Realizability Models for BLL-like languages

M. Hofmann*  P. J. Scott †

June 14, 2000

Abstract

We give a realizability model of Girard-Scedrov-Scott’s Bounded Linear Logic (BLL). This gives a new proof that all numerical functions representable in that system are polytime. Our analysis naturally justifies the design of the BLL syntax and suggests further extensions.

1 Introduction

Bounded Linear Logic (BLL) [3] was an early attempt to provide an intrinsic notion of polynomial time computation within a logical system. That is, the aim was not merely to express polynomial time computability in terms of provability of certain restricted formulas, but rather to provide a typed logical system in which computation via cut-elimination or proof normalization is inherently polytime. Since the appearance of this paper, several different typed functional systems for analyzing ptime computability have appeared in the literature [5, 4, 10, 11, 6, 7]. For deeper foundational purposes, we should mention Girard’s Light Linear Logic (LLL) [4] as a major improvement of the syntax of BLL, in that it eliminates the explicit polynomial I/O size-bounds, but at the expense of introducing more subtle typing distinctions. Moreover, while capturing the same extensional class of polytime functions, it appears to be less flexible than BLL in terms of expressing concrete algorithms. Furthermore BLL has its own merits: from the viewpoint of computer science, BLL is a natural polymorphically-typed functional language in which bounded storage can represent bounded calls to memory.

The main theorem in [3] is that the number-theoretic functions representable in BLL are polytime. The proof of this result used sophisticated techniques from the proof theory of linear logic, notably a very detailed analysis of normalization of proof nets with boxes. The normalization strategy itself was of a special kind, inspired from Girard’s Geometry of Interaction program. In this paper, we give a direct, semantic proof of this main result which does not involve any notion of reduction, term rewriting, or cut-elimination. Rather, we assign polytime algorithms to proofs in a compositional, syntax-directed manner. We use

*Research partially supported by EPSRC grant No. GR/N28436
†Research supported by an operating grant from the Natural Sciences and Engineering Research Council of Canada.
realizability to relate these algorithms to the intended set-theoretic meaning of the proofs themselves. All this is presented in the form of a concrete categorical model of BLL which interprets BLL-formulas as sets with some additional structure and proofs as functions witnessed by polytime algorithms operating on this additional structure. At the same time, our analysis gives a natural interpretation of the BLL syntax which justifies the fine points of its design and might suggest further extensions. For example, our analysis encompasses an affine variant of BLL.

Our proof is constructive, in the sense that it can be formalized in an extensional version of the Calculus of Inductive Constructions [2]. This provides a new compilation method for turning BLL proofs into equivalent polytime algorithms. Of course in practice one would not use such a formalization, but rather derive the compilation algorithm by hand directly from our proof.

Our interpretation bears an intriguing relationship to approaches based on finite model theory, such as [5]. Namely, the polytime functions we obtain are of the form of Goerdt-Gurevich’s total global functions: they take one extra argument which is interpreted as a bound on the size of all the actual inputs. However, unlike Goerdt’s system which is a finite model interpretation of Gödel’s system $T$—hence the successor function is not injective—BLL supports the usual semantics, including Peano’s axioms, thus can be seen as a meaning-preserving annotation to standard functional programming. We hope that this relationship can be used to transfer Goerdt’s characterizations of LOGSPACE and PSPACE to a similar setting.

2 Bounded Linear Logic

We introduce the theory BLL of bounded linear logic, first proposed in [3]

2.1 Resource Polynomials

Resource polynomials [3] are finite sums of products of binomial coefficients, i.e. $\sum_{j \leq q} \prod_{i \leq p} \binom{x_{ij}}{n_{ij}}$ where, for any fixed $j$, the variables $x_{ij}$ are distinct and $n_{ij}$ are non-negative integer constants.

Resource polynomials are closed under sum, product, and composition. Given such polynomials $p, q$ write $p \leq q$ to denote that $q - p$ is a resource polynomial. If $p \leq p'$ and $q \leq q'$, then their composites satisfy $q\cdot p \leq q'\cdot p'$.

