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Uncountable Limits and the Lambda Calculus

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Abstract. In this paper we address the problem of solving recursive domain equations using *uncountable limits* of domains. These arise for instance, when dealing with the ω_1-continuous function-space constructor and are used in the denotational semantics of programming languages which feature unbounded choice constructs. Surprisingly, the category of cpo's and ω_1-continuous embeddings is not ω_0-cocomplete. Hence the standard technique for solving reflexive domain equations fails. We give two alternative methods. We discuss also the issue of completeness of the $\lambda\beta\eta$-calculus w.r.t reflexive domain models. We show that among the reflexive domain models in the category of cpo's and ω_0-continuous functions there is one which has a minimal theory. We give a reflexive domain model in the category of cpo's and ω_1-continuous functions whose theory is precisely the $\boldsymbol{\lambda\beta\eta}$ theory. So ω_1-continuous λ-models are *complete* for the $\lambda\beta\eta$-calculus.

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Key words: countable non-determinism, denotational semantics, domain equations, lambda-calculus.

1. Introduction

The study of semantic models for fairness leads naturally to the study of countable non-determinism, Apt and Plotkin [1986], Plotkin [1982]. There non-continuity phenomena appear. So, rather than the usual notion of ω_0-continuity (the preservation of lubs of countable chains), one considers the weaker notion of ω_1-continuity (the preservation of lubs of ω_1-chains). It is

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then natural to consider a category CPO_1 in which the objects have lubs of both ω_0- and ω_1-chains (and a least element) but where the morphisms are the ω_1-continuous functions. In Section 3 we show that that the corresponding category of embeddings CPO_1^E is not ω_0-cocomplete (contrary to an assertion in Plotkin [1982]). Therefore the standard categorical techniques for solving recursive domain equations, based on embeddings, Smyth and Plotkin [1982], do not apply in this case. In Section 4 we show how to overcome this difficulty. In particular we give two methods which generalise appropriately the technique of Adámek and Koubek [1979] and allow us to solve recursive domain equations involving several domain constructors, including $\rightarrow_{\omega_1}$ (exponentiation in CPO_1) and $\mathcal{P}^\omega$ the countable powerdomain constructor (introduced in Plotkin [1982] where it is written as $\mathcal{P}_1$). Finally in Section 5 we discuss the issue of completeness of the $\lambda\beta\eta$-calculus w.r.t. the reflexive domain models contained in a standard category of continuous functions CPO and w.r.t. the reflexive domain models contained in the category for fairness CPO_1. We utilise an argument based on logical relations to show that among the reflexive domain models in CPO there is one which has a minimal theory. We utilise the second method for solution of domain equations to define a reflexive domain model in CPO_1 whose theory is exactly the theory $\boldsymbol{\lambda\beta\eta}$. We show therefore that the $\lambda\beta\eta$-calculus is complete w.r.t. the class of reflexive domain models based on ω_1-continuous functions. Before passing to these, quite technical matters, in Section 2 we discuss fairness and countable non-determinism in the setting of a simple guarded command language. This suffices to demonstrate the failure of ω_0-continuity and also the need to iterate through all the countable ordinals, when the use of ω_1-continuity is appropriate.

2. Unbounded non-determinism and fairness

Dijkstra's *guarded command language*, $\mathcal{GC}$, is an imperative language featuring a particular kind of command, the *guarded* command, that can be used to implement *finitary non-determinism.* See Appendix A for the definition and semantics of $\mathcal{GC}$.

The *random assignment* command $x := ?$ is added to $\mathcal{GC}$ in order to achieve *countable non-determinism*; it sets x to an arbitrary natural number. As is well-known, this command allows countable sets of possible outputs even under the assumption of program termination. König's lemma does not apply to this context; representing the computation history of a non-deterministic program using a generating tree, the random assignment command allows programs whose generating tree may not be finitely branching, and hence can have an infinite number of nodes without having an infinite path (all computations terminate).

The $\mathcal{GC}$-language has *weakly fair iteration* if, for every iteration cycle, each guard B_i that is continuously enabled will not be indefinitely postponed. Clearly the assumption of fairness implies unbounded non-determinism.

More precisely: under the assumption of fairness, we can construct an appropriate non-deterministic program which exhibits the same behaviour as the random assignment command:

$$\begin{array}{l} x := 0; b := \mathit{true}; \\ \mathbf{do}\ b \ \rightarrow \ x := x+1 \mid b \ \rightarrow \ b := \mathit{false}\ \mathbf{od} \end{array}$$

The simulation can also go the other way. By using the unbounded choice operator, we can simulate fair computations. More precisely, the weakly fair iteration command:

$$\mathbf{do}\, B_1 \rightarrow C_1 \mid \ \ldots \ \mid B_n \rightarrow C_n\, \mathbf{od}$$

is simulated by the following statement:

$$\begin{array}{lll} \mathbf{do}\ (B_1 \vee \ldots \vee B_n) \rightarrow & x_1 := ?; \ldots ; x_n := ?; & \\ & \mathbf{do} & x_1 \geq 0 \rightarrow x_1 := x_1 - 1;\ \mathbf{if}\, B_1 \rightarrow \ C_1\, \mathbf{fi} \\ & \mid & \ldots \\ & \mid & x_n \geq 0 \rightarrow x_1 := x_n - 1;\ \mathbf{if}\, B_n \rightarrow \ C_n\, \mathbf{fi} \\ & \mathbf{od} & \\ \mathbf{od} & & \end{array}$$

where $x_1, \ldots, x_n$ do not occur in the B_i or C_i.

This connection between fairness and unbounded (but countable) non-determinism motivates the study of non-deterministic languages with unbounded choice operators. Thus, generative semantics for fair processes (i.e. involving only and all fair computations) can be given by simulation via semantics for countable non-determinism, rather than through a difficult direct analysis of fairness properties.

In order to define the denotational semantics of the language $\mathcal{GC}$ we need to introduce a *domain of denotations* for non-deterministic computations. For this purpose we consider the *Plotkin powerdomain* of the flat cpo $S_\perp$, where S is a countable set of stores (states of computations); see Apt and Plotkin [1986] for more details.

DEFINITION 1. *i) The powerdomain $\mathcal{P}(S_\perp)$ is the set $\{A \subseteq S_\perp \mid A \neq \emptyset, A$ finite or $\perp \in A\}$ with the Egli-Milner order:*

$$A \sqsubseteq B \quad \textit{iff} \quad A = B \ \vee \ (\perp \in A \ \wedge \ A \setminus \{\perp\} \subseteq B)$$

ii) Given a function $f : S \rightarrow \mathcal{P}(S_\perp)$, its extension is $f^+ : \mathcal{P}(S_\perp) \rightarrow \mathcal{P}(S_\perp)$, where $f^+(A) = \bigcup_{a \in A \setminus \{\perp\}} f(a) \cup \{\perp \mid \perp \in A\}$.

The denotational semantics of $\mathcal{GC}$ is given by a function

$$\mathcal{C} : Com \rightarrow (S \rightarrow \mathcal{P}(S_\perp))$$

(See the Appendix for details) When we extend the language $\mathcal{GC}$ with the atomic command $x := ?$ we need to give a different powerdomain capable of accommodating unbounded (but countable) non-determinism. We need, in fact, a richer set of points including countable sets of total values.

DEFINITION 2. *Apt and Plotkin [1986] The powerdomain $\mathcal{P}^\omega(S_\perp)$ is the set $\{A \subseteq S_\perp \mid A \neq \emptyset\}$ partially ordered by:*
$A \sqsubseteq B$ *iff* $A = B \vee (\perp \in A \wedge A \setminus \{\perp\} \subseteq B)$.

It is easy to prove that $\mathcal{P}^\omega(S_\perp)$ is a cpo with $\{\perp\}$ as least element. The meaning function: $\mathcal{C} : Com \to (S \to \mathcal{P}^\omega(S_\perp))$ is the obvious extension of the previous one, using a least fixed-point for iteration and with the further clause:

$$\mathcal{C}[\![x := ?]\!]s = \{[x \mapsto n]s \mid n \in I\!N\}$$

As pointed out in Apt and Plotkin [1986] there are essential failures of Scott continuity in a (compositional) denotational semantics for this form of unbounded non-determinism. We now give an example of this phenomenon. Consider the command:

$$\textbf{do}\ \ x = 0 \ \to\ x :=?;\ x := x + 1 \ \mid\ x > 1 \ \to\ x := x - 1 \ \ \textbf{od}$$

Its semantics is given by the least fixed-point of an operator F which is monotone but not Scott-continuous; F is defined by:

$$F(f)(s) = \begin{cases} \{s\} & \text{if } s(x) = 1 \\ f^+(\{[x \mapsto n+1]s \mid n \in I\!N\}) & \text{if } s(x) = 0 \\ f^+(\{[x \mapsto x-1]s\}) & \text{if } s(x) > 1 \end{cases}$$

To check the non continuity of F observe that the n-th approximation of its fixed-point, with $n \geq 2$, is given by:

$$F^{(n)}(s) = \begin{cases} \{[x \mapsto 1]s\} \cup \{\perp\} & \text{if } s(x) = 0 \\ \{[x \mapsto 1]s\} & \text{if } 0 < s(x) \leq n \\ \{\perp\} & \text{otherwise} \end{cases}$$

The ω-th approximation of the fixed-point is therefore:

$$F^{(\omega)}(s) = \bigsqcup_{n<\omega} F^{(n)}(s) = \begin{cases} \{[x \mapsto 1]s\} \cup \{\perp\} & \text{if } s(x) = 0 \\ \{[x \mapsto 1]s\} & \text{if } 0 < s(x) \end{cases}$$

However $F^{(\omega)}$ is not the fixed-point of F; to obtain the fixed-point we need to add an $\omega+1$ step since: $F^{(\omega+1)} = F(F^{(\omega)}) = \lambda s.\{[x \mapsto 1]s\} = F(F^{(\omega+1)})$.

