To infinity – and beyond!

Julian Bradfield

LFCS, ICCS

Infinity and eternity

for ever and ever (idiom)

immer und ewig (idiom)

for ever and a day (Shakespeare)

nunc et semper et in saecula saeculorum (from Gk, probably from Aramaic idiom)

The King said, "The third question is, how many seconds of time are there in eternity." Then said the shepherd boy, "In Lower Pomerania is the Diamond Mountain, which is a league high, a league wide, and a league in depth; every hundred years a little bird comes and sharpens its beak on it, and when the whole mountain is worn away by this, then the first second of eternity will be over." (from Grimm)

Towards infinity

Mathematicians and other children often play the following game: We take turns naming numbers, and see who can name the largest one. This is a game in the psychological rather than the formal sense, since I might always just add one to your number, but my goal is to try to completely demolish your ego by transcending your number via some completely new principle.

Kenneth Kunen

Georg Cantor

'The father of set theory'

1845–1918

Martin-Luther-Universität Halle-Wittenberg

1874: the birth of set theory, and the discovery of different levels of infinity

1883: the theory of ordinal numbers

Ordinals and Cardinals

Language:

- cardinal numerals
  - *one*, *two*, . . .
  - “how many?”
- ordinal numerals
  - *first*, *second*, . . .
  - “where in a sequence?”

Ordinals and Cardinals

Language:

- cardinal numerals
  - *one*, *two*, . . .
  - "how many?"
- ordinal numerals
  - *first*, *second*, . . .
  - "where in a sequence?"

Mathematics:

- cardinal numbers
  - 0, 1, 2, . . .
  - "how many [in a set]?"
- ordinal numbers
  - 0, 1, 2, . . .
  - "where in a sequence?", also "how long [is a sequence]?"

The sequence a, b, a is 3 letters long, but contains 2 distinct letters.

Counting

0:
1: |
2: ||
10: ||||||||||

Counting

0:

1: |

2: ||

10: ||||||||||

Obviously we can keep counting 'for ever':

|||||||||||||||||||| •••

Counting

0:

1: |

2: ||

10: ||||||||||

Obviously we can keep counting ‘for ever’:

|||||||||||||||||||| •••

and why not then count some more?

|||||||||||||||||||| ••• |

Counting

0:

1: |

2: ||

10: ||||||||||

Obviously we can keep counting 'for ever':

ω: |||||||||||||||||||| •••

and why not then count some more?

$\omega + 1$: |||||||||||||||||||| ••• |

Write ω for the length of the infinite sequence.

Counting

0:
1: |
2: ||
10: ||||||||||

Obviously we can keep counting ‘for ever’:

ω: |||||||||||||||||||| •••

and why not then count some more?

$\omega + 1$: |||||||||||||||||||| ••• |

Write ω for the length of the infinite sequence.

To help visualization, compress the infinite sequence to

Addition of ordinals

Adding sequences is just putting one after the other:

$\omega + 1$: 

$\omega + 3$: 

$\omega + \omega$: 

Addition of ordinals

Adding sequences is just putting one after the other:

$\omega + 1$: 

$\omega + 3$: 

$\omega + \omega$: 

But $1 + \omega$:  $= \omega$

Multiplication of ordinals

Integer multiplication is just repeated addition: $2 \times 3 = 2 + 2 + 2$.

By convention, let's write $x \cdot y$ to mean y copies of x added together.

$2 \cdot 3$: || || ||

$\omega \cdot 3$: 

$2 \cdot \omega$: || || ••• =  $= \omega$

Multiplication of ordinals

Integer multiplication is just repeated addition: $2 \times 3 = 2 + 2 + 2$.

By convention, let's write $x \cdot y$ to mean y copies of x added together.

$2 \cdot 3$: || || ||

$\omega \cdot 3$: 

$2 \cdot \omega$: || || ••• $=$  $= \omega$

$\omega \cdot \omega$: 

which we might visualize as

Well-foundedness and induction

Key property of these ordinals: they are *well-founded*. Start with an infinite ordinal, and keep decreasing it: after a **finite** (but arbitrarily large) number of steps, you must hit zero.