2.2 Syntax of BLL

Formulae are given by the following general syntax:

$$A, B ::= \alpha(p) \mid A \otimes B \mid A \multimap B \mid \forall \alpha.A \mid !_{x \leq p}.A$$

Here atomic formulae have the form $\alpha(p_0, \ldots, p_{n-1})$, where $\alpha$ is a second-order variable of given finite positive arity $n$ and $p = (p_0, \ldots, p_{n-1})$ denotes a list of resource polynomials of length $n$. We assume that there are infinitely many second-order variables of each finite arity.
The formula $\forall \alpha.A$ denotes second order universal quantification, while $!_{x<p}A$ is \textit{bounded storage}, where $p$ is a resource polynomial not containing $x$ and $x$ is bound in $!_{x<p}A$.

Positive and negative occurrences of resource terms in formulae are defined by induction as usual: in $!_{x<p}A$, $p$ occurs negatively. The $p_i$ and their subterms occur positively in an atomic formula $\alpha(p)$. The connectives $\otimes$ and $\forall \alpha$ are monotone; the connective $\rightarrow$ is antitone in its first and monotone in its second argument so that for example, $p$ occurs positively in $\forall \alpha(!_{y<p}(\alpha(y) \rightarrow \alpha(y+1)) \otimes \alpha(0)) \rightarrow \alpha(p)$.

Let the free resource variables $x_0, \ldots, x_{n-1}$ occur only positively in $B$. Then $\lambda x_0, \ldots, x_{n-1}.B$ is a (second order) abstraction term, say $T$. $A[\alpha := T]$ denotes the result of substituting $T$ for $\alpha$ in $A$, i.e. of replacing the atoms $\alpha(p_0, \ldots, p_{n-1})$ in $A$ by $B[x_0:=p_0] \ldots [x_{n-1}:=p_{n-1}]$.

Given types $A$ and $A'$, write $A \leq A'$ if $A$ and $A'$ only differ in their choice of resource polynomials, and

(i) for any positive occurrence of resource polynomial $p$ in $A$, the homologous $p'$ in $A'$ is such that $p \leq p'$.

(ii) for any negative occurrence of resource polynomial $p$ in $A$, the homologous $p'$ in $A'$ is such that $p' \leq p$.

If $\Gamma$ and $\Gamma'$ are finite multisets of formulae, $\Gamma \leq \Gamma'$ iff it is true componentwise.

\textit{Proofs} are given by Gentzen sequents, as follows.

\subsection{2.3 BLL Sequents}

Sequents have the form $\Gamma \vdash B$, where $\Gamma$ is a finite (possibly empty) multiset of formulae. In order to avoid mentioning the permutation rule, the formulae in $\Gamma$ are considered indexed but not ordered. In what follows, $p, q, w$ (possibly with subscripts) range over resource polynomials.

\textit{Axiom (Waste of Resources)} \quad $A \vdash A'$, where $A \leq A'$ \quad (Special case: $A \vdash A$).

\textit{Cut}

\[ \frac{\Gamma \vdash A \quad \Delta, A \vdash B}{\Gamma, \Delta \vdash B} \]

\textit{\otimes L}

\[ \frac{\Gamma, A, B \vdash C}{\Gamma, A \otimes B \vdash C} \]

\textit{\otimes R}

\[ \frac{\Gamma \vdash A \quad \Delta \vdash B}{\Gamma, \Delta \vdash A \otimes B} \]

\textit{\rightarrow L}

\[ \frac{\Gamma \vdash A \quad \Delta, B \vdash C}{\Gamma, \Delta, A \rightarrow B \vdash C} \]

\textit{\rightarrow R}

\[ \frac{\Gamma, A \vdash B}{\Gamma \vdash A \rightarrow B} \]

\textit{\forall L}

\[ \frac{\Gamma, A[\alpha := T] \vdash B}{\Gamma, \forall \alpha.A \vdash B} \]

\textit{\forall R}

\[ \frac{\Gamma \vdash A}{\Gamma \vdash \forall \alpha.A} \]

(provided $\alpha$ is not free in $\Gamma$)

\textit{(!w)} \quad \textit{Weakening}

\[ \frac{\Gamma \vdash B}{\Gamma, !_{x<w}A \vdash B} \]

3
(!D) **Dereliction**

\[ \frac{\Gamma, A[x := 0] \vdash B}{\Gamma, \; \forall x < 1 + w A \vdash B} \]

(!C) **Contraction**

\[ \frac{\Gamma, \; \forall x < p A, \; \forall y < q A[x := p + y] \vdash B}{\Gamma, \; \forall x < p + q + w A \vdash B} \]

where \( p + y \) is free for \( x \) in \( A \).

