}}}(s,t) > e$, contradicting $d_{\overline{\mathcal{U} \cup \Gamma}}(s,t) \leq e$. ∎

The next theorem proves that the construction of $\mathbb{T}[M]$ is universal (in a categorical sense) with respect to all the quantitative algebras satisfying the quantitative equational theory $\mathcal{U}$. Specifically, $\mathbb{T}[M]$ has the universal mapping property for M to the (obvious) forgetful functor $U_{\mathbf{Set}} \colon \mathbb{K}(\Omega,\mathcal{U}) \to \mathbf{Set}$. Concretely, for any quantitative algebra $\mathcal{A} = (A, \Omega^{\mathcal{A}}, d^{\mathcal{A}}) \in \mathbb{K}(\Omega,\mathcal{U})$ and any set-map $\alpha \colon M \to A$, there exists a unique morphism of quantitative algebras $h : \mathbb{T}[M] \to \mathcal{A}$ that makes the following diagram commutative.

$$\begin{array}{cc} \text{in } \mathbf{Set} & \text{in } \mathbb{K}(\Omega,\mathcal{U}) \\ \begin{tikzcd} M \arrow[r, "\eta_M"] \arrow[dr, "\alpha"'] & \mathbb{T}[M] \arrow[d, "h"] \\ & A \end{tikzcd} & \begin{tikzcd} \mathbb{T}[M] \arrow[d, dashed, "h"] \\ \mathcal{A} \end{tikzcd} \end{array}$$

where $\eta_M \colon M \to \mathbb{T}[M]$ is the map given by $\eta_M(m) = m^{\cong}$. The map h is characterized as follows:

- for $m \in M$, $h(m^{\cong}) = \alpha(m)$;
- for $f : n \in \Omega$ and $p_1, \ldots, p_n \in \mathbb{T}M$,
$$h((f(p_1, \ldots, p_n))^{\cong}) = f^{\mathcal{A}}(h(p_1^{\cong}), \ldots, h(p_n^{\cong})).$$

THEOREM 5.3. *$(\mathbb{T}[M], \eta_M)$ is a universal arrow from $M \in \mathbf{Set}$ to $U_{\mathbf{Set}}$.*

Since X and $\mathcal{U}$ are arbitrarily chosen, Theorem 5.3 justifies calling $\mathbb{T}[X]$ the *free Ω-quantitative algebra* generated over X.

In standard universal algebras, the set of terms gives rise to a monad, the *term monad*. As one would expect, this is the case also for quantitative algebras, with the only difference that now terms are quotiented w.r.t. 0-provability in $\mathcal{U}$.

The free-construction above provides a functor $\mathbb{T}_{\mathcal{U}} \colon \mathbf{Set} \to \mathbf{Set}$ that maps objects $M \in \mathbf{Set}$ to the set $\mathbb{T}[M]$ of Ω-terms. $\mathbb{T}_{\mathcal{U}}$ is monadic, with unit and multiplication given by the natural transformations $\eta \colon Id \Rightarrow \mathbb{T}_{\mathcal{U}}$ and $\mu \colon \mathbb{T}_{\mathcal{U}}\mathbb{T}_{\mathcal{U}} \Rightarrow \mathbb{T}_{\mathcal{U}}$, characterized, for arbitrary $m \in M$, $t \in \mathbb{T}[M]$, $f : n \in \Omega$, $C_1, .., C_n \in \mathbb{T}[\mathbb{T}[M]]$ by:

$$\eta_M(m) = m^{\cong}\,, \quad \mu_M(t) = t\,,$$
$$\mu_M(f(C_1, .., C_n)^{\cong}) = f\big(\mu_M(C_1), .., \mu_M(C_n)\big)^{\cong}\,.$$

Note that this monad corresponds to the standard equational term monad constructed from the equational for universal algebras. In the next sections we show that quantitative equational theories are actually stronger then their non-quantitative counterparts, by allowing the construction of *metric term monads*.

6. Free Quantitative Algebras over Metric Spaces

Consider a quantitative equational theory $\mathcal{U}$ of type Ω over $\mathbb{T}X$, where X is the countable set of variables.

There is an obvious forgetful functor $U_{\mathbf{Met}} \colon \mathbb{K}(\Omega,\mathcal{U}) \to \mathbf{Met}$ from the category $\mathbb{K}(\Omega,\mathcal{U})$ of algebras satisfying $\mathcal{U}$ to the category of metric spaces and non-expansive maps. Similarly to Theorem 5.3, we aim to show that any metric space (M,d) generates a free quantitative algebra $\mathbb{T}^d[M]$ in $\mathbb{K}(\Omega,\mathcal{U})$.

Let $\Omega_M = \Omega \cup \{m : 0 \mid m \in M\}$ be the extension of Ω with additional constant symbols taken from M (assume that $\Omega \cap M = \emptyset$); and let $\mathcal{U}_M$ be the smallest quantitative equational theory of type Ω_M over X, containing $\mathcal{U}$ and satisfying, for all $m, n \in M$, the additional axioms $\emptyset \vdash m =_\epsilon n$, whenever $d(m,n) \leq \epsilon$.

The construction of $\mathcal{U}_M$ guarantees that any algebra in $\mathbb{K}(\Omega_M,\mathcal{U}_M)$ can be turned into an algebra in $\mathbb{K}(\Omega,\mathcal{U})$ simply by forgetting the interpretations of the constants in M. Conversely, given a non-expansive map $\alpha \colon M \to A$, any algebra $\mathcal{A} = (A, \Omega^{\mathcal{A}}, d^{\mathcal{A}}) \in \mathbb{K}(\Omega,\mathcal{U})$ can be turned into an algebra in $\mathbb{K}(\Omega_M,\mathcal{U}_M)$ just by interpreting each constant symbol $m : 0 \in M$ as $\alpha(m) \in A$.

This relation is functorial, and it gives the (forgetful) functor

$$U \colon \mathbb{K}(\Omega_M,\mathcal{U}_M) \to \mathbb{K}(\Omega,\mathcal{U})\,.$$

Consider $\mathbb{T}[\emptyset] \in \mathbb{K}(\Omega_M,\mathcal{U}_M)$, the free Ω_M-quantitative algebra generated over the empty set and define

$$\mathbb{T}^d[M] = U(\mathbb{T}[\emptyset]) \in \mathbb{K}(\Omega,\mathcal{U})$$

where $\mathbb{T}^d[M] = (\mathbb{T}^d[M], \Omega, d_M^{\cong})$.

The following theorem states that $\mathbb{T}^d[M]$ is the quantitative algebra in $\mathbb{K}(\Omega,\mathcal{U})$ freely generated from the metric space (M,d). Specifically, $\mathbb{T}^d[M]$ has the universal mapping property for $(M,d) \in \mathbf{Met}$ to the forgetful functor $U_{\mathbf{Met}} \colon \mathbb{K}(\Omega,\mathcal{U}) \to \mathbf{Met}$. This is described by the commutative diagram below:

$$\begin{array}{cc} \text{in } \mathbf{Met} & \text{in } \mathbb{K}(\Omega,\mathcal{U}) \\ \begin{tikzcd} (M,d) \arrow[r, "\eta_M"] \arrow[dr, "\alpha"'] & (\mathbb{T}^d[M], d_M^{\cong}) \arrow[d, "h"] \\ & (A, d^{\mathcal{A}}) \end{tikzcd} & \begin{tikzcd} \mathbb{T}^d[M] \arrow[d, dashed, "h"] \\ \mathcal{A} \end{tikzcd} \end{array}$$

where $\eta_M \colon M \to \mathbb{T}^d[M]$ is given by $\eta_M(m) = m^{\cong}$.

THEOREM 6.1. *$(\mathbb{T}^d[M], \eta_M)$ is a universal arrow from $(M,d) \in \mathbf{Met}$ to $U_{\mathbf{Met}}$.*

The free-construction described above gives rise to *the metric term monad*: given a quantitative equational theory $\mathcal{U}$, one can define the functor $\mathbb{T}_{\mathcal{U}} \colon \mathbf{Met} \to \mathbf{Met}$ that maps an object $(M,d) \in \mathbf{Met}$ to the metric space $(\mathbb{T}^d[M], d_M^{\cong})$ of Ω-terms constructed over M and quotiented w.r.t. 0-provability in $\mathcal{U}_M$, with metric $d_M^{\cong}$ induced by the equational theory $\mathcal{U}_M$. $\mathbb{T}_{\mathcal{U}}$ is monadic, with unit and multiplication being the natural transformations $\eta \colon Id \Rightarrow \mathbb{T}_{\mathcal{U}}$ and $\mu \colon \mathbb{T}_{\mathcal{U}}\mathbb{T}_{\mathcal{U}} \Rightarrow \mathbb{T}_{\mathcal{U}}$, defined for arbitrary $m \in M$, $t \in \mathbb{T}^d[M]$, $f : n \in \Omega$, and $C_1, .., C_n \in \mathbb{T}^d[\mathbb{T}^d[M]]$, by

$$\eta_M(m) = m^{\cong}\,, \quad \mu_M(t) = t\,,$$
$$\mu_M(f(C_1, .., C_n)^{\cong}) = f\big(\mu_M(C_1), .., \mu_M(C_n)\big)^{\cong}\,.$$

Unlike the monad described in the previous section, this monad lives in $\mathbf{Met}$ and the metrics associated with the set of terms are uniquely induced by the quantitative equational theories $\mathcal{U}$.

We conclude this section with a characterization of the consistency of $\mathcal{U}_M$ from a metric perspective.

We say that a metric space is *degenerate* if its support is empty or a singleton.

THEOREM 6.2. *If (M,d) is a non-degenerate metric space, then $\mathcal{U}_M$ is consistent iff the map $\eta_M : (M,d) \longrightarrow (\mathbb{T}^d[M], d^{\cong}_M)$ is an isometry*[2].

COROLLARY 6.3. *If (M,d) is non-degenerate, then $\mathcal{U}_M$ is inconsistent iff $\mathcal{U}$ is inconsistent.*

7. Free models over complete metric spaces

A basic result that we will sketch in this section is the following: if one takes a quantitative theory and forms its free algebra in the category of metric spaces and then takes its metric completion (suitably extending the operations) then that is the free algebra in the category of complete metric spaces. This gives a general characterization of the monad on complete metric spaces; though, of course, for specific examples one can give much better characterizations. The corresponding result fails for dcpos.

Recall that our metrics take values in the extended positive reals. The category of such metric spaces and non-expansive maps has coproducts and products, whereas the usual metric spaces only have finite products and, in general, no coproducts. One defines *components* of a metric space by defining an equivalence relation $x \sim y$ if $d(x,y) < \infty$. The equivalence classes are ordinary metric spaces, these are the components. A metric space is the coproduct of its components.

The usual metric completion $\mathbb{C}(M)$ of an ordinary metric space M is universal in the category of extended metric spaces. The extension of a non-expansive map $f : M \longrightarrow N$ to $\overline{f}$ is determined by the equation:

$$\overline{f}(\lim x_i) = \lim f(x_i).$$

One can now form the metric completion of any space M in the usual way. We note that it is exactly the coproduct of the completions of its components; thus one has a universal completion of any space. The usual metric completion of a finite product of ordinary spaces is the product of their metric completions, so the finite product of the universal completions is a universal completion. This argument extends to components. One has then the expected extensions of n-ary functions to completions.

One can extend this completion to algebras. Given an algebra A on a metric space M one obtains an algebra $\overline{A}$ on a complete metric space by taking the completion of M and then extending the operations on M to the completion as we described above. One can readily verify that $\overline{A}$ is the universal completion of A.

Now we introduce the continuous equation scheme to capture the idea that equations depend on their variables in a continuous way.

DEFINITION 7.1 (Continuous equation scheme). *Let Ω be an algebraic similarity type. A set*

$$\{\{x_1 =_{e_1} y_1, .., x_n =_{e_n} y_n\} \vdash s =_{f(e_1,..,e_n)} t \mid e_1, .., e_n \in \mathbb{R}_+\}$$

of basic quantitative inference over $\mathbb{T}X$ such that f is a continuous function in all variables is called a continuous equation scheme on $\mathbb{T}X$.

We say that a quantitative algebra satisfies a continuous equation scheme if it satisfies all the elements of the continuous equation scheme.

[2] By isometry in this context we mean a distance-preserving map, since η is obviously not a bijection.

PROPOSITION 7.2. *If a quantitative algebra $\mathcal{A}$ satisfies a continuous equation scheme, so does its completion $\overline{\mathcal{A}}$.*

From the above, one obtains the following results.

THEOREM 7.3. *Consider a quantitative equational theory $\mathcal{U}$ axiomatized by continuous equation schemes and a metric space (M,d). The freely generated quantitative algebra $\mathbb{T}^{\overline{d}}[\overline{M}]$ over the completion $(\overline{M},\overline{d})$ of (M,d) is isomorphic to the completion $\overline{\mathbb{T}^d[M]}$ of the quantitative algebra $\mathbb{T}^d[M]$.*

COROLLARY 7.4. *Consider a quantitative equational theory axiomatized by continuous equation schemes, over a signature with countably many operation symbols. Then the free model over a complete separable metric space M is separable, with countable set of generators being the least subalgebra containing any countable set of generators of M.*

8. Left-Invariant Barycentric Algebras

In this section we present a first example of quantitative universal algebra, the *left-invariant barycentric algebra*, and demonstrate that the freely generated one is, in this case, the algebra of probability distributions with finite support over the set of generators and the metric space is induced by the total-variation distance between distributions.

Consider the algebraic similarity type

$$\mathcal{B} = \{+_e : 2 \mid e \in [0,1]\}$$

containing, for each $e \in [0,1]$, a binary operator $+_e$. We call it the *barycentric signature*.

DEFINITION 8.1 (Left-Invariant Barycentric Equational Theory). *This theory is given by the following axiom schemata, where $x, x', x'' \in X$ (X is the countable set of variables) and $e, e' \in [0,1]$:*

(B1) $\vdash x +_1 x' =_0 x$

(B2) $\vdash x +_e x =_0 x$

(SC) $\vdash x +_e x' =_0 x' +_{1-e} x$

(SA) $\vdash (x +_e x') +_{e'} x'' =_0 x +_{ee'} (x' +_{\frac{e-ee'}{1-ee'}} x'')$ *provided that* $e, e' \in (0,1)$

(LI) $\vdash x' +_e x =_\epsilon x'' +_e x$ *where* $e \leq \epsilon \in \mathbb{Q}_+$

(SC) stands for *skew commutativity* and (SA) for *skew associativity*. We call (LI) the *left-invariance axiom schema*. Observe that if $e \in \mathbb{Q}$, (LI) takes the simpler form:

$$\vdash x' +_e x =_e x'' +_e x.$$

The algebras satisfying left-invariant barycentric equational theories are called *left-invariant barycentric algebras* or LIB algebras for short.

Hereafter we focus on the the class $\mathbb{K}(\mathcal{B},\mathcal{U}^{LI})$ defined by the left-invariant barycentric equational theory $\mathcal{U}^{LI}$.

8.1 The Freely-Generated Algebra

If (S,Σ) is a measurable space and $\Delta[S,\Sigma]$ is the class of probability measures over (S,Σ), the *total variation distance* between probability measures is defined, for arbitrary $\mu,\nu \in \Delta[S,\Sigma]$ by

$$T(\mu,\nu) = \sup_{E\in\Sigma} |\mu(E) - \nu(E)|.$$

Let now M be a set and $\mathbb{T}[M]$ be the LIB algebra in $\mathbb{K}(\mathcal{B}, \mathcal{U}^{LI})$ freely generated from M. By Theorem 5.3, $\mathbb{T}[M]$ has the universal mapping property for M to $U_{\mathbf{Set}} \colon \mathbb{K}(\mathcal{B}, \mathcal{U}^{LI}) \to \mathbf{Set}$.

Denote by $\Pi[M]$ the set of *finitely-supported* discrete probability distributions on M. Next we will show that $\Pi[M]$ endowed with the total-variation distance can be organized as a LIB algebra in $\mathbb{K}(\mathcal{B}, \mathcal{U}^{LI})$ having the universal mapping property for M to $U_{\mathbf{Set}} \colon \mathbb{K}(\mathcal{B}, \mathcal{U}^{LI}) \to \mathbf{Set}$. From the uniqueness of the universal arrows, we will get that $\Pi[M]$ and $\mathbb{T}[M]$ are isomorphic.

We can organize $\Pi[M]$ as an algebra of type $\mathcal{B}$ by interpreting each operator $+_e : 2 \in \mathcal{B}$, for arbitrary $\mu, \nu \in \Pi[M]$, as follows

$$\mu +_e \nu = e\mu + (1-e)\nu \,,$$

We can further regard $\Pi[M]$ as a quantitative algebra by taking the total-variation distance as a metric on $\Pi[M]$.

THEOREM 8.2. $\Pi[M] = (\Pi[M], \mathcal{B}, T) \in \mathbb{K}(\mathcal{B}, \mathcal{U}^{LI})$.

The next theorem shows that $\Pi[M]$ has the universal mapping property for M to $U_{\mathbf{Set}}$, with universal arrow $(\Pi[M], \delta_M)$, where $\delta_M \colon M \to \Pi[M]$ maps $m \in M$ to $\delta_m \in \Pi[M]$ —the Dirac measure with probability mass concentrated at $m \in M$.

THEOREM 8.3. $(\Pi[M], \delta_M)$ *is an universal arrow from* $M \in \mathbf{Set}$ *to* $U_{\mathbf{Set}}$.

The next result follows directly by Theorem 5.3 and 8.3.

COROLLARY 8.4. *The quantitative* $\mathcal{B}$*-algebras* $\Pi[M]$ *and* $\mathbb{T}[M]$ *are isomorphic with bijective isometry* $h \colon \mathbb{T}^d[M] \to \Pi[M]$ *given, for* $m \in M$ *and* $t, s \in \mathbb{T}M$ *by*

$$h(m^{\cong}) = \delta_m \,, \quad h((t +_e s)^{\cong}) = e h(t^{\cong}) + (1-e) h(s^{\cong}) \,.$$

Consequently, the metric induced by the quantitative equational theory $\mathcal{U}^{LI}$ coincides with the total variation distance on $\Pi[M]$. Thus we say that $\mathcal{U}^{LI}$ axiomatizes the total variation distance.

9. Quantitative Semilattices with a zero

In this section we provide a first example of free quantitative algebra over metric spaces. We discuss the case of the quantitative semilattices with a zero and show how their axiomatization induces Hausdorff distances both in the finitary and in the continuous case.

Recall the definition of Hausdorff metric: let (M, d) be a metric space. The *Hausdorff metric induced by* d on the set of all compact subsets of M, is defined, for arbitrary compact sets $A, B \subseteq M$ by

$$H_d(A, B) = \max \left\{ \sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A) \right\} \,,$$

where, $d(m, N) = \inf_{n \in N} d(m, n)$ denotes the distance from an element $m \in M$ to a set $N \subseteq M$.

Consider the signature of (bounded join-) semilattices with a zero

$$\mathcal{S} = \{+ : 2, 0 : 0\}$$

containing one binary operator $+$ and one constant 0.

DEFINITION 9.1 (Quantitative Semilattice Equational Theory). *This theory is given by the following axiom schemata where* $x, x', x'', y, y' \in X$ *(*X *is the countable set of variables) and* $\epsilon, \epsilon' \in [0, 1]$*:*

(S0) $\vdash x + 0 =_0 x$

(S1) $\vdash x + x =_0 x$

(S2) $\vdash x + x' =_0 x' + x$

(S3) $\vdash (x + x') + x'' =_0 x + (x' + x'')$

(S4) $\{x =_\epsilon y, x' =_{\epsilon'} y'\} \vdash x + x' =_\delta y + y'$*, where* $\delta = \max\{\epsilon, \epsilon'\}$.

In this section we focus on the algebras satisfying quantitative semilattice equational theories; we call these *quantitative semilattices with a zero*.

Hereafter we focus on the class $\mathbb{K}(\mathcal{S}, \mathcal{U}^{\mathcal{S}})$ defined by the quantitative semilattice equational theory $\mathcal{U}^{\mathcal{S}}$.

9.1 The Finitary Case

Fix a metric space (M, d). Let $\mathbb{T}^d[M]$ be the quantitative semilattice with a zero in $\mathbb{K}(\mathcal{S}, \mathcal{U}^{\mathcal{S}})$ freely generated from (M, d). By Theorem 6.1, $\mathbb{T}^d[M]$ has the universal mapping property for (M, d) to $U_{\mathbf{Met}} \colon \mathbb{K}(\mathcal{S}, \mathcal{U}^{\mathcal{S}}) \to \mathbf{Met}$.

Denote by $\mathbb{F}[M]$ the set of all *finite* subsets of M. In what follows we show that $\mathbb{F}[M]$ can be organized as a quantitative semilattice with a zero in $\mathbb{K}(\mathcal{S}, \mathcal{U}^{\mathcal{S}})$ where the metric structure is defined by the Hausdorff metric H_d; moreover, $\mathbb{F}[M]$ has the universal mapping property for (M, d) to $U_{\mathbf{Met}} \colon \mathbb{K}(\mathcal{S}, \mathcal{U}^{\mathcal{S}}) \to \mathbf{Met}$. This will prove that $\mathbb{F}[M]$ and $\mathbb{T}^d[M]$ are isomorphic $\mathcal{S}$-quantitative algebras.

We organize $\mathbb{F}[M]$ as an universal algebra of type $\mathcal{S}$ by defining, for arbitrary $A, B \in \mathbb{F}[M]$, $A + B = A \cup B$, $0 = \emptyset$. We can further organize $\mathbb{F}[M]$ as a quantitative algebra by taking the Hausdorff metric H_d induced by d.

THEOREM 9.2. $\mathbb{F}[M] = (\mathbb{F}[M], \mathcal{S}, H_d) \in \mathbb{K}(\mathcal{S}, \mathcal{U}^{\mathcal{S}})$.

The next theorem states that $\mathbb{F}[M]$ has the universal mapping property for (M, d) to $U_{\mathbf{Met}}$, with universal arrow $(\mathbb{F}[M], \chi_M)$, where $\chi_M \colon M \to \mathbb{F}[M]$ is the map that assigns to arbitrary $m \in M$, the singleton set $\chi_M(m) = \{m\}$. Note that, $H_d(\{m\}, \{n\}) = d(m, n)$, hence χ_M is non-expansive.

THEOREM 9.3. $(\mathbb{F}[M], \chi_M)$ *is an universal arrow from* $(M, d) \in \mathbf{Met}$ *to* $U_{\mathbf{Met}}$.

Theorem 6.1 and Theorem 9.3 prove the following corollary.

COROLLARY 9.4. *The quantitative* $\mathcal{S}$*-algebras* $\mathbb{F}[M]$ *and* $\mathbb{T}^d[M]$ *are isomorphic with bijective isometry* $h \colon \mathbb{T}^d[M] \to \mathbb{F}[M]$ *given, for* $m \in M$ *and* $t, s \in \mathbb{T}M$ *by*

$$h(m^{\cong}) = \{m\} \,, \qquad h((t + s)^{\cong}) = h(t^{\cong}) \cup h(s^{\cong}) \,.$$

Hence, the distance induced by the quantitative equational theory $\mathcal{U}^{\mathcal{S}}$ extended with the axioms relative to the generator (M, d) is the Hausdorff metric induced by d. Thus we say that $\mathcal{U}^{\mathcal{S}}_M$ axiomatizes the Hausdorff distance.

9.2 The Continuous Case

We now focus on the class of the compact subsets of a complete separable metric space and prove that it can be organized as a quantitative semilattice with a zero. It turns out that this is the freely generated algebra in the category of quantitative semilattices with a zero over complete separable metric spaces. As might be expected, the proofs here are more analytic in contrast with the combinatorial proofs of the previous subsection.

Consider a complete separable metric space (M, d). Let $\mathbb{G}[M]$ be the set of the compact subsets of M in the open-ball topology of d. We show that by interpreting $+$ by $\cup$, 0 by $\emptyset$ and endowing $\mathbb{G}[M]$ with the Hausdorff metric H_d, we obtain a quantitative semilattice with a zero that satisfies $\mathcal{U}^{\mathcal{S}}$.

As shown in the previous section, we can also construct the freely generated quantitative semilattice with a zero $\mathbb{T}^d[M] = (\mathbb{T}^d[M], \mathcal{S}, d^{\cong}_M)$, which is isomorphic to $\mathbb{F}[M] = (\mathbb{F}[M], \mathcal{S}, H_d)$.

However, $(\mathbb{T}^d[M], d^{\cong}_M)$ is separable (with countable dense subset given by $\mathbb{T}^d[D]$, where D is the countable dense set in M) but it is not a complete metric space.

Consider $(\overline{\mathbb{T}^d[M]}, \overline{d^{\cong}_M})$, the completion of $(\mathbb{T}^d[M], d^{\cong}_M)$. Since $\mathbb{T}^d[M]$ is isomorphic to $\mathbb{F}[M]$, their completions must be isomorphic metric spaces.

Let $\overline{\mathbb{K}}_{\mathcal{S}}$ be the subcategory of quantitative semilattices with a zero over complete separable metric spaces. We prove that $\mathbb{G}[M] = (\mathbb{G}[M], \mathcal{S}, H_d)$ and $\overline{\mathbb{T}^d[M]} = (\overline{\mathbb{T}^d[M]}, \mathcal{S}, \overline{d^{\cong}_M})$ are isomorphic quantitative semilattices with a zero.

THEOREM 9.5. *If (M, d) is a complete separable metric space, then $\mathbb{G}[M] \in \overline{\mathbb{K}}_{\mathcal{S}}$. Moreover, $\mathbb{G}[M]$ is isomorphic to $\overline{\mathbb{T}^d[M]}$.*

Proof. Verifying the axioms of the quantitative semilattices with a zero for $\mathbb{G}[M]$ is routine. What we need to prove further is that $(\mathbb{G}[M], H_d)$ is a complete separable metric space.

Let $D \subseteq M$ be a countable dense subset of M (its existence is guaranteed by the fact that (M, d) is a separable space). $\mathbb{F}[D]$ is countable and we now show that it is dense in $\mathbb{G}[M]$.

Consider an arbitrary compact set $C \in \mathbb{G}[M]$. The set $S = D \cap C$ is countable and dense in C. Suppose that $S = \{s_1, s_2, \ldots\}$. Then, the sets $S_i = \{s_1, \ldots, s_i\}$, $i \in \mathbb{N}$ are all compact, hence elements of $\mathbb{F}[M]$, and their sequence converges to C in $(\mathbb{G}[M], H_d)$. Hence, $\mathbb{F}[D]$ is dense in $\mathbb{G}[M]$.

Previously, we have shown that $d^{\cong}_M = H_d$ on $\mathbb{F}[M]$, hence also on $\mathbb{F}[D]$. Since the completion of $\mathbb{F}[D]$ is unique and it gives us $(\mathbb{G}[M], H_d)$, we obtain the isomorphism between $\overline{\mathbb{T}^d[M]}$ and $\mathbb{G}[M]$; hence also an isomorphism of metric spaces. ∎

Next we state that $\overline{\mathbb{T}^d[M]}$ is the quantitative algebra in $\overline{\mathbb{K}}_{\mathcal{S}}$ freely generated from the complete separable metric space (M, d). Specifically, $\overline{\mathbb{T}^d[M]}$ has the universal mapping property for $(M, d) \in$ $\mathbf{CSMet}$ (the category of complete separable metric spaces with non-expansive maps) to the forgetful functor

$$U_{\mathbf{CSMet}} \colon \overline{\mathbb{K}}_{\mathcal{S}} \to \mathbf{CSMet}\,.$$

This situation is described by the commutative diagram below (*cf.* Definition 3.2):

$$\begin{array}{cc} \text{in } \mathbf{CSMet} & \text{in } \overline{\mathbb{K}}_{\mathcal{S}} \\ \begin{tikzcd} (M,d) \arrow[r, "\eta_M"] \arrow[dr, "\alpha"'] & (\overline{\mathbb{T}^d[M]}, \overline{d^{\cong}_M}) \arrow[d, "h"] \\ & (A, d^{\mathcal{A}}) \end{tikzcd} & \begin{tikzcd} \overline{\mathbb{T}^d[M]} \arrow[d, dashed, "h"] \\ \mathcal{A} \end{tikzcd} \end{array}$$

THEOREM 9.6. *$(\overline{\mathbb{T}^d[M]}, \eta_M)$ is a universal morphism from $(M, d) \in$ $\mathbf{CSMet}$ to $U_{\mathbf{CSMet}}$.*

10. Interpolative Barycentric Algebras

In this section we study a variation of quantitative barycentric algebras, which is similar to the left-invariant barycentric algebra discussed in Section 8 but with one slightly stronger axiom than (LI). The signature remains the same but the axioms though, superficially, only slightly different give a very different metric. Instead of axiomatizing the total variation distance, we get an axiomatization of the p-Wasserstein metric for $p \geq 1$, both in the finitary and the continuous cases. For $p = 1$ this reduces to the Kantorovich metric. We call these algebras *interpolative barycentric* algebras or p-IB algebras for short. The new axiom is a kind of interpolation axiom.

Consider the barycentric signature $\mathcal{B} = \{+_e : 2 \mid e \in [0, 1]\}$ from Section 8.

DEFINITION 10.1 (p-IB Equational Theory). *This theory is given by the axiom-schemata (B1), (B2), (SC), (SA) from Definition 8.1) and the following axiom-scheme (IB$_p$), where $\epsilon_1, \epsilon_2 \in [0, 1]$ and $\delta \in \mathbb{Q}_+ \cap [0, 1]$:*

(IB$_p$) $\{x =_{\epsilon_1} y, x' =_{\epsilon_2} y'\} \vdash x +_e x' =_\delta y +_e y'$, *where* $(e\epsilon_1^p + (1-e)\epsilon_2^p)^{1/p} \leq \delta$.

Note that (IB$_p$) is not an unconditional quantitative inference as are the previous examples. Moreover, it is stronger than the axiom (LI) in Definition 8.1 for 1-bounded metrics, in the sense that (LI) is than just an instantiation of (IB$_p$).

Hence, this new proof system can prove more basic quantitative equations.

If we state (IB$_1$), we get the axiom below.

(IB$_1$) $\{x =_{\epsilon_1} y, x' =_{\epsilon_2} y'\} \vdash x +_e x' =_\delta y +_e y'$, where $e\epsilon_1 + (1-e)\epsilon_2 \leq \delta$.

In this section we focus on the class $\mathbb{K}(\mathcal{B}, \mathcal{U}^{IB})$ defined by the p-IB barycentric equational theory $\mathcal{U}^{IB}$.

Kantorovich-Wasserstein Duality

Let (M, d) be a metric space and $p \geq 1$. The *p-Wasserstein metric induced by d* on the set $\Delta[M]$ of Borel probability measures over M, is defined, for arbitrary $\mu, \nu \in \Delta[M]$ as

$$W^p_d(\mu, \nu) = \sup \left\{ \left| \int f^p \, \mathrm{d}\mu - \int f^p \, \mathrm{d}\nu \right|^{1/p} \right\},$$

where $f \colon M \to \mathbb{R}_+$ is non-expansive. In particular, for $p = 1$, one gets the Kantorovich metric induced by d on $\Delta[M]$:

$$K_d(\mu, \nu) = \sup \left\{ \left| \int f \, \mathrm{d}\mu - \int f \, \mathrm{d}\nu \right| \right\}.$$

Like the total variation distance, the p-Wasserstein metrics satisfy a dual characterization, based on the notion of coupling:

LEMMA 10.2 (Kantorovich-Wasserstein Duality (Villani 2008)). *Let (M, d) be a metric space. Then, for arbitrary Borel probability measures $\mu, \nu \in \Delta[M]$*

$$W^p_d(\mu, \nu)^p = \min \left\{ \int d^p \, \mathrm{d}\omega \mid \omega \in \mathcal{C}(\mu, \nu) \right\}.$$

Notice that an optimal coupling for W^p_d, i.e., the one that attains the minimum in the characterization above, always exists.

One can show that the total variation distance is just a particular case of the Kantorovich metric, namely, $T(\mu,\nu) = K_{1_{\neq}}(\mu,\nu)$, where $1_{\neq}$ is the metric that assigns distance 1 to all distinct pairs of points.

10.1 The Finitary Case

Fix a metric space (M, d). Let $\mathbb{T}^d[M]$ be the p-IB algebra in $\mathbb{K}(\mathcal{B},\mathcal{U}^{IB})$ freely generated from (M, d), as constructed in Section 6. By Theorem 6.1, $\mathbb{T}^d[M]$ has the universal mapping property for (M, d) to $U_{\mathbf{Met}} \colon \mathbb{K}(\mathcal{B},\mathcal{U}^{IB}) \to \mathbf{Met}$.

THEOREM 10.3. *If (M, d) is a non-degenerate metric space then $\mathbb{T}^d[M]$ is a non-degenerate p-IB algebra. In particular, $\mathcal{U}^{IB}$ is a consistent quantitative theory.*

Denote by $\Pi[M]$ the set of finitely supported Borel probability measures on M —i.e., those that can be represented as finite convex combinations of Dirac distributions δ_m, for $m \in M$. Next we will show that $\Pi[M]$ can be organized as a p-IB algebra in $\mathbb{K}(\mathcal{B},\mathcal{U}^{IB})$, with metric given by the p-Wasserstein metric W^p_d. Moreover, we show that this algebra enjoys the universal mapping property for (M, d) to $U_{\mathbf{Met}} \colon \mathbb{K}(\mathcal{B},\mathcal{U}^{IB}) \to \mathbf{Met}$; consequently $\Pi[M]$ and $\mathbb{T}^d[M]$ are isomorphic $\mathcal{B}$-algebras.

Similarly to Section 8, we regard $\Pi[M]$ as a universal algebra of type $\mathcal{B}$ by interpreting each operator $+_e : 2 \in \mathcal{B}$, for arbitrary $\mu, \nu \in \Pi[M]$, as

$$\mu +_e \nu = e\mu + (1-e)\nu \,,$$

However, unlike the situation in Section 8, $\Pi[M]$ will be viewed as a quantitative algebra by taking as a metric the p-Wasserstein metric W^p_d induced by d, rather then the total variation distance.

THEOREM 10.4. *$\Pi[M] = (\Pi[M], \mathcal{B}, W^p_d) \in \mathbb{K}(\mathcal{B},\mathcal{U}^{IB})$, i.e., $\Pi[M] \models \mathcal{U}^{IB}$.*

The next theorem shows that $\Pi[M]$ has the universal mapping property for (M, d) to $U_{\mathbf{Met}}$, with universal arrow $(\Pi[M], \delta_M)$, where $\delta_M \colon M \to \Pi[M]$ maps $m \in M$ to $\delta_m \in \Pi[M]$ — the Dirac measure with probability mass in $m \in M$. Note that, $W^p_d(\delta_m, \delta_n) = d(m, n)$, hence δ_M is non-expansive.

THEOREM 10.5. *$(\Pi[M], \delta_M)$ is an universal arrow from $(M, d) \in \mathbf{Met}$ to $U_{\mathbf{Met}}$.*

The next result follows directly by Theorem 6.1 and 10.5.

COROLLARY 10.6. *The quantitative $\mathcal{B}$-algebras $\Pi[M]$ and $\mathbb{T}^d[M]$ are isomorphic with bijective isometry $h \colon \mathbb{T}^d[M] \to \Pi[M]$ characterized, for $m \in M$ and $t, s \in \mathbb{T}M$ by*

$$h(m^{\cong}) = \delta_m \,, \quad h((t +_e s)^{\cong}) = e h(t^{\cong}) + (1-e) h(s^{\cong}) \,.$$

This means that the quantitative equational theory $\mathcal{U}^{IB}$, further extended with the axioms relative to the metric space (M, d), axiomatizes the p-Wasserstein metric induced by d; and for $p = 1$ it characterizes the Kantorovich metric.

10.2 The Continuous Case

We now focus on the class of Borel probability measures over a complete separable metric space and prove that it forms a p-IB algebra. In this case we are not restricting to finitely-supported distributions. It turns out that this is the freely-generated algebra in the category of the p-IB algebras defined for complete separable metric spaces.

Consider a complete separable metric space (M, d). Let $\Delta[M]$ the set of all Borel probability measures on M endowed with the signature $\mathcal{B}$, where we define for arbitrary $\mu, \nu \in \Delta[M]$ and $r \in [0, 1]$,

$$\mu +_r \nu = r\mu + (1-r)\nu.$$

As shown previously, $\mathbb{T}^d[M] = (\mathbb{T}^d[M], \mathcal{B}, d^{\cong}_M)$ is a barycentric algebra isomorphic to $\Pi[M] = (\Pi[M], \mathcal{B}, W_d)$. However, we prove below that $(\mathbb{T}^d[M], d^{\cong}_M)$ is separable but it is not a complete metric space.

Consider the metric space $(\overline{\mathbb{T}^d[M]}, \overline{d^{\cong}_M})$ obtained by the completion of $(\mathbb{T}^d[M], d^{\cong}_M)$.

We need now to recall a series of definitions and results that relates the concept of weak topology and the p-Wasserstein distance.

DEFINITION 10.7. *The* ***weak topology*** *on $\Delta[M]$ is the topology such that convergence of the sequence of measures ν_i to ν means that for all bounded continuous real-valued functions f we have*

$$\int f \mathrm{d}\nu_i \to \int f \mathrm{d}\nu.$$

A ***base*** *for the topology consists of sets of the form*

$$U_{f_1,\dots,f_k,\varepsilon}(\nu) = \left\{ \mu : \left| \int_M f_i \,\mathrm{d}\nu - \int_M f_i \,\mathrm{d}\mu \right| < \varepsilon, i = 1, \dots, k \right\},$$

where the f_i are bounded continuous functions, ν is a probability measure and $\varepsilon > 0$.

If (M, d) is a Polish space[3], then it is known that p-Wasserstein W^p_d metrizes the weak topology on $\Delta[M]$ (see Theorem 6.9 and Corollary 6.13 in (Villani 2008)).

The following lemma is well known (see Theorem 6.18 in (Villani 2008)).

PROPOSITION 10.8. *Let M be a Polish space and let $\{c_i\}^k_{i=1}$ be positive real numbers such that $\sum^k_{i=1} c_i = 1$. Let $\{m_i\}^k_{i=1}$ be points in M. Then measures of the form $\sum^k_{i=1} c_i\delta_{m_i}$ are weakly dense in $\Delta[M]$.*

Let $\overline{\mathbb{K}}_{\mathcal{B}}$ be the class of IB algebras with complete separable metric spaces. We prove that $\Delta[M] = (\Delta[M], \mathcal{B}, W^p_d)$ is isomorphic, as a barycentric algebra, to $\overline{\mathbb{T}^d[M]} = (\overline{\mathbb{T}^d[M]}, \mathcal{B}, \overline{d^{\cong}_M})$.

THEOREM 10.9. *If (M, d) is a complete separable metric space, then $\Delta[M] \in \overline{\mathbb{K}}_{\mathcal{B}}$. Moreover, $\Delta[M]$ is isomorphic to $\overline{\mathbb{T}^d[M]}$.*

Proof. Verifying the axioms of the barycentric algebras for $\Delta[M]$ is routine and follows closely the proof of Theorem 10.4. What we need to prove further is that $\Delta[M]$ is a complete separable metric space.

Let $D \subseteq M$ be a countable dense subset of M (its existence is guaranteed by the fact that (M, d) is a separable space). Now $\Pi[D]$ is of course not countable but we can take all distributions that assign only rational measures to points and get a countable set. We call this $\mathcal{P}[D]$ for short. We now show that it is dense in $\Delta[M]$.

Let $\rho \in \Delta[M]$. Since (M, d) is Polish, W^p_d metrizes the weak-topology on $\Delta[M]$, which is also a Polish space (Corollary 6.13

[3] A Polish space is the topological space underlying a complete separable metric space.

in (Villani 2008)). Moreover, $\Pi[M]$ is dense in $\Delta[M]$ with respect to this topology by Prop. 10.8. Hence, there exists a sequence $(\rho_i)_{i\in\mathbb{N}} \subseteq \Pi[M]$ of distributions with finite support on M that converges to ρ. Since D is dense in M and the rationals are dense in $[0, 1]$, for any sequence $(\epsilon_i)_{i\in\mathbb{N}} \in [0, 1]$ that converges to 0, we can find a sequence $(\rho'_i)_{i\in\mathbb{N}} \subseteq \mathcal{P}[D]$ such that $W^p_d(\rho_i, \rho'_i) < \epsilon_i$. Thus, $\{\rho_i \mid i \in \mathbb{N}\} \cup \{\rho'_i \mid i \in \mathbb{N}\}$ is a Cauchy sequence in $\Pi[M]$ and since $(\rho_i)_{i\in\mathbb{N}}$ converges to ρ and $\Pi[M]$ is complete, also $(\rho'_i)_{i\in\mathbb{N}}$ converges to ρ. And this proves that $\mathcal{P}[D]$ is dense in $\Delta[M]$.

In the previous section we have shown that $d^{\cong}_M = W^p_d$ on $\Pi[M]$, hence also on $\Pi[D]$. Since the completion of $\Pi[D]$ is unique and it gives us $(\Delta[M], d^{\cong}_M)$, we obtain the isomorphism between $\overline{\mathbb{T}^d[M]}$ and $\Delta[M]$; hence, also the isomorphism of metric spaces. ∎

Next we show that $\overline{\mathbb{T}^d[M]}$ is the quantitative algebra in $\overline{\mathbb{K}}_{\mathcal{B}}$ freely generated from the complete separable metric space (M, d). Specifically, $\overline{\mathbb{T}^d[M]}$ has the universal mapping property for $(M, d) \in \mathbf{CSMet}$ (the category of complete separable metric spaces with non-expansive maps) to the forgetful functor

$$U_{\mathbf{CSMet}} \colon \overline{\mathbb{K}}_{\mathcal{B}} \to \mathbf{CSMet}\,.$$

This situation is described by the commutative diagram below (*cf.* Definition 3.2):

$$\begin{tikzcd}
 & \text{in } \mathbf{CSMet} & \text{in } \overline{\mathbb{K}}_{\mathcal{B}} \\
(M,d) \arrow[r, "\eta_M"] \arrow[dr, "\alpha"'] & (\overline{\mathbb{T}^d[M]}, \overline{d^{\cong}_M}) \arrow[d, "h"] & \overline{\mathbb{T}^d[M]} \arrow[d, dashed, "h"] \\
 & (A, d^{\mathcal{A}}) & \mathcal{A}
\end{tikzcd}$$

THEOREM 10.10. *$(\overline{\mathbb{T}^d[M]}, \eta_M)$ is a universal morphism from $(M, d) \in \mathbf{CSMet}$ to $U_{\mathbf{CSMet}}$.*

All the results of this section can be readily extended to the case of subprobability measures by introducing a new constant in the signature where the "missing mass" can reside. These are called *pointed* barycentric algebras. Similar results with the ones presented in this section can be proven for the case of pointed barycentric algebras.

11. Related work

The closest related work is by van Breugel et al. (van Breugel et al. 2007) and by Adamek et al. (Adámek et al. 2012) both of which were important precursors to the present work. The first paper really shows why the Hausdorff and Kantorovich metrics are canonical. The second one shows the finitary natures of these monads. In the paper by van Breugel et al. (van Breugel et al. 2007) it was shown that the Kantorovich functor is left adjoint to a forgetful functor from a suitable algebraic category (mean-value algebras) to complete metric spaces. Similarly they show that a suitable Hausdorff functor can be treated in a similar way. Their results are intended to exhibit the power of an approach to solving recursive equations using the theory of accessible categories. Adamek et al. (Adámek et al. 2012) have studied the finitary versions of the same functors and have given equational presentations.

A fairly important difference with the present work is that we use the barycentric axioms rather than the mean value axioms. The major difference, however, is our use of quantitative equations that capture the idea of approximate equality.

The difference between the mean-value axiomatization and the barycentric axiomatization may seem unimportant but we feel that barycentric algebras are more fundamental. They allow *all* binary choices to be directly available; they are of course all definable from the mean-value *if you allow infinite terms but certainly not if you want everything to be finitary.* The barycentric algebras are the axioms for abstract convex spaces and arise widely in mathematics; see the historical remarks in (Keimel and Plotkin 2015). Barycentric algebras work very well in other settings too. For example, if one takes the free pointed barycentric algebras in other categories like sets or cpos one gets the structures one expects: finite probability distributions for the case of sets and the valuation powerdomain for the case of continuous dcpos.

We do not see as yet how all this fits with the program being pursued by Bart Jacobs and his group at Nijmegen where they have a general notion of quantitative logic based on structures that they call an "effectus."(Cho et al. 2015). There are many intriguing possibilities but we must defer a proper comparison until we have digested effectus theory more deeply. One of the motivating strands of that work was various dualities involving convex structures so there certainly should be connections.

12. Future work

There is clearly much more to do both in the general theory and in specific examples. A fundamental task is to understand how to combine effects just as in the non-quantitative case; many of the basic results (Hyland et al. 2006, 2007) apply.

We are actively looking at Markov processes as an example; this could benefit from a many-sorted extension of the basic theory or could alternatively use recursive domain equations. As far as we know, an equational presentation of Markov processes does not exist. Other possible examples are general distributions coming from a suitable axiomatization of cones or an axiomatization of Choquet capacities.

Acknowledgements

Prakash Panangaden is supported by a grant from Natural Sciences and Engineering Research Council of Canada. Radu Mardare is supported by a grant from The Danish Council for Independent Research. We thank Giorgio Bacci, Bartek Klin, Dexter Kozen and Kim G. Larsen for invaluable discussions.

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