Well-foundedness and induction

Key property of these ordinals: they are *well-founded*. Start with an infinite ordinal, and keep decreasing it: after a **finite** (but arbitrarily large) number of steps, you must hit zero.

This means that ordinals can generalize *proof by induction*:
If

- $P(\alpha)$ holds for $\alpha = 0$, and
- **if** $P(\beta)$ holds for all $\beta < \alpha$, **then** $P(\alpha)$ holds,

then we can conclude that $P(\alpha)$ holds for all ordinals α.
(Often restricted to ordinals less than some fixed α_0.)

Example: why does Ackermann terminate?

The *Ackermann* function (of two integer arguments) $A(x, y)$ is defined recursively thus:

$$
\begin{aligned}
A(0, y) &= y + 1 \\
A(x, 0) &= A(x-1, 1) && \text{for } x > 0 \\
A(x, y) &= A(x-1, A(x, y-1)) && \text{for } x, y > 0
\end{aligned}
$$

Is it obvious that this recursive computation ever finishes on, e.g., $A(4, 4)$?

Example: why does Ackermann terminate?

The *Ackermann* function (of two integer arguments) $A(x, y)$ is defined recursively thus:

$$\begin{aligned} A(0, y) &= y + 1 \\ A(x, 0) &= A(x - 1, 1) && \text{for } x > 0 \\ A(x, y) &= A(x - 1, A(x, y - 1)) && \text{for } x, y > 0 \end{aligned}$$

Is it obvious that this recursive computation ever finishes on, e.g., $A(4, 4)$?

In each recursive call, *either* x gets smaller, *or* x stays the same and y gets smaller.

This is an induction on $\omega \cdot \omega$.

The Ackermann function grows quite fast – see later . . .

Ordinal Exponentiation

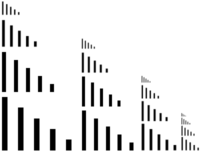
Integer exponentiation is just repeated multiplication:
$2^3 = 2 \times 2 \times 2$.

I.e., we write x^y for y copies of x multiplied together.

ω^2: $\omega \cdot \omega$

2^ω: $2 \cdot 2 \cdot 2 \cdot \ldots = \omega$

ω^3: $\omega \cdot \omega \cdot \omega = \omega^2 \cdot \omega$



Ordinal Exponentiation

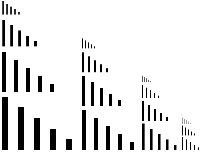
Integer exponentiation is just repeated multiplication:
$2^3 = 2 \times 2 \times 2$.

I.e., we write x^y for y copies of x multiplied together.

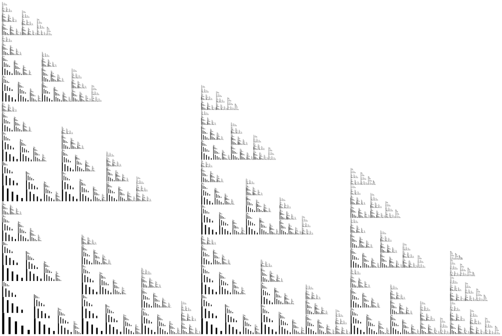
ω^2: $\omega \cdot \omega$

2^ω: $2 \cdot 2 \cdot 2 \cdot \ldots = \omega$

ω^3: $\omega \cdot \omega \cdot \omega = \omega^2 \cdot \omega$



ω^ω: $\omega \cdot \omega \cdot \omega \cdot \ldots$



A little puzzle ...

A number is written in *hereditary base* b if it's a sum of powers of b, with all the exponents also written in hereditary base b. E.g. with $b = 2$

$$1030 = 2^{10} + 2^2 + 2 = 2^{2^{2+1}+2} + 2^2 + 2$$

or with $b = 3$

$$1030 = 3^{3\times 2} + 3^{3+2} + 3^3 \times 2 + 3 + 1$$

A little puzzle ...

A number is written in *hereditary base* b if it's a sum of powers of b, with all the exponents also written in hereditary base b. E.g. with $b = 2$

$$1030 = 2^{10} + 2^2 + 2 = 2^{2^{2+1}+2} + 2^2 + 2$$

or with $b = 3$

$$1030 = 3^{3\times 2} + 3^{3+2} + 3^3 \times 2 + 3 + 1$$

Think of a number n. Write it in h.b. 2. Now replace 2 by 3 and evaluate. Subtract 1. Write the result in h.b. 3. Replace 3 by 4 and evaluate. Subtract 1. And so on ... until you hit zero.

Let $G(n)$ be the length of this process – if it finishes!

For example: $G(3)$

$$
\begin{aligned}
3 &=_2 2+1 \\
&\to \quad 3+1 =_3 4 \\
4-1 = 3 &=_3 3 \\
&\to \quad 4 \qquad =_4 4 \\
4-1 = 3 &=_4 3 \\
&\to \quad 3 \qquad =_5 3 \\
3-1 = 2 &=_5 2 \\
&\to \quad 2 \qquad =_6 2 \\
2-1 = 1 &=_6 1 \\
&\to \quad 1 \qquad =_7 1 \\
1-1 = 0 &
\end{aligned}
$$

So $G(3) = 6$

$G(4) =$

6895080803092620165736389959611509956957749875802973658965164942362743495979724871888253075446277672715412687741
341962942747540246239451654234208474169773799114638335526912932007323504513073113341532147327644310055744993250515006661770697335697266822986380629230539311939473732984321896450580873694731773419752295124184084011737329946623658351712664276240439034396836403624670678602112542697422457548590135058038973996050222167215602290558339854433645828496215789123866817088207178861702990104860943042983193831330062353799303221914424434721561312309414317693867571586750377935644459245645595556087522305546773436198470323324254077850839610789585961963878972971045818445757715767775120696734632762541346561350694765538438030750865233130216216163628621847422406626611936799943353915562435591389508007250783177878077707466959752689545447264715003524185987439101105888291148209914347554185098618554

We now skip 135278 slides . . .

30187441025940056292813034386280177679544034517464936619
31073504178981468557969472279965424657699404235005937914
72930883993663027161466688743717253581936410274739801296
83060714286243899305063868650864112105238061406944808189
08304913462509086531545100380965533413343423478836091833
53220182680722735478679352859535040769913825815484931187
10726329608316620305483302616305150324000876272357296528
10182401729583610978448023254665651115973448118179302336
79234929512268465106495927833854484067484182464486747555
62975216019453924341023727286959093404563639409013246678
20328593203290715635768149137536972887886088038810819894
08291784060416318863529224353808259669206267357619658951
446422310193135419323844928197722374143

Or, more comprehensibly, about 7×10^{121210694}; or about $2^{2^{29}}$.

$G(5) =$

$$10^{10^{\cdot^{\cdot^{\cdot^{10}}}}}$$

$G(5) =$

$$\left.10^{10^{\cdot^{\cdot^{\cdot^{10}}}}}\right\} 10^{10^{\cdot^{\cdot^{\cdot^{10}}}}}$$

$G(5) =$

$$\left. 10^{10^{\cdot^{\cdot^{\cdot^{10}}}}} \right\} \left. 10^{10^{\cdot^{\cdot^{\cdot^{10}}}}} \right\} 10^{10^{\cdot^{\cdot^{\cdot^{10}}}}}$$

$G(5) =$

$$\left.10^{10^{\cdot^{\cdot^{\cdot^{10}}}}}\right\}\left.10^{10^{\cdot^{\cdot^{\cdot^{10}}}}}\right\}\left.10^{10^{\cdot^{\cdot^{\cdot^{10}}}}}\right\}10^{10^{10^{21}}}$$

$G(5) =$

$$\left.10^{10^{\cdot^{\cdot^{\cdot^{10}}}}}\right\} \left.10^{10^{\cdot^{\cdot^{\cdot^{10}}}}}\right\} \left.10^{10^{\cdot^{\cdot^{\cdot^{10}}}}}\right\} 10^{10^{10^{21}}}$$

or to put it in binary,

$$\left.2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}\right\} \left.2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}\right\} \left.2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}\right\} 2^{2^{2^{2^{6}}}}$$

Why does G always terminate?