(S!) **Storage**

\[ \frac{\; \forall z < q_1(x) A_1[y := v_1(x) + z], \ldots, \forall z < q_n(x) A_n[y := v_n(x) + z] \vdash B}{\; \forall y < v_1(p) + w_1 A_1, \ldots, \forall y < v_n(p) + w_n A_n \vdash \; \forall x < p B,} \]

where \( v_i(x) + z \) is free for \( y \) in \( A_i \), where \( v_i(x) = \sum_{z < x} q_i(z) \) and where all formulae to the left of the \( \vdash \) have the indicated form.

**Remark 2.1**

- The rules of BLL are written in such a way that given any proof \( \pi \) of a sequent \( \Gamma \vdash A \) and given any \( \Gamma' \leq \Gamma \) and \( A \leq A' \) then a simple change of resource parameters will yield a proof \( \pi' \) of \( \Gamma' \vdash A' \) **without altering the structure of the proof**.

Note that the “waste” \( w \) in each of the rules associated with storage can without loss of generality be assumed 0 as the general case can be recovered by cutting with appropriate axioms. In this paper we are not interested in cut elimination, therefore, we will adopt this simplification.

- We also introduce a unit for \( \otimes \), denoted \( I \). The ordinary LL rules for \( I \) are as follows:

\[ \frac{\Gamma \vdash B}{\Gamma, I \vdash B} \]

- The data type of tally natural numbers of size at most \( x \) is:

\[ \text{N}_x = \forall \alpha \{ y < x (\alpha(y) \rightarrow \alpha(y + 1)) \rightarrow (\alpha(0) \rightarrow \alpha(x))\} \]

Moreover, there are proofs \( \vdash 0 : \text{N}_0 \) and \( S : \text{N}_x \vdash \text{N}_{x+1} \) representing “zero” and “successor”, resp. (see [3]).

- The data type of dyadic lists of size at most \( x \) is:

\[ \text{N}_x^2 = \forall \alpha \{ y < x (\alpha(y) \rightarrow \alpha(y + 1)) \rightarrow \forall y < x (\alpha(y) \rightarrow \alpha(y + 1)) \rightarrow (\alpha(0) \rightarrow \alpha(x))\} \]

There are proofs \( \epsilon : \text{N}_0^2 \) and \( S_0, S_2 : \text{N}_x^2 \vdash \text{N}_{x+1}^2 \) representing zero and the two successor functions, resp. (see [3]).

The following two rules are not contained in the definition of BLL [3], and as far as we can see are not admissible in BLL.

(Func)

\[ \frac{A \vdash B}{\forall x < p A \vdash \forall x < p B} \]

(Mon)

\[ \forall x < p A \otimes ! x < p B \vdash \forall x < p (A \otimes B) \]
These rules express functoriality and monoidalness of \(!_{x<p}\). Note that they can be subsumed under the following generalization of \((\text{Func} \otimes)\):

\[
\frac{A_1, \ldots, A_n \vdash B}{!_{x<p}A_1, \ldots, !_{x<p}A_n \vdash !_{x<p}B}
\]

Our semantics validates these rules, so as a result their addition does not increase the computational strength of BLL. We note that in their presence the storage rule can be replaced by the following axiom:

\[
!_{y<\sum_{x<y} q(x)} A \vdash !_{x<p} !_{x<q(y)} A \| y := z + \sum_{\xi < x} q(\xi)
\]