The command considered always terminates, but there is no finite bound on the number of iterations necessary to finish the command when the input is 0. Since $F^{(\omega)}$ is the pointwise sup of the finite approximations also $F^{(\omega)}$ can diverge for input 0.

It is possible to construct examples of iterative commands whose semantics are obtained iterating the application of the associated functionals through any recursive ordinal. For example the semantics of the command:

$$\textbf{do}\ \ x < y \ \to\ y := y - 1 \ \mid\ 1 < x \ \to\ x := x - 1;\ y :=? \ \ \textbf{od}$$

is obtained after $\omega \times \omega$ steps.

These are instances of the many non-Scott-continuity phenomena which arise when dealing with unbounded choice. Other examples are: the non-continuity of $\lambda f.f^+$ or the fact that there is no continuous compositional semantics for this language, see Apt and Plotkin [1986]. Therefore, in order to discuss denotational semantics for unbounded choice we follow the solution Apt and Plotkin and consider spaces of functions which satisfy a weaker continuity condition, ω_1-continuity. All functionals arising in the problematic cases are in fact continuous w.r.t. to this weaker notion of continuity.

3. ω_1-continuous functions and recursive domain equations

In this section we define the notion of ω_1 continuity and the category, CPO_1, whose morphisms are ω_1-continuous functions. We discuss reflexive domain equations in CPO_1 and the difficulties in applying the traditional categorical techniques for solving them. Finally, we outline how to overcome these difficulties in the special case of $D \cong [D \to D]$.

DEFINITION 3. *i) Let A be a partial order, and let κ be a cardinal number. A* κ-chain *in A is a monotone map from κ to A.*
ii) A partial order A is κ-complete *if it has lubs of all κ-chains.*
iii) Let A, B be partial orders and let $f : A \to B$ be a function. Then f is κ-continuous *if it preserves lubs of κ-chains.*

We are led to study the following categories:

DEFINITION 4. *The* fairness *category* CPO_1 *has as objects the ω_0- and ω_1-complete partial orders with a least element, and as morphisms the ω_1-continuous functions. We will use also the subcategory* CPO *having the same objects as* CPO_1 *and as morphisms the functions which are both ω_0- and ω_1-continuous.*

Both CPO_1 and CPO are Cartesian closed categories with the usual Cartesian product and pointwise-ordered function spaces. In CPO_1 least fixed-points of morphisms, such as the F considered above, are obtained by iteration through all the countable ordinals:

PROPOSITION 1. *Let $f : D \to D$ be a morphism in* CPO_1 *For every ordinal $\beta \leq \omega_1$ define $f^{(\beta)}$ in D by $f^{(0)} = \perp_D$, $f^{(\beta+1)} = f(f^{(\beta)})$, $f^{(\lambda)} = \bigsqcup_{\beta<\lambda} f^{(\beta)}$ (λ a limit ordinal). Then $f^{(\omega_1)}$ is the least fixed-point of f.*

If we want to give a denotational semantics for a language featuring higher-order procedures together with unbounded choice, we need to solve reflexive equations on CPO_1 involving $\to_{\omega_1}$ as domain constructor. Also if we consider languages with a parallel constructor, to be interpreted via interleaving and resumptions, we need to solve reflexive domain equations, involving the powerdomain constructor $\mathcal{P}^\omega$ for countable non-determinism extending that considered above on flat cpos. This powerdomain has been introduced in Plotkin [1982] where is called $\mathcal{P}_1$. We are therefore led to investigate the theory of reflexive domain equations in CPO_1. Traditionally, reflexive domain equations are handled in one of two ways.

The first method works in any category $\mathbf{C}$ with colimits of ω_0-chains. It proceeds by analogy with the solution of fixed-point equations $x = f(x)$ in cpos, given by $Fix_f = \bigsqcup_n f^n(\bot)$ and justified by the existence of chain-lubs and by the continuity of f. What one does in the categorical setting is to construct the solution of the equation $X \cong F(X)$ as $Fix_F = colim\ \Delta$ where $\Delta = \langle F^n(X_0), F^n(e)\rangle_{n\in\omega}$ is the ω-chain constructed starting from an object X_0 and a morphism $e : X_0 \to F(X_0)$. This is justified by the existence of colimits and by the continuity of the endofunctor F (meaning that it preserves the colimits of ω_0-chains), see Smyth and Plotkin [1982]. (Strictly speaking, the analogy between categories and partial orders would lead us to take X_0 as the initial object. This is often done, but here the extra generality will prove useful.)

The second method for solving recursive domain equations in category theory has been developed in Adámek and Koubek [1979] and used in Plotkin [1982]. It allows one to solve a larger class of equations than the first method and it is based on the fact that, under a stronger assumption of cocompleteness of the category, a weaker requirement for the functor suffices. In particular, this method allows one to find a fixed-point solution also for functors which are only ω_1-continuous, i.e. which preserve the colimits of ω_1-chains. The least fixed-point is obtained in this case as the colimit of a suitable ω_1-chain.

Both approaches are based on the subcategory of embeddings:

DEFINITION 5. *Let* $\mathrm{CPO}^{\mathrm{E}}$ *(*$\mathrm{CPO}_1^{\mathrm{E}}$*) be the subcategory of* CPO *(*CPO_1*) having the same objects and whose morphisms from D to E are the* embedding *functions from D to E. Embeddings are the morphisms $e : D \to_{(\omega_1)} E$ for which there exists a morphism $p : E \to_{(\omega_1)} D$, called a* projection *such that: $p \circ e = id_D$ and $e \circ p \sqsubseteq id_E$; $\langle e, p\rangle$ is an (ω_1-)*embedding-projection *pair (or (ω_1-)*ep *pair).*

Given an embedding e there exists a unique function p satisfying the above conditions and conversely. Restricting to embeddings allows one to consider locally monotone functors of mixed variance, such as $\to_{\omega_1}$, as covariant functors on the subcategory of embeddings, see Smyth and Plotkin [1982] and Plotkin [1982] for more details.

Not surprisingly if we consider the category $\mathrm{CPO}_1^{\mathrm{E}}$ the first method, above, fails to produce a solution of the simple reflexive equation $D \cong [D \to_{\omega_1} D]$, as the $\to_{\omega_1}$ constructor is *not* continuous. But, much more surprisingly, the second method also fails since the category $\mathrm{CPO}_1^{\mathrm{E}}$ is not ω_0-cocomplete (i.e. it does not have colimit for every ω_0-chain), contrary to what has been claimed in Plotkin [1982]. For a counterexample, first define $\mathbf{0}_\alpha$ to be partial order of all the ordinals strictly smaller than α. Now take

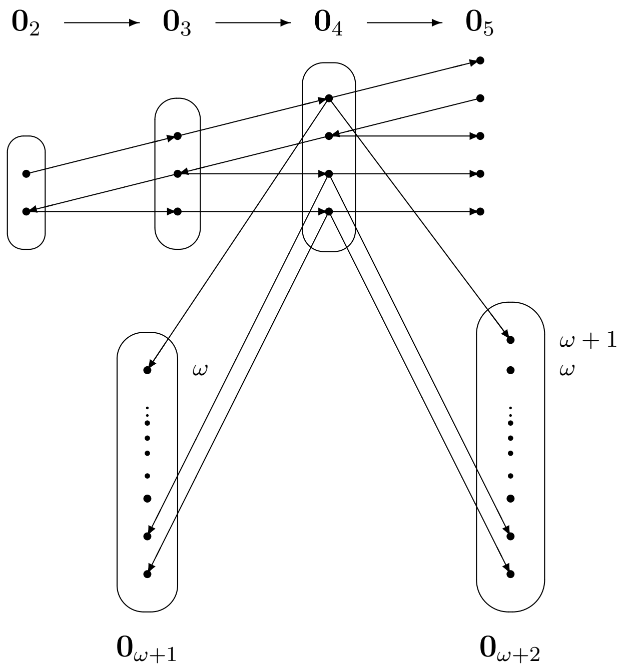


Fig. 1: Counterexample

the ω_0-chain $\Delta = \langle \mathbf{0}_n, e_n \rangle_{1<n<\omega_0}$ in the category $\mathrm{CPO}_1^{\mathrm{E}}$, see Fig.1, where we define the embeddings $e_n : \mathbf{0}_n \to_{\omega_1} \mathbf{0}_{n+1}$ by:

$$e_n(m) = \begin{cases} m & \text{if } m < n-1 \\ n & \text{if } m = n-1 \end{cases}$$

Notice that the chain Δ would be the one constructed according to both methods outlined above in order to solve in $\mathrm{CPO}_1^{\mathrm{E}}$ the recursive domain equation $D \cong D_\perp$, starting from the domains $\mathbf{0}_2$ and $\mathbf{0}_3 \cong (\mathbf{0}_2)_\perp$ and the morphism $e_2 : \mathbf{0}_2 \to_{\omega_1} \mathbf{0}_3$ defined as above.

The chain Δ does not have a colimit. For, consider the two cones having as vertices $\mathbf{0}_{\omega+1}$ and $\mathbf{0}_{\omega+2}$, respectively, and whose embeddings $e^1 : \Delta \to \mathbf{0}_{\omega+1}$ and $e^2 : \Delta \to \mathbf{0}_{\omega+2}$, are defined by:

$$e_n^1(m) = \begin{cases} m & \text{if } m < n-1 \\ \omega & \text{if } m = n-1 \end{cases} \qquad e_n^2(m) = \begin{cases} m & \text{if } m < n-1 \\ \omega+1 & \text{if } m = n-1 \end{cases}$$

Observe that the projection p_n^1 corresponding to the embedding e_n^1 is such that $p_n^1(m) < n-1$ for each finite natural m and $p_n^1(\omega) = n-1$, therefore p_n^1 is not ω_0 continuous and the cone $e^1_n : \langle \mathbf{0}_n, e_n\rangle_{1<n<\omega_0} \to \mathbf{0}_{\omega+1}$ is not contained in CPO$^{\mathrm{E}}$. Now suppose, for the sake of contradiction, that $\mu : \Delta \to D$ is the colimit of the above chain. Then there is a mediating embedding $e : D \to \mathbf{0}_{\omega+1}$, we show that e has to be an isomorphism: let p be the corresponding projection, we show that $e \circ p = id_{\mathbf{0}_{\omega+1}}$. For every natural number n we have: $e \circ p(n) = e \circ p \circ e^1_{n+2} \circ p^1_{n+2}(n) = e \circ p \circ e \circ \mu_{n+2} \circ p^1_{n+2}(n)$ $= e \circ \mu_{n+2} \circ p^1_{n+2}(n) = e^1_{n+2} \circ p^1_{n+2}(n) = n$, and at ω, $e \circ p(\omega) \sqsupseteq \bigsqcup_n e^1_n \circ p^1_n(\omega) =$ $\omega \sqsupseteq e \circ p(\omega)$. So, if the colimit exists it is $e^1 : \Delta \to \mathbf{0}_{\omega+1}$. Therefore there must be a mediating embedding from $\mathbf{0}_{\omega+1}$ to $\mathbf{0}_{\omega+2}$. Unfortunately, such a mediating embedding does not exist. In fact there is only one mediating function e' from $\mathbf{0}_{\omega+1}$ to $\mathbf{0}_{\omega+2}$, e' is defined by $e'(n) = n$ and $e'(\omega) = \omega + 1$. But e' is not an embedding, as it does not preserve the sup $\omega = \bigsqcup_{n<\omega} n$ (all the left adjoints between partial orders preserve any existing suprema).

Nevertheless the classical domain equation: $D \cong [D \to_{\omega_1} D]$, can still be solved non-trivially. The mathematical construction might seem ad hoc, but it amounts to the construction of an inverse limit. It will be fully generalised and put on firm categorical ground in the next section.

In order to provide a non-trivial solution we start from a given cpo D_0 and an ω_1-ep pair $\langle e_{0,1}, p_{1,0}\rangle$, where $e_{0,1} : D_0 \to_{\omega_1} [D_0 \to_{\omega_1} D_0]$ and $p_{1,0} :$ $[D_0 \to_{\omega_1} D_0] \to_{\omega_1} D_0$.

A chain $\langle D_\beta, \langle e_{\alpha,\beta}, p_{\beta,\alpha}\rangle\rangle_{\alpha<\beta\leq\omega_1+1}$ of cpos and ω_1-ep pairs is defined by induction on β in the following way:

- For $\beta = 1$, $D_1 = [D_0 \to_{\omega_1} D_0]$ and $e_{0,1}$, $p_{0,1}$ are as given above.
- For $\beta = \beta' + 2$ define:
  - $D_\beta = [D_{\beta'+1} \to_{\omega_1} D_{\beta'+1}]$
  - $e_{\beta'+1,\beta} = \lambda\, d_{\beta'+1}.\, e_{\beta',\beta'+1} \circ d_{\beta'+1} \circ p_{\beta'+1,\beta'}$
  - $p_{\beta,\beta'+1} = \lambda\, d_\beta.\, p_{\beta'+1,\beta'} \circ d_\beta \circ e_{\beta',\beta'+1}$

    and for $\alpha \leq \beta'$ define:
  - $e_{\alpha,\beta} = e_{\beta'+1,\beta} \circ e_{\alpha,\beta'+1}$
  - $p_{\beta,\alpha} = p_{\beta'+1,\alpha} \circ p_{\beta,\beta'+1}$
- For β be a limit ordinal define:
  - $D_\beta = \{\langle d_0, \ldots, d_\alpha, \ldots\rangle_{\alpha<\beta} |\, d_\alpha \in D_\alpha \,\&\, \forall\, \alpha' < \alpha\; d_{\alpha'} = p_{\alpha,\alpha'}(d_\alpha)\}$
  - $e_{\alpha,\beta} = \lambda d_\alpha. \langle p_{\alpha,0}(d_\alpha), \ldots, d_\alpha, e_{\alpha,\alpha+1}(d_\alpha), \ldots\rangle$
  - $p_{\beta,\alpha} = \lambda\, \vec{d}\, .\, d_\alpha$
- For $\beta = \lambda + 1$ with λ a limit ordinal define:
  - $D_\beta = [D_\lambda \to_{\omega_1} D_\lambda]$
  - $e_{\lambda,\beta} = \lambda\, \vec{d}\, .\, \lambda\, \vec{d'}\, . \bigsqcup_{\alpha<\lambda} e_{\alpha,\lambda}(d_{\alpha+1}(d'_\alpha))$
  - $p_{\beta,\lambda} = \lambda f. \langle f^{(0)}, \ldots, f^{(\alpha)}, \ldots\rangle$

    where we define:
  - $f^{(\alpha+1)} = p_{\lambda,\alpha} \circ f \circ e_{\alpha,\lambda}$
  - $f^{(\gamma)} = p_{\gamma+1,\gamma}(f^{(\gamma+1)})$ for $\gamma = 0$ or γ a limit ordinal smaller than λ

    and for $\alpha < \lambda$ define:

– $e_{\alpha,\lambda+1} = e_{\lambda,\lambda+1} \circ e_{\alpha,\lambda}$
– $p_{\lambda+1,\alpha} = p_{\lambda,\alpha} \circ p_{\lambda+1,\lambda}$

Note. Observe that for λ a limit ordinal $\langle e_{\beta,\lambda}\rangle_\beta : \langle D_\beta, e_{\alpha,\beta}\rangle_{\alpha<\beta<\lambda} \to D_\lambda$ is a cone in the category $\mathrm{CPO}_1^{\mathrm{E}}$ but it is not necessarily the colimit.

THEOREM 1. *The embedding* $e_{\omega_1,\omega_1+1} : D_{\omega_1} \cong [D_{\omega_1} \to_{\omega_1} D_{\omega_1}]$ *is an isomorphism with inverse* p_{ω_1+1,ω_1} *and therefore* D_{ω_1} *is a non-trivial solution of the recursive domain equation* $D \cong [D \to_{\omega_1} D]$.

The previous theorem is a straightforward consequence of Theorem 3 below.

4. Two methods for solving domain equations

In this section we discuss two general methods which can be used to solve a large class of domain equations obtained using non-ω_0-continuous domain constructors which satisfy some ω_1-continuity conditions.

The first method is an appropriate application of the method in Adámek and Koubek [1979], and consists in *changing the category* under consideration and then applying the following theorem:

THEOREM 2. *Let* $\mathbf{C}$ *be a category having colimits of both* ω_0- *and* ω_1-*chains and let* $F : \mathbf{C} \to \mathbf{C}$ *be an* ω_1-*continuous functor. Suppose* $e : D \to F(D)$ *is a morphism. Then a chain* $\langle D_\beta, e_{\alpha,\beta}\rangle_{\alpha<\beta\leq\omega_1+1}$ *can be constructed by induction on* β *as follows:*

- $D_0 = D$, $D_1 = F(D_0)$ *and* $e_{0,1} = e$
- *for* $\beta = \beta' + 2$
  - $D_\beta = F(D_{\beta'+1})$
  - $e_{\beta'+1,\beta} = F(e_{\beta',\beta'+1})$
  - $e_{\alpha,\beta} = e_{\beta'+1,\beta} \circ e_{\alpha,\beta'+1}$ *(for* $\alpha \leq \beta'$*)*
- *for* β *a limit ordinal,* $\langle e_{\alpha,\beta}\rangle_\alpha : \langle D_\alpha, e_{\alpha,\gamma}\rangle_{\alpha<\gamma<\beta} \to D_\beta$ *is a colimiting cone*
- *for* $\beta = \lambda + 1$ *with* λ *a limit ordinal,*
  - $D_\beta = F(D_\lambda)$ *and*
  - $e_{\lambda,\lambda+1}$ *is the mediating morphism between the colimiting cone* $\langle e_{\alpha+1,\lambda}\rangle_\alpha : \langle D_{\alpha+1}, e_{\alpha+1,\gamma+1}\rangle_{\alpha<\gamma<\lambda} \to D_\lambda$ *and the cone* $\langle F(e_{\alpha,\lambda})\rangle_\alpha : \langle D_{\alpha+1}, e_{\alpha+1,\gamma+1}\rangle_{\alpha<\gamma<\lambda} \to F(D_\lambda)$
  - $e_{\alpha,\lambda+1} = e_{\lambda,\lambda+1} \circ e_{\alpha,\lambda}$ *(for* $\alpha < \lambda$*)*

We have that $e_{\omega_1,\omega_1+1} : D_{\omega_1} \to F(D_{\omega_1})$ *is an isomorphism and*
$e_{0,\omega_1} = e^{-1}_{\omega_1,\omega_1+1} \circ F(e_{0,\omega_1}) \circ e$

The idea in this case is to solve recursive domain equations using the category $\mathrm{CPO}^{\mathrm{E}}$ instead of the category $\mathrm{CPO}_1^{\mathrm{E}}$. This is possible because, although we use non-continuous functions, it is not always necessary to consider non-continuous ep-pairs in order to solve a recursive domain equation.

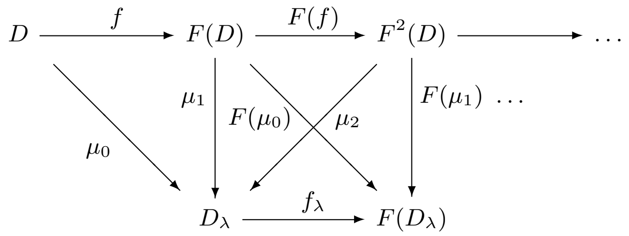


Fig. 2: Diagram for condition A

The category CPO has ω_0- and ω_1-colimits and so, as shown in Plotkin [1982], $\mathrm{CPO}^{\mathrm{E}}$ has ω_0- and ω_1-colimits and locally ω_1-continuous functors in CPO yield ω_1-continuous functors on $\mathrm{CPO}^{\mathrm{E}}$. All the functors we use in the construction of recursive domains, i.e. $+, \times, \rightarrow_{\omega_1}, \mathcal{P}^{\omega}$, are locally ω_1-continuous and restrict to functors on CPO and so we may solve recursive domain equations using them in $\mathrm{CPO}^{\mathrm{E}}$.

Using this first method, therefore, we have to restrict ourselves to embeddings contained in $\mathrm{CPO}^{\mathrm{E}}$. This condition, however, is too restrictive if we want to define a non-initial solution of $D \cong [D \rightarrow_{\omega_1} D]$ obtained starting from a domain D_0 and an embedding $e_0 : D_0 \rightarrow_{\omega_1} [D_0 \rightarrow_{\omega_1} D_0]$ not contained in $\mathrm{CPO}^{\mathrm{E}}$. Such a recursively defined domain will be used in Section 5 in order to give a minimal model for the λ-calculus. Hence, in order to produce a general construction which includes that solution, we present a second method for defining fixed-points of functors, which yields limits obtained also starting from non-continuous embeddings.

This second method can be seen as a generalisation of the solution proposed by Adámek and Koubek. With respect to their solution we ask for a weaker requirement on the category and functors. In particular we can apply this construction also to categories such as $\mathrm{CPO}^{\mathrm{E}}_1$ which do not have colimits of ω_0-chains.

Definition 6. *Given a category* $\mathbf{C}$ *we say that a functor* $F : \mathbf{C} \rightarrow \mathbf{C}$ satisfies condition A *if for every chain* Δ *in* $\mathbf{C}$*, such that* $\Delta = \langle D_\alpha, f_{\alpha,\beta}\rangle_{\alpha<\beta<\lambda}$*, where* λ *is a countable limit ordinal,* $D_{\alpha+1} = F(D_\alpha)$ *and* $f_{\alpha+1,\beta+1} = F(f_{\alpha,\beta})$ *there exists a cone* $\mu : \Delta \rightarrow D_\lambda$ *and a morphism* $f_\lambda : D_\lambda \rightarrow F(D_\lambda)$ *such that for all* $\alpha < \lambda$ $f_\lambda \circ \mu_{\alpha+1} = F(\mu_\alpha)$*, i.e. the diagram in Fig.2 commutes.*

Proposition 2. *In a category* $\mathbf{C}$ *having colimits for every* ω_1*-chain, if a functor* $F : \mathbf{C} \rightarrow \mathbf{C}$ *is* ω_1*-continuous, satisfies condition* A *and there is an object* D_0 *and a morphism* $e_0 : D_0 \rightarrow F(D_0)$*, then* F *has a fixed-point* $i : D \cong F(D)$ *and there exists a morphism* $e : D_0 \rightarrow D$ *such that:*
$e = i^{-1} \circ F(e) \circ e_0$.

Proof. Under the specified conditions it is possible to construct an ω_1-chain $\langle D_\beta, e_{\alpha,\beta}\rangle_{\alpha<\beta<\omega_1}$ in the following way:

- $D_1 = F(D_0)$ and $e_{0,1} = e_0$
- for $\beta = \beta' + 2$
  - $D_\beta = F(D_{\beta'+1})$
  - $e_{\beta'+1,\beta} = F(e_{\beta',\beta'+1})$
  - $e_{\alpha,\beta} = e_{\beta'+1,\beta} \circ e_{\alpha,\beta'+1}$ (for $\alpha \leq \beta$)
- for β a limit ordinal, $\langle e_{\alpha,\beta}\rangle_\alpha : \langle D_\alpha, e_{\alpha,\gamma}\rangle_{\alpha<\gamma<\beta} \to D_\beta$ is the cone whose existence is assured by condition A.
- for $\beta = \lambda + 1$ with λ a limit ordinal,
  - $D_\beta = F(D_\lambda)$ and
  - $e_{\lambda,\lambda+1}$ is the morphism whose existence is assured by condition A.
  - $e_{\alpha,\lambda+1} = e_{\lambda,\lambda+1} \circ e_{\alpha,\lambda}$ (for $\alpha < \lambda$)

If we consider now a colimiting cone $\langle e_{\alpha,\omega_1}\rangle_\alpha : \langle D_\alpha, e_{\alpha,\beta}\rangle_{\alpha<\beta<\omega_1} \to D_{\omega_1}$ and the unique morphism i between the cones $\langle e_{\alpha+1,\omega_1}\rangle_{\alpha<\omega_1}$ and $\langle F(e_{\alpha,\omega_1})\rangle_{\alpha<\omega_1}$ it is easy to prove that $i : D_{\omega_1} \to F(D_{\omega_1})$ is an isomorphism and obviously the equation $e_{0,\omega_1} = i^{-1} \circ F(e_{0,\omega_1}) \circ e_0$ holds. □

We now want to apply Proposition 2 in order to prove that the functors which are unary compositions of basic functors $+, \times, \to_{\omega_1}, \mathcal{P}^\omega$ admit a fixed-point. In order to do so we now consider the category $\mathrm{CPO}_1^{\mathrm{E}}$. As shown in Plotkin [1982] $\mathrm{CPO}_1^{\mathrm{E}}$ has colimits for every ω_1-chain and the domain constructors $+, \times, \to_{\omega_1}, \mathcal{P}^\omega$ and their compositions, are functors preserving ω_1-colimits (since they are locally ω_1-continuous). So it remains to prove that the unary compositions of the basic functors satisfy condition A.

To do that we need to establish some properties concerning ω_0-chains.

Notation i) We denote by $\mathrm{CPO}^{\mathrm{P}}$ ($\mathrm{CPO}_1^{\mathrm{P}}$) the category having cpo's as objects and ω_0- and ω_1-continuous (ω_1-continuous) projections as morphisms.

ii) Given an embedding $f : A \to B$ (a projection $g : A \to B$) we denote by $f^P : B \to A$ ($g^E : B \to A$) the corresponding projection (embedding).

iii) Given a functor $F : \mathrm{CPO}^{\mathrm{E}} \to \mathrm{CPO}^{\mathrm{E}}$ ($F : \mathrm{CPO}_1^{\mathrm{E}} \to \mathrm{CPO}_1^{\mathrm{E}}$) we denote by $F^P : \mathrm{CPO}^{\mathrm{P}} \to \mathrm{CPO}^{\mathrm{P}}$ ($F^P : \mathrm{CPO}_1^{\mathrm{P}} \to \mathrm{CPO}_1^{\mathrm{P}}$) the corresponding functor on projections, that is F^P behaves like F on objects, and on projections it is defined by: $F^P(g) = (F(g^E))^P$.

Proposition 3. *The category* CPO_1 *has all limits of* ω_0- *and* ω_1 *inverse chains.*

Proof. Let $\Delta = \langle D_\alpha, f_{\beta,\alpha}\rangle_{\alpha<\beta<\kappa}$ with $\kappa = \omega_0$ or $\kappa = \omega_1$ be an inverse chain in CPO_1. The limit of Δ is defined as:

$$\lim \Delta = \{\langle d_0, \ldots, d_\alpha, \ldots\rangle_{\alpha<\kappa} \mid d_\alpha \in D_\alpha \ \wedge\ \forall d_\alpha, d_\beta \,.\, d_\alpha = f_{\beta,\alpha}(d_\beta)\}$$

and the order on $\lim \Delta$ is the pointwise order.

The cone functions $\phi_\alpha : \lim \Delta \to D_\alpha$ are the projections: $\phi_\alpha(\langle d_0, \ldots, d_\alpha, \ldots\rangle) = d_\alpha$. We need to prove that $\lim \Delta$ is an ω_0- and ω_1-cpo. Let $\langle d_0, \ldots, d_\gamma \ldots\rangle_{\gamma<\kappa'}$ be a κ'-chain of elements in $\lim \Delta$. If $\kappa' = \omega_1$ the limit of the chain d is the sequence $l = \langle \bigsqcup_{\gamma<\kappa'} d_{\gamma,0}, \ldots, \bigsqcup_{\gamma<\kappa'} d_{\gamma,\alpha}, \ldots \rangle_{\alpha<\kappa}$. If $\kappa' = \omega_0$ the sequence l does not necessarily belong to $\lim \Delta$ since the functions $f_{\beta,\alpha}$ need not be ω_0-continuous.

To find the lub of d we need to define the following sequence d'_λ $(\lambda \leq \omega_1)$ of sequences:

- $d'_0 = l$
- $d'_{\gamma+1} = \langle \bigsqcup_{\beta<\kappa} f_{\beta,0}(d_{\gamma,\beta}), \ldots, \bigsqcup_{\beta<\kappa} f_{\beta,\alpha}(d_{\gamma,\beta}), \ldots \rangle_{\alpha<\kappa}$
- $d'_\lambda = \langle \bigsqcup_{\gamma<\lambda} d'_{\gamma,0}, \ldots, \bigsqcup_{\gamma<\lambda} d'_{\gamma,\alpha}, \ldots \rangle_{\alpha<\kappa}$
  for λ a limit ordinal.

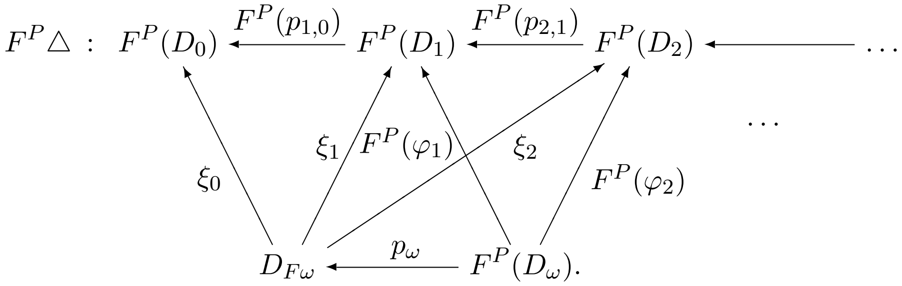
By the ω_1-continuity of the functions $f_{\beta,\alpha}$, the sequence d'_{ω_1} is an element of $\lim \Delta$ and it is easy to prove that it is in fact the lub of d. Observe that in the case $\kappa = \omega_1$ we do not need to define the whole sequence since already d'_1 is an element of $\lim \Delta$.

Given any other cone $\langle D', \xi_\alpha \rangle_{\alpha<\kappa}$ for Δ, the only function $\theta : D' \to \lim \Delta$ making the cones commute is defined by: $\theta(d') = \langle \xi_0(d') \ldots \xi_\alpha(d') \ldots \rangle_{\alpha<\kappa}$. □

Proposition 4. *In* CPO_1 *the countable limit construction preserves projections. That is, suppose* $\Delta = \langle D_n, p_{m,n} \rangle_{n<n<\omega_0}$ *is a chain of morphisms and* $\xi : D_\omega \to \Delta$ *is its limit in* CPO_1. *Then if each* $p_{m,n}$ *is a projection each* ξ_n *is also a projection.*

Proof. The result follows from the construction of the limit in CPO_1 stated in the proof of proposition 3. The embedding functions $\xi_i^E : D_i \to D_\omega$ are defined by: $\xi_i^E(d_i) = \langle p_{i,0}(d_i) \ldots d_i \ldots p_{j,i}^E(d_i) \ldots \rangle_i$. □

Lemma 1. *For every pair of natural numbers* j, k, *for every functor* $F : (\mathrm{CPO}_1^{\mathrm{E}})^{\mathrm{j}} \to (\mathrm{CPO}_1^{\mathrm{E}})^{\mathrm{k}}$ *composition of the basic functors and for every* ω_0*-chain of projections* $\Delta = \langle D_n, p_n \rangle_{n<\omega_0}$, *in* $(\mathrm{CPO}_1)^{\mathrm{j}}$ *let* $F^P\Delta$ *indicate the* ω_0*-chain* $\langle F^P(D_n), F^P(p_n) \rangle_{n<\omega_0}$ *and let* $\varphi : D_\omega \to \Delta$ *and* $\xi : D_{F\omega} \to F^P\Delta$ *be the limits in* $(\mathrm{CPO}_1)^{\mathrm{j}}$ *and* $(\mathrm{CPO}_1)^{\mathrm{k}}$ *of* Δ *and* $F^P\Delta$ *respectively. Clearly* $F^P(\varphi) : F^P(D_\omega) \to F^P\Delta$ *is a cone. Let* p_ω *be the unique function* $p_\omega : F^P(D_\omega) \to D_{F\omega}$ *mediating between the two cones:*



Then the following properties hold:

i) $p_\omega = \bigsqcup_{n<\omega_0} \xi_n^E \circ F^P(\varphi_n)$

ii) $F^P(\varphi_m) \circ \bigsqcup_{n<\omega_0} ((F^P(\varphi_n))^E \circ \xi_n) = \xi_m$

iii) p_ω *is a projection and* $p_\omega^E = \bigsqcup_{n<\omega_0} (F^P(\varphi_n))^E \circ \xi_i$

PROOF. Point *i)* follows from the construction of the limit.

We prove that point *ii)* is satisfied for all the basic functors. We start considering $\to_{\omega_1}: (\mathrm{CPO}_1^{\mathrm{E}})^2 \to \mathrm{CPO}_1^{\mathrm{E}}$. Let $\Delta = \langle D_n \times D'_n, p_{m,n}, p'_{m,n}\rangle_{n<m<\omega_0}$ be an ω_0-chain in $(\mathrm{CPO}_1^{\mathrm{P}})^2$, let $\langle D_\omega \times D'_\omega, \varphi_n, \varphi_n\rangle_{n<\omega_0}$ and $\langle D_{(\to_{\omega_1})^P}, \xi_n\rangle_{n<\omega_0}$ be the limits of Δ and $(\to_{\omega_1})^P\Delta$ in $(\mathrm{CPO}_1)^2$ and CPO_1 respectively and let f be an element in $D_{(\to_{\omega_1})^P}$ and d_m an element of D_m. Note that f is a sequence of functions $\langle f_0, f_1, \ldots, f_i, \ldots\rangle$ with $f_n : D_n \to_{\omega_1} D'_n$ and such that $f_n = p'_{m,n} \circ f_m \circ p^E_{m,n}$.

The following equalities hold:

$$
\begin{aligned}
&((\to_{\omega_1})^P(\varphi_m, \varphi'_m) \circ \textstyle\bigsqcup_{n<\omega_0}(((\to_{\omega_1})^P(\varphi_n, \varphi'_n))^E \circ \xi_n))(f)(d_m)\\
&= (\to_{\omega_1})^P(\varphi_m, \varphi'_m)(\textstyle\bigsqcup_{n<\omega_0}((\to_{\omega_1})^P(\varphi_n, \varphi'_n))^E(\xi_n(f)))(d_m)\\
&= (\to_{\omega_1})^P(\varphi_m, \varphi'_m)(\textstyle\bigsqcup_{n<\omega_0} \varphi'^E_n \circ \xi_n(f) \circ \varphi_n)(d_m)\\
&= (\varphi'_m \circ (\textstyle\bigsqcup_{n<\omega_0} \varphi'^E_n \circ f_n \circ \varphi_n) \circ \varphi^E_m)(d_m)\\
&= \varphi'_m(\textstyle\bigsqcup_{n>m}(\varphi'^E_n \circ f_n \circ \varphi_n \circ \varphi^E_m)(d_m))\\
&= \varphi'_m(\textstyle\bigsqcup_{n>m}(\varphi'^E_n \circ f_n \circ p^E_{n,m})(d_m)) \qquad \text{by commutativity of the diagrams}\\
&= \varphi'_m(\textstyle\bigsqcup_{n>m}\langle(p'_{n,0} \circ f_n \circ p^E_{n,m})(d_m), \ldots, (p'_{n,m} \circ f_n \circ p^E_{n,m})(d_m), \ldots,\\
&\qquad (p'_{n,l} \circ f_n \circ p^E_{n,m})(d_m), \ldots, (f_n \circ p^E_{n,m})(d_m), \ldots\rangle)\\
&= \varphi'_m(\textstyle\bigsqcup_{n>m}\langle(p'_{m,0} \circ f_m)(d_m), \ldots, f_m(d_m), \ldots, (f_l \circ p^E_{l,m})(d_m), \ldots\rangle)\\
&= \varphi'_m(\langle(p'_{m,0} \circ f_m)(d_m), \ldots, f_m(d_m), \ldots, (f_l \circ p^E_{l,m})(d_m), \ldots\rangle) = f_m(d_m)
\end{aligned}
$$

But also $\xi_m(f) = f_m$ and therefore we have:

$$(\to_{\omega_1})^P(\langle\varphi_{m,1}, \varphi_{m,2}\rangle) \circ \textstyle\bigsqcup_{n<\omega_0} (((\to_{\omega_1})^P(\langle\varphi_{n,1}, \varphi_{n,2}\rangle))^E \circ \xi_n) = \xi_m$$

For what concerns the functor $\mathcal{P}^\omega$ the proof follows easily from the fact that for any function f in CPO_1, the function $\mathcal{P}^\omega(f)$ is ω_0-continuous (see Plotkin [1982]). The functors $+$ and $\times$ preserve the limit of the ω_0 chain and so the proof is immediate. It is also easy to prove that the property in point *ii)* is preserved by composition of functors and therefore the thesis holds.

To prove point *iii)* we prove that the pair $\langle p^E_\omega, p_\omega\rangle$ is an ep-pair. Let d be an element in $F^P(D_\omega)$, we have:

$$
\begin{aligned}
&(p^E_\omega \circ p_\omega)(d)\\
&= p^E_\omega(\textstyle\bigsqcup_{n<\omega_0} \xi^E_n(F^P(\varphi_n))(d))\\
&= p^E_\omega(\textstyle\bigsqcup_{n<\omega_0}\langle F^P(\varphi_0)(d), \ldots F^P(\varphi_n)(d), \ldots ((F^P(p_{m,n}))^E \circ F^P(\varphi_n))(d), \ldots\rangle)
\end{aligned}
$$

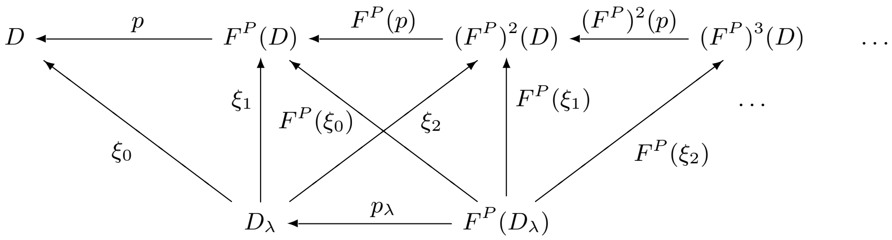


Fig. 3: Diagram of projections

$= \bigsqcup_{n<\omega_0}(F^P(\varphi_n))^E \circ \xi_n)(\langle F^P(\varphi_0))(d), \ldots, F^P(\varphi_m))(d), \ldots\rangle_{m<\omega_0})$
$= \bigsqcup_{n<\omega_0}(F^P(\varphi_n))^E(F^P(\varphi_n)(d)) \sqsubseteq d$
and:
$p_\omega \circ p_\omega^E = \bigsqcup_{m<\omega_0}(\xi_m^E \circ F^P(\varphi_m)) \circ \bigsqcup_{n<\omega_0}((F^P(\varphi_n))^E \circ \xi_n)$
$= \bigsqcup_{m<\omega_0}(\xi_m^E \circ F^P(\varphi_m) \circ (\bigsqcup_{n<\omega_0}(F^P(\varphi_n))^E \circ \xi_n))$ (by *ii*)
$= \bigsqcup_{m<\omega_0} \xi_m^E \circ \xi_m = id_{D_{F_\omega}}$ □

Proposition 4 and Lemma 1 immediately yield the corresponding results for all countable ordinals.

Using the lemma it is now easy to prove the following proposition.

PROPOSITION 5. *Every unary composition* $F : \mathrm{CPO}_1^{\mathrm{E}} \to \mathrm{CPO}_1^{\mathrm{E}}$ *of the basic functors satisfies the following property: given a chain of projections* $\Delta = \langle D_\alpha, p_{\beta,\alpha}\rangle_{\alpha<\beta<\lambda}$ *such that* λ *is a countable limit ordinal,* $D_{\alpha+1} = F(D_\alpha)$ *and* $p_{\beta+1,\alpha+1} = F(p_{\beta,\alpha})$ *let* $\xi : D_\lambda \to \Delta$ *be a limiting cone in* CPO_1 *and let* $p_\lambda : F^P(D_\lambda) \to D_\lambda$ *be the unique morphism making the diagram in Fig.3 commute. Then we have:*

i) $p_\lambda = \bigsqcup_{\alpha<\lambda} \xi_{\alpha+1}^E \circ F^P(\xi_\alpha)$

ii) p_λ *is a projection and* $p_\lambda^E = \bigsqcup_{\alpha\in\lambda} (F^P(\xi_\alpha))^E \circ \xi_{\alpha+1}$

By using the previous propositions one has immediately that:

COROLLARY 1. *In the category* $\mathrm{CPO}_1^{\mathrm{E}}$ *the basic functors and their compositions satisfy condition A.*

We can finally conclude:

THEOREM 3. *For every unary composition* $F : \mathrm{CPO}_1^{\mathrm{E}} \to \mathrm{CPO}_1^{\mathrm{E}}$ *of the basic functors, for every cpo* D_0 *and* ω_1*-continuous embedding* $e_0 : D_0 \to F(D_0)$ *there exists a fixed-point* $i : D \cong F(D)$ *for* F *and an* ω_1*-continuous embedding* $e : D_0 \to D$ *such that:* $e = i^{-1} \circ F(e) \circ e_0$.

5. Completeness results for the λ-calculus

In this section we discus the completeness of the $\lambda\beta\eta$-calculus relative to reflexive domain models in CPO and CPO_1. A reflexive domain model D for the $\lambda\beta\eta$-calculus is a domain isomorphic to the domain of its own endomorphisms. By *completeness* we mean that the class of λ-equalities which hold in all such models is exactly the theory $\boldsymbol{\lambda\beta\eta}$; see Ronchi and Honsell [1992] for a more detailed account of this issue. The "completeness" problem for reflexive domain models based on Scott-continuous functions is a longstanding open problem. In Theorem 4 below, we show that among the reflexive domain models in CPO there is one which has a minimal theory, and it is an open problem whether this minimal theory is exactly the theory $\boldsymbol{\lambda\beta\eta}$.

For reflexive domain models in CPO_1 we are able to prove completeness. We utilise the construction of Section 3 in order to define a cpo D_{ω_1} satisfying the equation $D_{\omega_1} \cong [D_{\omega_1} \to_{\omega_1} D_{\omega_1}]$, which provides a model for the theory $\boldsymbol{\lambda\beta\eta}$. That is, an equation $M = N$ is true in the model if and only if M and N are $\beta\eta$-convertable.

We start defining a minimal reflexive domain model in CPO. The construction relies on logical relations.

Let Δ be the set of $\lambda\beta\eta$-theories (since $\lambda\beta\eta$-theories are expressed using a countable set of variables and no constant there are at most 2^{ω_0} such a theories) for which there exists a reflexive domain model in CPO. Using the axiom of choice we can select a reflexive domain model C_δ in CPO for each theory $\delta \in \Delta$. Let i_δ indicate the isomorphism $i_\delta : C_\delta \simeq [C_\delta \to C_\delta]$. We shall define a reflexive domain model whose theory is $\bigcap \Delta$. A cpo D_0 is defined as $D_0 = \Pi_{\delta\in\Delta} C_\delta$; for each C_δ there is an obvious ω_0- ω_1-continuous ep-pair $\langle e_\delta, p_\delta\rangle : C_\delta \to D_0$. An ω_0- ω_1-continuous ep-pair $\langle e_0, p_0\rangle : D_0 \to [D_0 \to D_0]$ is defined by:

$e_0(\langle c_\delta\rangle_{\delta\in\Delta}) = \lambda\langle c'_\delta\rangle_{\delta\in\Delta} \,.\, \langle i_\delta(c_\delta)(c'_\delta)\rangle_{\delta\in\Delta}$

$p_0(f) = \langle i_\delta^{-1}(p_\delta \circ f \circ e_\delta)\rangle_{\delta\in\Delta}$.

Using the standard method to solve domain equations in the category CPO one can build a chain $\langle D_n, \langle e_{n,m} p_{m,n}\rangle\rangle_{n<m\leq\omega_0+1}$ such that: $e_{\omega_0,\omega_0+1} : D_{\omega_0} \simeq [D_{\omega_0} \to D_{\omega_0}]$. We will use also the symbol i to indicate the isomorphism e_{ω_0,ω_0+1}. In the following we will prove that D_{ω_0} is a minimal reflexive domain model, in CPO, for the $\lambda\beta\eta$-calculus.

Definition 7. *Define the relations $R_n \subset D_0 \times D_n$, $n \leq \omega_0$ by the induction:*

$R_0(d_0, d'_0) \;\equiv\; d_0 = d'_0$

$R_{n+1}(d_0, d_{n+1}) \;\equiv\; \forall d'_0, d'_n \,.\, R_n(d'_0, d'_n) \Rightarrow R_n(e_0(d_0)(d'_0), d_{n+1}(d'_n))$

$R_{\omega_0}(d_0, d_{\omega_0}) \;\equiv\; \forall\, n < \omega_0 \,.R_n(d_0,\, p_{\omega_0,n}(d_{\omega_0}))$

Proposition 6. *The relations R_n satisfy the following closure properties*

i) For each $n \leq \omega_0$ R_n is closed under lub's of ω_0-chains.

ii) For each $n < m \leq \omega_0$ if $R_n(d_0, d_n)$ then $R_m(d_0, e_{n,m}(d_n))$

iii) For each $n < m \leq \omega_0$ if $R_m(\langle d_0, d_m)$ then $R_n(d_0,\, p_{m,n}(d_m))$

PROOF. All the properties are proved by induction. The proof of property i) is straightforward.

For properties ii) and iii) it is enough to prove by induction on $n < \omega_0$ that:
a) if $R_n(d_0, d_n)$ then $R_{n+1}(d_0, e_{n,n+1}(d_n))$ and
b) if $R_{n+1}(d_0, d_{n+1})$ then $R_n(d_0, p_{n+1,n}(d_{n+1}))$.

Basic steps: point a) follows immediately from the definition.
b) if $R_1(d_0, d_1)$ then, by Definition 7, for all $d'_0 \in D_0$, $e_0(d_0)(d'_0) = d_1(d'_0)$, by extensionality $d_1 = e_0(d_0)$ and so $p_0(d_1) = d_0$ and $R_0(d_0, p_0(d_1))$.

Inductive steps:
a) if $R_n(d_0, d_n)$ then by inductive hypothesis and by Definition 7,
$R_{n+1}(d_0, e_{n-1,n} \circ d_n \circ p_{n,n-1})$ and by definition of $e_{n,n+1}$, $R_{n+1}(d_0, e_{n,n+1}(d_n))$,
b) if $R_{n+1}(d_0, d_{n+1})$ then by inductive hypothesis and by Definition 7,
$R_n(d_0, p_{n,n-1} \circ d_{n+1} \circ e_{n-1,n})$, by definition of $p_{n+1,n}$, $R_n(d_0, p_{n+1,n}(d_{n+1}))$
□

PROPOSITION 7. *For each $d_0 \in D_0$ and $d_{\omega_0} \in D_{\omega_0}$, $R_{\omega_0}(d_0, d_{\omega_0})$ if and only if for every $d'_0 \in D_0$ and $d'_{\omega_0} \in D_{\omega_0}$ we have that:*
$R_{\omega_0}(d'_0, d'_{\omega_0}) \Rightarrow R_{\omega_0}(e_0(d_0)(d'_0), i(d_{\omega_0})(d'_{\omega_0}))$.

PROOF. ($\Rightarrow$) Suppose $R_{\omega_0}(d_0, d_{\omega_0})$ and $R_{\omega_0}(d'_0, d'_{\omega_0})$ then by Proposition 6 and by Definition 7 we have:
$R_{\omega_0}(e_0(d_0)(d'_0), \bigsqcup_{n<\omega_0} e_{n,\omega_0}(p_{\omega_0,n+i}(d_{\omega_0})(p_{\omega_0,n}(d'_{\omega_0}))))$ therefore,
$R_{\omega_0}(e_0(d_0)(d'_0),\ i(d_{\omega_0})(d'_{\omega_0}))$.

($\Leftarrow$) To prove $R_{\omega_0}(d_0, d_{\omega_0})$ it suffices to prove that for all $n < \omega_0$,
$R_{n+1}(d_0, p_{\omega_0,n+1}(d_{\omega_0}))$ that is for all n, d'_0, d'_n,
$R_n(d'_0, d'_n) \Rightarrow R_n(e_0(d_0)(d'_0), p_{\omega_0,n+1}(d_{\omega_0})(d'_n))$.
Now, if $R_n(d'_0, d'_n)$ then $R_{\omega_0}(d'_0, e_{n,\omega_0}(d'_n))$, by hypothesis
$R_{\omega_0}(e_0(d_0)(d'_0), i(d_{\omega_0})(e_{n,\omega_0}(d'_n)))$
and since $p_{\omega_0,n}(i(d_{\omega_0})(e_{n,\omega_0}(d'_n))) = p_{\omega_0,n+1}(d_{\omega_0})(d'_n)$ we have
$R_n(e_0(d_0)(d'_0), p_{\omega_0,n+1}(d_{\omega_0})(d'_n))$. □

DEFINITION 8. *Given a λ-term M let $[\![M]\!]^\delta$ (respectively $[\![M]\!]^{\omega_0}$) indicate the interpretation of M in the model C_δ (respectively D_{ω_0}). The interpretation of M in the cpo D_0 relative to an environment $\rho : Var \to D_0$ is defined by $[\![M]\!]^0_\rho = \langle [\![M]\!]^\delta_{p_\delta \circ \rho} \rangle_\delta$.*

PROPOSITION 8. *For any λ-term M, given two environments $\rho : Var \to D_0$ and $\rho' : Var \to D_{\omega_0}$ if for all variables $x \in Var$ $R_{\omega_0}(\rho(x), \rho'(x))$ then $R_{\omega_0}([\![M]\!]^0_\rho, [\![M]\!]^{\omega_0}_{\rho'})$.*

PROOF. By induction on the structure of the term M. The proof follows immediately from the previous proposition and from the observation that for every environment $\rho : Var \to D_0$ we have $[\![MN]\!]^0_\rho = e_0([\![M]\!]^0_\rho)([\![N]\!]^0_\rho)$ and $e_0([\![\lambda x.M]\!]^0_\rho) = \lambda d_0 \in D_0.[\![M]\!]^0_{\rho[d_0/x]}$. □

We can finally establish the theorem.

THEOREM 4. *The cpo D_ω is a minimal reflexive domain model in* CPO*, in the sense that if two λ-terms are equated in D_ω then they are equated in all reflexive domain models in* CPO*.*

PROOF. Since two λ-terms are equated in a model if the respective closure are equated, we limit ourselves to consider closed λ terms. Let M and N be two closed λ-terms which are equated in D_{ω_0}, we shall prove that M and N are equated in any other reflexive domain model C in CPO. Let δ be the theory of the model C. To prove the theorem it is sufficient to prove that M and N are equated in C_δ. By the previous proposition $R_{\omega_0}([\![N]\!]^0, [\![N]\!]^{\omega_0})$, and so by Definition 7 $R_0([\![N]\!]^0, p_{\omega_0,0}([\![N]\!]^{\omega_0}))$, and so $[\![N]\!]^0 = p_{\omega_0,0}([\![N]\!]^{\omega_0})$. The same equality holds for the term M. Therefore:
$[\![N]\!]^\delta = p_\delta([\![N]\!]^0) = p_\delta(p_{\omega_0,0}([\![N]\!]^{\omega_0})) = p_\delta(p_{\omega_0,0}([\![M]\!]^{\omega_0})) = p_\delta([\![M]\!]^0) = [\![M]\!]^\delta$ □

We now turn to the construction of a reflexive domain model in CPO_1 for the theory $\boldsymbol{\lambda\beta\eta}$.

DEFINITION 9. *i) Let T be the term model for the pure $\lambda\beta\eta$-calculus, i.e. the elements of T are the equivalence classes $[M]$ of pure λ-terms under $\beta\eta$-conversion.*
ii) Let D_0 be the flat cpo obtained by adding $\perp$ to the set T with the obvious order relation.
iii) Let $D_1 = D_0 \to_{\omega_1} D_0$ $(= D_0 \to_\omega D_0)$,
iv) Let $\langle e_0, p_0\rangle$ be the ω_1-ep pair from D_0 to D_1 given by:

$$e_0([M])([N]) = [MN], \quad e_0([M])(\perp) = \perp \quad \textit{and } e_0(\perp)(d_0) = \perp$$

The function $p_0 : D_1 \to_{\omega_1} D_0$ is uniquely determined by e_0:

$$p_0(f) = \bigsqcup\{d_0 \mid \forall d_0'.\ e_0(d_0)(d_0') \sqsubseteq f(d_0')\}$$

in other words: $p_0(f) = [M]$ if $\forall[N].f([N]) = [MN]$ and $p_0(f) = \perp$ if there is no such term $[M]$.

The function p_0 is not continuous but only ω_1-continuous. This is the only place where ω_1-continuity plays an essential role.

Using the machinery presented in Section 3 it is now possible to build a chain $\langle D_\beta, \langle e_{\alpha,\beta}, p_{\beta,\alpha}\rangle\rangle_{\alpha<\beta\leq\omega_1+1}$ such that $e_{\omega_1,\omega_1+1} : D_{\omega_1} \cong [D_{\omega_1} \to_\omega D_{\omega_1}]$.

Given an environment $\rho : \mathit{Var} \to D_{\omega_1}$ we write $[\![M]\!]_\rho$ for the denotation of M in D_{ω_1} (relative to ρ).

Notation. For every function $\sigma : \text{Var} \to \Lambda$ and λ-term M with free variables $\{x_1, \ldots, x_n\}$, we write M_σ for the term $M[\sigma(x_1)/x_1, \ldots, \sigma(x_n)/x_n]$, moreover we write $[\sigma]$ for the environment such that $[\sigma](x) = e_{0,\omega_1}([\sigma(x)])$.

PROPOSITION 9. *For every λ-term M and for every function $\sigma : \mathit{Var} \to \Lambda$ the following equality holds: $p_{\omega_1,0}([\![M]\!]_{[\sigma]}) = [M_\sigma]$*

PROOF. By induction on the structure of M. In the proof we use equalities that can be easily derived from the explicit construction of D_{ω_1} given in Section 3.
i) the case where M is a variable is immediate,
ii) for an application, $M \equiv NP$ we calculate:

$$\begin{array}{rcl} p_{\omega_1,0}(\llbracket NP\rrbracket_{[\sigma]}) & = & p_{\omega_1,0}(e_{\omega_1,\omega_1+1}(\llbracket N\rrbracket_{[\sigma]})(\llbracket P\rrbracket_{[\sigma]})) \\ & = & p_{\omega_1,0}(\bigsqcup_{\alpha<\omega_1} e_{\alpha,\omega_1}(p_{\omega_1,\alpha+1}(\llbracket N\rrbracket_{[\sigma]})(p_{\omega_1,\alpha}(\llbracket P\rrbracket_{[\sigma]})))) \\ & \sqsubseteq & p_{\omega_1,0}(e_{0,\omega_1}(p_{\omega_1,1}(\llbracket N\rrbracket_{[\sigma]})(p_{\omega_1,0}(\llbracket P\rrbracket_{[\sigma]})))) \\ & = & p_{\omega_1,1}(\llbracket N\rrbracket_{[\sigma]})(p_{\omega_1,0}(\llbracket P\rrbracket_{[\sigma]})) \\ & \sqsubseteq & e_0(p_{\omega_1,0}([N_\sigma]))(p_{\omega_1,0}([P_\sigma])) \\ & = & e_0([N_\sigma])([P_\sigma]) \quad \text{(by induction hypothesis)} \\ & = & [(NP)_\sigma] \end{array}$$

and since D_0 is flat, equality holds.
iii) For an abstraction, $M \equiv \lambda x.N$ we have:

$$\begin{array}{rcl} p_{\omega_1,0}(\llbracket \lambda x.N\rrbracket_{[\sigma]}) & = & p_{\omega_1,0}(p_{\omega_1+1,\omega_1}(\lambda d : D_{\omega_1}.\llbracket N\rrbracket_{[\sigma][d/x]})) \\ & = & p_{\omega_1,0}(\langle p_0(p_{\omega_1,0} \circ (\lambda d : D_{\omega_1}.\llbracket N\rrbracket_{[\sigma][d/x]}) \circ e_{\omega_1,0}) \ldots\rangle) \\ & = & p_0(f) \end{array}$$

where $f = \lambda d_0 : D_0.p_{\omega_1,0}(\llbracket N\rrbracket_{[\sigma][e_{0,\omega_1}(d_0)/x]})$. But now, for any term P we have:

$$\begin{array}{rcl} f([P]) & = & p_{\omega_1,0}(\llbracket N\rrbracket_{[\sigma][e_{0,\omega_1}([P])/x]}) \\ & = & p_{\omega_1,0}(\llbracket N\rrbracket_{[\sigma[P/x]]}) \\ & = & [N_{\sigma[P/x]}] \quad \text{(by induction hypothesis)} \\ & = & [(\lambda x.N)_\sigma P] \end{array}$$

And from this $[(\lambda x.N)_\sigma] = p_0(f)$. □

We now have:

THEOREM 5. *The $\lambda\beta\eta$-theory induced by D_{ω_1} is the minimal λ-calculus theory, i.e. the theory* $\boldsymbol{\lambda\beta\eta}$.

PROOF. We need to prove that for every pair of terms M, N:

$$\vdash_{\beta\eta} M = N \quad \text{iff} \quad \forall\, \rho.\llbracket M\rrbracket_\rho = \llbracket N\rrbracket_\rho$$

Since D_{ω_1} is a model, the implication from left to right is straightforward. Suppose, instead, that $\forall\, \rho.\llbracket M\rrbracket_\rho = \llbracket N\rrbracket_\rho$. Then taking $\rho(x) = x$ for all x in Var, and using Proposition 9 we calculate:

$$[M] = [M_\rho] = p_{\omega_1,0}\,(\llbracket M\rrbracket_{[\rho]}) = p_{\omega_1,0}\,(\llbracket N\rrbracket_{[\rho]}) = [N_\rho] = [N]$$

and so $\vdash_{\beta\eta} M = N$. □

The previous theorem *cannot* be generalised to arbitrary theories. There is no reflexive domain model in CPO_1 for the theory considered in Ronchi and Honsell [1992]; this can be shown by a similar argument to that given there for the Scott-continuous case.

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Appendix: The language $\mathcal{GC}$

The language $\mathcal{GC}$ is a language featuring assignment, command composition, and two non-deterministic commands: non-deterministic selection and non-deterministic multitest loop, which iterates as long as at least one of its guards is true. The set *Com* of *commands* of the language $\mathcal{GC}$, ranged over by C, is generated by the following abstract syntax grammar:

$$C ::= x := E \mid C_1; C_2 \mid \mathbf{if}\, G\, \mathbf{fi} \mid \mathbf{do}\, G\, \mathbf{od}$$

$$G ::= G_1 \mathrel{[]} G_2 \mid B \to C$$

Here G ranges over the set *GCom* of *guarded commands* and x ranges over identifiers. The set *Exp* of *expressions*, ranged over by E, and the set *BExp* of *boolean expressions*, ranged over by B are assumed given The grammar is extended with extra clause $C ::= x :=?$, when dealing with unbounded non-determinism.

The denotational semantics of $\mathcal{GC}$ is defined as follows, assuming functions $\mathcal{E} : Exp \to (S \to Val)$ and $\mathcal{B} : BExp \to (S \to \{tt, ff\})$[1]:

$\mathcal{C}[\![x := E]\!]s = \{[\mathcal{E}[\![E]\!]s \mapsto x]s\}$

$\mathcal{C}[\![C_1; C_2]\!] = (\mathcal{C}[\![C_2]\!])^+ \circ \mathcal{C}[\![C_1]\!]$

[1] For simplicity, the semantics of non-deterministic selection command does not cause failure when each guard fails.

$$\mathcal{C}[\![\textbf{if}\, B_1 \to C_1 \mid \ldots \mid B_n \to C_n\, \textbf{fi}]\!]s$$

$$= \begin{cases} \{s\} & \text{if } \mathcal{B}[\![B_i]\!]s = \mathit{ff}, \text{ for } i = 1, \ldots, n \\ \bigcup\{\mathcal{C}[\![C_i]\!]s \mid \mathcal{B}[\![B_i]\!]s = tt\} & \text{otherwise} \end{cases}$$

$$\mathcal{C}[\![\textbf{do}\, B_1 \to C_1 \mid \ldots \mid B_n \to C_n\, \textbf{od}]\!] = f$$

where $f : S \to \mathcal{P}(S_\perp)$ is the least function satisfying the recursive specification:

$$f(s) = \begin{cases} \{s\} & \text{if } \mathcal{B}[\![B_i]\!]s = \mathit{ff}, \text{ for } i = 1, \ldots, n \\ f^+(\bigcup\{\mathcal{C}[\![C_i]\!]s \mid \mathcal{B}[\![B_i]\!]s = tt\}) & \text{otherwise} \end{cases}$$