A slight variation of the description:

Think of a number n. Write it in h.b. 2, and replace 2 by ω; let $b = 2$. Increment b, and subtract 1, expanding ω to b (only) when necessary; repeat until zero.

$$
\begin{array}{rll}
4 & = \omega^\omega & b = 2 \\
\to 26 & = \omega^\omega - 1 & b = 3 \\
 & = \omega^3 - 1 & b = 3 \\
 & = \omega^2 \cdot 2 + \omega \cdot 2 + 2 & b = 3 \\
\to 41 & = \omega^2 \cdot 2 + \omega \cdot 2 + 1 & b = 4 \\
\to 60 & = \omega^2 \cdot 2 + \omega \cdot 2 & b = 5 \\
\to 83 & = \omega^2 \cdot 2 + \omega + 5 & b = 6
\end{array}
$$

The ordinal always decreases, even while its evaluation with $\omega = b$ is increasing. This is ordinal induction.

Fast-growing functions . . .

$G(n)$ grows fast. So also does Ackermann $A(x, y)$:

$x \backslash y$	0	1	2	3	4
0	1	2	3	4	5
1	2	3	4	5	6
2	3	5	7	9	11
3	5	13	29	61	125

Fast-growing functions . . .

$G(n)$ grows fast. So also does Ackermann $A(x, y)$:

$x \backslash y$	0	1	2	3	4
0	1	2	3	4	5
1	2	3	4	5	6
2	3	5	7	9	11
3	5	13	29	61	125
4	13	65533	$\sim 2^{65533}$	$\sim 2^{2^{65533}}$	$\sim 2^{2^{2^{65533}}}$

Fast-growing functions ...

$G(n)$ grows fast. So also does Ackermann $A(x, y)$:

$x \backslash y$	0	1	2	3	4
0	1	2	3	4	5
1	2	3	4	5	6
2	3	5	7	9	11
3	5	13	29	61	125
4	13	65533	$\sim 2^{65533}$	$\sim 2^{2^{65533}}$	$\sim 2^{2^{2^{65533}}}$

Let $A(n)$ mean $A(n, n)$. It looks as if $A(n) > G(n)$:

n	$A(n)$	$G(n)$
0	1	1
1	3	2
2	7	5
3	61	6
4	$2^{2^{2^{65533}}}$	$2^{2^{29}}$

. . . via iterating exponentiation . . .

Multiplication $2 \cdot n$ is iterated addition $2 + 2 + 2 + \cdots + 2$.

Exponentiation 2^n or $2\hat{}n$ is iterated multiplication $2 \times 2 \times 2 \times \cdots \times 2$.

Tetration ${}^n2$ or $2\hat{}\hat{}n$ is iterated exponentiation $2\hat{}2\hat{}2\hat{}\ldots\hat{}2$.

and so on . . .

. . . via iterating exponentiation . . .

Multiplication $2 \cdot n$ is iterated addition $2 + 2 + 2 + \cdots + 2$.

Exponentiation 2^n or 2ˆn is iterated multiplication $2 \times 2 \times 2 \times \cdots \times 2$.

Tetration ${}^n 2$ or 2ˆˆn is iterated exponentiation 2ˆ2ˆ2ˆ . . . ˆ2.

and so on . . .

Call a number *small* if it's . . . small . . .

. . . and *1-big* if it's 2ˆ(small) . . .

. . . and *2-big* if it's 2ˆˆ(small) . . . and so on.

. . . via iterating exponentiation . . .

Multiplication $2 \cdot n$ is iterated addition $2 + 2 + 2 + \cdots + 2$.

Exponentiation 2^n or 2ˆn is iterated multiplication $2 \times 2 \times 2 \times \cdots \times 2$.

Tetration ${}^n2$ or 2ˆˆn is iterated exponentiation 2ˆ2ˆ2ˆ . . . ˆ2.

and so on . . .

Call a number *small* if it's . . . small . . .

. . . and *1-big* if it's 2ˆ(small) . . .

. . . and *2-big* if it's 2ˆˆ(small) . . . and so on.

Let's continue the Ackermann – Goodstein comparison:

n	$A(n)$	$G(n)$
4	2-big	2-big
5	3-big	3-big

...via iterating exponentiation ...

Multiplication $2 \cdot n$ is iterated addition $2 + 2 + 2 + \cdots + 2$.

Exponentiation 2^n or 2^n is iterated multiplication $2 \times 2 \times 2 \times \cdots \times 2$.

Tetration ${}^n2$ or 2^^n is iterated exponentiation 2^2^2^...^2.

and so on ...

Call a number *small* if it's ...small ...

... and *1-big* if it's 2^(small) ...

... and *2-big* if it's 2^^(small) ... and so on.

Let's continue the Ackermann – Goodstein comparison:

n	$A(n)$	$G(n)$
4	2-big	2-big
5	3-big	3-big
6	4-big	5-big
7	5-big	7-big

. . . via iterating exponentiation . . .

Multiplication $2 \cdot n$ is iterated addition $2 + 2 + 2 + \cdots + 2$.

Exponentiation 2^n or 2ˆn is iterated multiplication $2 \times 2 \times 2 \times \cdots \times 2$.

Tetration ${}^n 2$ or 2ˆˆn is iterated exponentiation 2ˆ2ˆ2ˆ . . . ˆ2.

and so on . . .

Call a number *small* if it's . . . small . . .

. . . and *1-big* if it's 2ˆ(small) . . .

. . . and *2-big* if it's 2ˆˆ(small) . . . and so on.

Let's continue the Ackermann – Goodstein comparison:

n	$A(n)$	$G(n)$
4	2-big	2-big
5	3-big	3-big
6	4-big	5-big
7	5-big	7-big
8	6-big	$G(4)$-big

. . . and so *ad infinitum*

Call a number *1-huge* if it's (1-big)-big, *2-huge* if it's (1-huge)-big, etc.

Call a number *1-humungous* if it's (1-huge)-huge, etc.

etc. etc. etc. etc. till your brain explodes.

. . . and so *ultra infinitum*

Play the iterated exponentiation game with ordinals, and don't stop at infinity!

$$\omega, \omega^{\omega}, \omega^{\omega^{\omega}}, \ldots$$

After this we need a new symbol $\epsilon_0 = \omega^{\omega^{\omega^{\cdot^{\cdot^{\cdot}}}}}$ – note that $\omega^{\epsilon_0} = \epsilon_0$, just as $\omega^{\omega} = \omega \cdot \omega^{\omega}$ and $\omega \cdot \omega = \omega + \omega \cdot \omega$.

ϵ_0 is the first *fixed point* of the function $\alpha \mapsto \omega^{\alpha}$.

. . . and so *ultra infinitum*

Play the iterated exponentiation game with ordinals, and don't stop at infinity!

$$\omega, \omega^{\omega}, \omega^{\omega^{\omega}}, \dots$$

After this we need a new symbol $\epsilon_0 = \omega^{\omega^{\omega^{\cdot^{\cdot^{\cdot}}}}}$ – note that $\omega^{\epsilon_0} = \epsilon_0$, just as $\omega^{\omega} = \omega \cdot \omega^{\omega}$ and $\omega \cdot \omega = \omega + \omega \cdot \omega$.

ϵ_0 is the first *fixed point* of the function $\alpha \mapsto \omega^{\alpha}$.

Then ϵ_1 is the second fixed point (and is the end of $\epsilon_0 + 1, \omega^{\epsilon_0+1}, \omega^{\omega^{\epsilon_0+1}}, \dots$).

. . . and so *ultra infinitum*

Play the iterated exponentiation game with ordinals, and don't stop at infinity!

$$\omega, \omega^{\omega}, \omega^{\omega^{\omega}}, \ldots$$

After this we need a new symbol $\epsilon_0 = \omega^{\omega^{\omega^{\cdot^{\cdot^{\cdot}}}}}$ – note that $\omega^{\epsilon_0} = \epsilon_0$, just as $\omega^{\omega} = \omega \cdot \omega^{\omega}$ and $\omega \cdot \omega = \omega + \omega \cdot \omega$.

ϵ_0 is the first *fixed point* of the function $\alpha \mapsto \omega^{\alpha}$.

Then ϵ_1 is the second fixed point (and is the end of $\epsilon_0 + 1, \omega^{\epsilon_0+1}, \omega^{\omega^{\epsilon_0+1}}, \ldots$).

Then there's $\epsilon_{\epsilon_{\epsilon_{\ddots}}}$, the first fixed point of $\alpha \mapsto \epsilon_{\alpha}$.

... and so *ultra infinitum*

Play the iterated exponentiation game with ordinals, and don't stop at infinity!

$$\omega, \omega^\omega, \omega^{\omega^\omega}, \ldots$$

After this we need a new symbol $\epsilon_0 = \omega^{\omega^{\omega^{\cdot^{\cdot^{\cdot}}}}}$ – note that $\omega^{\epsilon_0} = \epsilon_0$, just as $\omega^\omega = \omega \cdot \omega^\omega$ and $\omega \cdot \omega = \omega + \omega \cdot \omega$.

ϵ_0 is the first *fixed point* of the function $\alpha \mapsto \omega^\alpha$.

Then ϵ_1 is the second fixed point (and is the end of $\epsilon_0 + 1, \omega^{\epsilon_0+1}, \omega^{\omega^{\epsilon_0+1}}, \ldots$).

Then there's $\epsilon_{\epsilon_{\epsilon_{\ddots}}}$, the first fixed point of $\alpha \mapsto \epsilon_\alpha$.

Then start counting fixed points of the functions that list the fixed points: the *Veblen* hierarchy. The end of this process is called Γ_0.

... and so *ultra infinitum*

Play the iterated exponentiation game with ordinals, and don't stop at infinity!

$$\omega, \omega^{\omega}, \omega^{\omega^{\omega}}, \ldots$$

After this we need a new symbol $\epsilon_0 = \omega^{\omega^{\omega^{\cdot^{\cdot^{\cdot}}}}}$ – note that $\omega^{\epsilon_0} = \epsilon_0$, just as $\omega^{\omega} = \omega \cdot \omega^{\omega}$ and $\omega \cdot \omega = \omega + \omega \cdot \omega$.

ϵ_0 is the first *fixed point* of the function $\alpha \mapsto \omega^{\alpha}$.

Then ϵ_1 is the second fixed point (and is the end of $\epsilon_0 + 1, \omega^{\epsilon_0+1}, \omega^{\omega^{\epsilon_0+1}}, \ldots$).

Then there's $\epsilon_{\epsilon_{\epsilon_{\cdot_{\cdot_{\cdot}}}}}$, the first fixed point of $\alpha \mapsto \epsilon_{\alpha}$.

Then start counting fixed points of the functions that list the fixed points: the *Veblen* hierarchy. The end of this process is called Γ_0.

Then it starts getting complicated ...

Why do (some) computer scientists care?

The working theoretical computer scientist needs ordinals to do inductions. However, generally speaking, up to ω^ω is as far as we need to go.

Proof theorists (logicians, but sometimes found in CS depts!) need much bigger ordinals.

The strength of a theory (how much that is true, can it prove?) can be measured by how long are the inductions it can do.

E.g. *Primitive Recursive Arithmetic* can't do ω^ω inductions.

Peano Arithmetic can't do ϵ_0 induction – so can't prove that G terminates!

Proof Theory may be of interest to *Theorem Provers* . . .

Envoi

All these ordinals are small!

The real Cantorian revolution was about *cardinals* – we have not gone beyond the first infinite cardinal.

But that's for another talk.