3 Main Result

We shall assume that our ambient set theory is constructive, so that we shall have a set (of sets) \(\mathcal{U}\) containing the natural numbers \(\mathbb{N}\), closed under product, function space, and \(\mathcal{U}\)-indexed products [9]. We discuss this point in more detail below in Section 5. This allows us to interpret types as sets in the following way: given a formula \(A\) and an environment \(\rho\) which assigns sets to type-variables, we obtain a set-theoretic interpretation \([A]_{\rho}\) as follows:

\[
\begin{align*}
\alpha(\vec{p})_{\rho} &= \rho(\alpha) \\
A \times B_{\rho} &= [A]_{\rho} \times [B]_{\rho} \\
A \Rightarrow B_{\rho} &= [A]_{\rho} \Rightarrow [B]_{\rho} \\
\forall A_{\rho} &= \prod_{C \in \mathcal{U}} [A]_{\rho(C)} \\
!_{x<p} A_{\rho} &= [A]_{\rho}
\end{align*}
\]

Notice that this interpretation of types ignores the resource polynomials.

To every proof \(\pi\) of a sequent \(A_1, \ldots, A_n \vdash B\) and environment \(\rho\), we can assign a set-theoretic function

\([\pi]_{\rho} : [A_1 \otimes \ldots \otimes A_n]_{\rho} \rightarrow [B]_{\rho}\)

by induction on derivations, in the obvious way. Observe that

\([N_p] = \prod_{C \in \mathcal{U}} \text{(C} \Rightarrow \text{C}) \Rightarrow \text{(C} \Rightarrow \text{C})\)

There is a pair of functions \(\phi : \mathbb{N} \rightarrow [N_p]\) and \(\psi : [N_p] \rightarrow \mathbb{N}\) satisfying \(\psi \circ \phi = id_{\mathbb{N}}\), defined as follows:

\[
\begin{align*}
\phi(n)_{C}(f, z) &= f^n(z) \\
\psi(x) &= x_{\mathbb{N}}(S)(0)
\end{align*}
\]

where in the definition of \(\psi\), the symbols \(S\) and 0 are the usual successor and zero on \(\mathbb{N}\). Note that \(\psi[0] = 0\) and \(\psi[S] \phi = S\).

Our main goal is to give a new proof of the following theorem, which is equivalent to Theorem 5.4 in [3]:

5
Theorem 3.1 Let $\pi$ be a proof of $\vdash N_x \rightarrow N_{p(x)}$, where $p$ does not contain any other free
resource variables except $x$. Let $f : \mathbb{N} \rightarrow \mathbb{N}$ be the function:
\[
f(n) = \psi([\pi](\phi(n)))
\]
Then: $f(n)$ is computable in polynomial time in $n$. Moreover, $f(n) \leq p(n)$. and an algorithm
for $f$ can be effectively obtained from the proof $\pi$.

An analogous result holds for the type of dyadic lists, as in Theorem 5.3 of [3].

4 A Realizability Model for BLL

We now introduce a refined model $\mathcal{B}$ for BLL based on realizability and ideas from [6].
This will allow us to obtain the above theorem as a direct corollary of soundness of the
interpretation. See the proof after Theorem 4.21.

4.1 Preliminaries

For $x \in \mathbb{N}$, we write $|x| = \lceil \log_2 (x + 1) \rceil$ for the binary length of $x$. We fix a linear time
computable pairing function $\langle \cdot, \cdot \rangle : \mathbb{N}^2 \rightarrow \mathbb{N}$ satisfying $|\langle x, y \rangle| = |x| + |y| + O(\log(|x|))$.

We should also remark that the inverses of the pairing function are assumed to be linear
time computable.

Let $X$ be a finite set of variables. We write $\mathcal{V}(X)$ for $\mathbb{N}^X$—the elements of $\mathcal{V}(X)$ are
called valuations (over $X$). If $\eta \in \mathcal{V}(X)$ and $c \in \mathbb{N}$ then $\eta[x \mapsto c]$ denotes the valuation which
maps $x$ to $c$ and acts like $\eta$ otherwise. We assume some reasonable encoding of valuations
as integers allowing them to be passed as arguments to algorithms.

We write $\mathcal{P}(X)$ for the set of resource polynomials over $X$. If $p \in \mathcal{P}(X)$ and $\eta \in \mathcal{V}(X)$
we write $p(\eta)$ for the number obtained by evaluating $p$ with $x \mapsto \eta(x)$ for each $x \in X$.

Let $X, Y$ be finite sets of variables. A substitution $\sigma : X \rightarrow \mathcal{P}(X)$. We may write a substitution from $X$ to $Y = \{y_0, \ldots, y_{n-1}\}$ in the form $\sigma = [X; y_0 := p_0, y_1 := p_1, \ldots, y_{n-1} := p_{n-1}]$. This is defined if $p_i \in \mathcal{P}(X)$ and in this case we have
$\sigma(y_i) = p_i$. If the domain $X$ is clear from the context, we may simply write $\sigma = [y_0 := p_0, y_1 := p_1, \ldots, y_{n-1} := p_{n-1}]$.

A substitution $\sigma$ from $X$ to $Y$ induces functions $\sigma(-) : \mathcal{V}(X) \rightarrow \mathcal{V}(Y)$ and
$\sigma[\cdot] : \mathcal{P}(Y) \rightarrow \mathcal{P}(X)$ in the obvious way, i.e., $(\sigma(\eta))(y) \overset{\text{def}}{=} \sigma(y)(\eta)$ and $p[\sigma] \overset{\text{def}}{=} p[y_0 := \sigma(y_0)] \ldots [y_{n-1} := \sigma(y_{n-1})]$.

We assume known the untyped lambda calculus as defined e.g. in [1]. An untyped
lambda term is affine linear if each variable (free or bound) appears at most once (up
to $\alpha$-congruence). E.g. $\lambda x \lambda y.yx$ and $\lambda x \lambda y.yx$ and $\lambda x. xx$ are affine linear; the term $\lambda x. xx$ is
not. Notice that such a term $t$ is strongly normalisable in less than $|t|$ steps where $|t|$ is the
size of the term. The runtime of the computation leading to the normal form is therefore
$O(|t|^2)$. We will henceforth use the expression affine lambda term for an untyped affine linear
lambda term which is in normal form. If $s, t$ are affine lambda terms then their application
$st$ is defined as the normal form of the lambda term $st$. Notice that the application $st$ can
be computed in time $O((|s| + |t|)^2)$.  

6
If \( s, t \) are affine lambda terms we write \( s \otimes t \) for the affine lambda term \( \lambda f. f s t \). If \( t \) is an affine lambda term possibly containing the free variables \( x, y \) then we write \( \lambda x \otimes y. t \). Notice that \((\lambda x \otimes y. t)(u \otimes v) = t[x:=u][y:=v]\).

More generally, if \((t_i)_{i<n}\) is a family of affine lambda terms, we write \( \otimes_{i<n} t_i \) for \( \lambda f.f t_0 \ldots t_{n-1} \) and \( \lambda \otimes_{i<n} x_i. t \) for \( \lambda u.u(\lambda x_0 \lambda x_1 \ldots \lambda x_{n-1}. t) \). Again, \((\lambda \otimes_{i<n} x_i. t)(\otimes_{i<n} t_i) = t[x_0:=t_0] \ldots [x_{n-1}:=t_{n-1}]\).

Tally natural numbers may be encoded as affine lambda terms by \( \gamma 0^\gamma = \lambda xy.x \otimes \lambda xy.x \) and \( \gamma n + 1^\gamma = (\lambda xy.y) \otimes \gamma n^\gamma \). Dyadic lists may be encoded as \( \gamma e^\gamma = \lambda xyz.x \otimes \lambda xy.x \) and \( \gamma 0w^\gamma = \lambda xyz.y \otimes \gamma w^\gamma \) and \( \gamma 1w^\gamma = \lambda xyz.z \otimes \gamma w^\gamma \).

We notice that \( \gamma n^\gamma \), resp. \( \gamma w^\gamma \) can be computed in linear time from \( n \), resp. \( w \), and vice versa. Of course, only a few functions \( f \) on natural numbers or dyadic lists can be represented by affine terms \( t \) in the sense that \( t^\gamma w^\gamma = \gamma f(w)\).

We write \( \Lambda_n \) for the set of closed affine lambda terms. