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THE λ-CALCULUS IS ω-INCOMPLETE

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§1. Introduction. The ω-rule in the λ-calculus (or, more exactly, the λK-β, η calculus) is

$$\frac{MZ = NZ \quad \text{(all closed terms } Z)}{M = N}.$$

In [1] it was shown that this rule is consistent with the other rules of the λ-calculus. We will show the rule cannot be derived from the other rules; that is, we will give closed terms M and N such that $MZ = NZ$ can be proved without using the ω-rule, for each closed term Z, but $M = N$ cannot be so proved. This strengthens a result in [4] and answers a question of Barendregt.

§2. Definitions. The language of the λ-calculus has an *alphabet* containing denumerably many *variables* $a, b, c, \ldots$ (which have a standard listing $e_1, e_2, \ldots$), *improper symbols* λ, (,) and a single *predicate* symbol $=$ for equality.

Terms are defined inductively by the following:

(1) A variable is a term.

(2) If M and N are terms, so is (MN); it is called a *combination.*

(3) If M is a term and x is a variable, $(\lambda\, x\, M)$ is a term; it is called an *abstraction.*

We use $\equiv$ for syntactic identity of terms.

If M and N are terms, $M = N$ is a *formula.*

$BV(M)$, the set of bound variables in M, and $FV(M)$, its free variables, are defined inductively by

$$BV(x) = \varnothing; \quad BV((MN)) = BV(M) \cup BV(N);$$
$$BV((\lambda x M)) = BV(M) \cup \{x\};$$
$$FV(x) = \{x\}; \quad FV((MN)) = FV(M) \cup FV(N);$$
$$FV((\lambda x M)) = FV(M)\backslash\{x\}.$$

A term M is *closed* iff $FV(M) = \varnothing$.

$[M/x]N$, the result of substituting M for x throughout N, is defined inductively by

$$[M/x]x \equiv M, \qquad [M/x]y \equiv y \quad (x \not\equiv y),$$
$$[M/x](NN') \equiv ([M/x]N[M/x]N'), \qquad [M/x](\lambda x N) \equiv (\lambda x N),$$
$$[M/x](\lambda y N) \equiv (\lambda z[M/x][z/y]N) \quad (x \not\equiv y)$$

where z is the variable defined by

(1) if $x \notin FV(N)$ or $y \notin FV(M)$, $z \equiv y$,

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(2) otherwise z is the first variable in the list $e_1, e_2, \ldots$ such that $z \notin FV(N) \cup FV(M)$.

That this is a good definition is shown in [2] where other properties of the substitution prefix can be found.

Rules.

(I)

$$1.\ (\lambda x M) = (\lambda y [y/x] M) \qquad (y \notin FV(M));$$
$$2.\ ((\lambda x M) N) = [N/x] M;$$
$$3.\ (\lambda x M x) = M \qquad (x \notin FV(M)).$$

(II)

$$1.\ M = M.$$
$$2.\ \frac{M = N}{N = M}.$$
$$3.\ \frac{M = N\ N = L}{M = L}.$$

(III)

$$\frac{M = M'}{(NM) = (NM')},\quad \frac{M = M'}{(MN) = M'N)},\quad \frac{M = M'}{(\lambda x M) = (\lambda x M')}.$$

We will use $M = N$ to mean that $M = N$ can be proved by the above rules. In addition, $M =_\alpha N$ (α-equivalence) is to mean that $M = N$ can be proved using (I)1, (II) and (III); $M \geq_\beta N$ (β-reduction) is to mean that $M = N$ can be proved using (I)1,2, (II)1,3 and (III); $M \geq_{\beta\eta} N$ ($\beta\eta$-reduction) is to mean that $M = N$ can be proved using (I), (II)1,3 and (III); $M \geq_\eta N$ (η-reduction) is to mean that $M = N$ can be proved using (I)3, (II)1,3 and (III).

Clearly, if $M \geq_\eta N$ or $M =_\alpha N$ then $FV(M) = FV(N)$.

It is shown in [2] that if $M = N$ then there is a Z, such that $M \geq_{\beta\eta} Z$ and $N \geq_{\beta\eta} Z$ (Church-Rosser theorem). Further if $M \geq_{\beta\eta} N$ then, for some Z, $M \geq_\beta Z \geq_\eta N$.

By $M \to N$ we mean that there are terms $M_1, \ldots, M_m$ and a variable x ($m \geq 2$) such that $M \equiv (\lambda x M_1) M_2 \cdots M_m$ and $N \equiv ([M_2/x] M_1) M_3 \cdots M_m$.

The transitive closure, $\to^+$, of $\to$ is called *head reduction.*

Standard reduction sequences (s.r. sequences) are defined inductively by the following:

(1) x is a s.r. sequence for any variable x.

(2) If $M_1, \ldots, M_m$ and $N_1, \ldots, N_n$ are s.r. sequences, so is $(M_1 N_1), \ldots, (M_m N_1), \ldots, (M_m N_n)$.

(3) If $M_1, \ldots, M_m$ is a s.r. sequence, so is $\lambda x M_1, \ldots, \lambda x M_m$ for any variable x.

(4) If $M_1, \ldots, M_m$ and $N_1, \ldots, N_n$ are s.r. sequences, M_m is the first abstraction in $M_1, \ldots, M_m$ and $(M_m N) \to N_1$ then $(M_1 N), \ldots, (M_m N), N_1, \ldots, N_n$ is a s.r. sequence.

This is a reformulation of the definition given in [2] where it is shown that if $M \geq_\beta N$ then for some $N' =_\alpha N$ there is a s.r. sequence from M to N' (standardisation theorem). If M_m is the first abstraction in a s.r. sequence $M_1, \ldots, M_m$ then $M_1 \to M_2 \to \cdots \to M_m$ and M_m is uniquely determined by M_1. A term M is of *order* 0 iff there is no abstraction N such that $M \geq_\beta N$ or, equivalently, if there is no abstraction N such that $M \rightarrow^+ N$, or $M \equiv N$.

If n is an integer, by $\boldsymbol{n}$ is meant the term

$$\lambda f \lambda x \underbrace{f(\cdots f}_{n \text{ times}}(x)\cdots) \quad \text{(where } n, f \text{ are distinct variables).}$$

For any term M let Y_M be $(\lambda x M(xx))(\lambda x M(xx))$ where $x \notin FV(M)$; then $Y_M = M(Y_M)$ and, indeed, $Y_M \rightarrow M(Y_M)$.

Let $\text{Succ} \equiv \lambda n \lambda f \lambda x(nf(fx))$ (with n, f and x distinct). Then $\text{Succ}\,\boldsymbol{n} = \boldsymbol{n+1}$. From [3] we see that there is a closed term Gd^{-1} such that, for any closed term Z, $Gd^{-1}\boldsymbol{n} = Z$ for some n.

Finally, we define the terms M and N which provide a counterexample to ω-completeness via intermediate definitions of terms H_1, H, G_1, G and F:

$$\begin{aligned}
H_1 &\equiv \lambda h \lambda g \lambda n \lambda x \lambda y((hg)n((hg)(\text{Succ}\, n)(g(\text{Succ}\, n))yx)(Gd^{-1}n)),\\
H &\equiv (Y_{H_1}),\\
G_1 &\equiv \lambda g \lambda n((Hg)(\text{Succ}\, n)(g(\text{Succ}\, n))(Gd^{-1}(\text{Succ}\, n))(gn)),\\
G &\equiv (Y_{G_1}),\\
F &\equiv (HG),\\
M &\equiv (F\boldsymbol{0}(G\boldsymbol{0})),\\
N &\equiv \lambda x(M(\lambda xx)) \quad \text{(with } h, g, n, x, y \text{ distinct variables).}
\end{aligned}$$

§3. LEMMA 1. *For all terms U, V, W,*

(1) $FUVW \rightarrow^+ FU(F(\text{Succ}\, U)(G(\text{Succ}\, U))WV)(Gd^{-1}U)$,

(2) $GU = F(\text{Succ}\, U)(G(\text{Succ}\, U))(Gd^{-1}(\text{Succ}\, U))(GU)$.

PROOF.

$$\begin{aligned}
FUVW &\equiv Y_{H_1}GUVW \rightarrow^+ H_1HGUVW\\
&\rightarrow^+ (HG)U((HG)(\text{Succ}\, U)(G(\text{Succ}\, U))WV)(Gd^{-1}U)\\
&\equiv FU(F(\text{Succ}\, U)(G(\text{Succ}\, U))WV)(Gd^{-1}U).
\end{aligned} \tag{1}$$

$$\begin{aligned}
GU &\equiv Y_{G_1}U = G_1GU\\
&= (HG)(\text{Succ}\, U)(G(\text{Succ}\, U))(Gd^{-1}(\text{Succ}\, U))(GU)\\
&\equiv F(\text{Succ}\, U)(G(\text{Succ}\, U))(Gd^{-1}(\text{Succ}\, U))(GU).
\end{aligned} \tag{2}$$

It follows immediately that $FUVW$ has order 0 for all terms U, V and W.

The terms F and G were actually found as solutions to the double recursion equations given in Lemma 1. We could not simplify this to two single recursions.

LEMMA 2. *For all $m, n \geq 0$, $F\boldsymbol{n}(G\boldsymbol{n})(Gd^{-1}\boldsymbol{n}) = F\boldsymbol{n}(G\boldsymbol{n})(Gd^{-1}(\boldsymbol{m} + \boldsymbol{n}))$.*

PROOF. By induction on m. For $m = 0$, the result is obvious. Otherwise,

$$\begin{aligned}
&F\boldsymbol{n}(G\boldsymbol{n})(Gd^{-1}\boldsymbol{n})\\
&\quad = F\boldsymbol{n}(F(\text{Succ}\, \boldsymbol{n})(G(\text{Succ}\, \boldsymbol{n}))(Gd^{-1}(\text{Succ}\, \boldsymbol{n}))(G\boldsymbol{n}))(Gd^{-1}\boldsymbol{n}) \quad \text{(by Lemma 1.2)}\\
&\quad = F\boldsymbol{n}(F\boldsymbol{n+1}(G\boldsymbol{n+1})(Gd^{-1}\boldsymbol{n+1})(G\boldsymbol{n}))(Gd^{-1}\boldsymbol{n})\\
&\quad = F\boldsymbol{n}(F\boldsymbol{n+1}(G\boldsymbol{n+1})(Gd^{-1}\boldsymbol{m+n})(G\boldsymbol{n}))(Gd^{-1}\boldsymbol{n})\\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{(by the induction hypothesis)}\\
&\quad = F\boldsymbol{n}(G\boldsymbol{n})(Gd^{-1}\boldsymbol{m+n}) \quad \text{(by Lemma 1.1).}
\end{aligned}$$

LEMMA 3. *For all closed terms Z, Z', $MZ = MZ'$.*

PROOF. Choose n, n' such that $Z = Gd^{-1}\mathbf{n}$ and $Z' = Gd^{-1}\mathbf{n}'$. Then

$$\begin{aligned} MZ &= F\mathbf{0}(G\mathbf{0})(Gd^{-1}\mathbf{n}) \\ &= F\mathbf{0}(G\mathbf{0})(Gd^{-1}\mathbf{0}) \quad \text{(by Lemma 2)} \\ &= F\mathbf{0}(G\mathbf{0})(Gd^{-1}\mathbf{n}') \quad \text{(by Lemma 2)} \\ &= MZ'. \end{aligned}$$

LEMMA 4. *If $FUVW, \ldots, Z$ is a s.r. sequence of length l where $y \in FV(V)$ but $y \notin FV(Z)$ then there is a s.r. sequence of length $\leq l$ from V to a term V' such that $y \notin FV(V')$.*

PROOF. Suppose otherwise. Let $FUVW, \ldots, Z$ be a s.r. sequence of minimal length l among those s.r. sequences from a term of the form $FUVW$, where $y \in FV(V)$, to a term Z, where $y \notin FV(Z)$; and, for all V', if $V, \ldots, V'$ is a s.r. sequence of length $\leq l$ then $y \in FV(V')$.

Case (a). The s.r. sequence is of the type given in clause 4 of the definition of a s.r. sequence. In this case it must have the form $FUVW, \ldots, (\lambda w N_1)W, N_2, \ldots, Z$ where $FUVW \rightarrow^+ N_2$ and $N_2, \ldots, Z$ is a s.r. sequence of length $l' < l$. This determines N_2 and we find that $N_2 \equiv FU(F(\text{Succ } U)(G(\text{Succ } U))WV)(Gd^{-1}U)$ from the proof of Lemma 1. By the minimality of l, there is a s.r. sequence of length $\leq l'$ from $F(\text{Succ } U)(G(\text{Succ } U))WV$ to a term Z' where $y \notin FV(Z')$. But $F(\text{Succ } U)(G(\text{Succ } U))W$ is of order 0. Therefore this s.r. sequence must be of the type given in clause 2 of the definition of a s.r. sequence. So there is a s.r. sequence of length $\leq l'$ from V to a term V' such that $y \notin FV(V')$, a contradiction.

Case (b). The s.r. sequence is of the type given in clause 2 of the definition of a s.r. sequence. Then there is a s.r. sequence of length $l' \leq l$ from FUV to a term Z' such that $y \notin FV(Z')$, which must have the form (as $y \in FV(V)$) $FUV, \ldots, (\lambda v N_1)V, N_2, \ldots, Z'$ where $FUV \rightarrow^+ N_2$ and $N_2, \ldots, Z'$ is a s.r. sequence of length $l'' < l'$. This determines N_2 and $N_2 \equiv \lambda w FU(F(\text{Succ } U)(G(\text{Succ } U)wV)(Gd^{-1}U)$ for some $w \notin FV(U) \cup FV(V)$. This must be of the type given in clause 3 of the definition of a s.r. sequence and there is a s.r. sequence of length l'' from $FU(F(\text{Succ } U)(G(\text{Succ } U))wV)(Gd^{-1}U)$ to a term Z'' such that $y \notin FV(Z'')$. This leads to a contradiction as in Case (a).

LEMMA 5. *If $FUVW, \ldots, Z$ is a s.r. sequence such that $y \in FV(W)$ but $y \notin FV(Z)$ then, for some W', $W \geq_\beta W'$ and $y \notin FV(W')$.*

PROOF. Suppose otherwise and let $FUVW, \ldots, Z$ be a s.r. sequence having minimal length l among those s.r. sequences from a term of the form $FUVW$ to a term Z where $y \in FV(W)$, $y \notin FV(Z)$ and, for all W', if $W \geq_\beta W'$ then $y \in FV(W')$.

This s.r. sequence must have the form $FUVW, \ldots, (\lambda w N_1)W, N_2, \ldots, Z$ where $FUVW \rightarrow^+ N_2$ and $N_2, \ldots, Z$ is a s.r. sequence of length $l' < l$. We find that $N_2 \equiv FU(F(\text{Succ } U)(G(\text{Succ } U))WV)(Gd^{-1}U)$. By Lemma 4 there is a s.r. sequence of length $\leq l'$ from $F(\text{Succ } U)(G(\text{Succ } U))WV$ to a term Z' such that $y \notin FV(Z')$. Now $F(\text{Succ } U)(G(\text{Succ } U))W$ is of order 0. Therefore this last s.r. sequence must be of the type described in clause 2 of the definition of a s.r. sequence and so there is a s.r. sequence of length $\leq l'$ from $F(\text{Succ } U)(G(\text{Succ } U))W$ to a term Z'' such that $y \notin FV(Z'')$. Hence, by the minimality of l, $W \geq_\beta W'$ for some term W' such that $y \notin FV(W')$, a contradiction.

LEMMA 6. *If $x \not\equiv y$ then $Mx \neq My$.*

Suppose that $x \not\equiv y$ and $Mx = My$. Then, by the Church-Rosser theorem there is a Z'' such that $Mx \geq_{\beta\eta} Z''$ and $My \geq_{\beta\eta} Z''$. Next, for some Z', $My \geq_{\beta} Z' \geq_{\eta} Z''$ and finally, for some $Z =_{\alpha} Z'$, there is a s.r. sequence from My to Z. But $y \notin FV(Mx) \supseteq FV(Z'') = FV(Z') = FV(Z)$. As $My \equiv F\mathbf{0}(G\mathbf{0})y$, it follows from Lemma 5 that, for some term W', $y \geq_{\beta} W'$ and $y \notin FV(W')$, which contradicts the hypothesis.

THEOREM. *The ω-rule is not derivable.*

PROOF. If Z is any closed term,

$$\begin{aligned} MZ &= M(\lambda xx) \quad \text{(Lemma 2)} \\ &= NZ. \end{aligned}$$

However, if $M = N$ then $Mx = Nx = M(\lambda xx) = Ny = My$, for any variables x and y, contradicting Lemma 6.

This result is not peculiar to the λK-$\beta\eta$ calculus. It can be obtained for any λK-$\beta\eta\delta$ calculus if there is a term Con^{-1} such that for every constant a there is an n such that $\mathrm{Con}^{-1}\, \boldsymbol{n} = a$; the result can also be obtained for the λI-$\beta\eta$ calculus in an analogous way.

A term M is a *universal generator* iff every closed term is a subterm of some term to which M $\beta\eta$-reduces. It is shown in [1] that if $MZ = NZ$ for all closed Z and neither M nor N are universal generators then $M = N$. Is it the case that if $M = N$ can be proved using the ω-rule and M is not a universal generator then $M = N$ can be proved without the ω-rule? Notice that in the counterexample given above both M and N are universal generators.

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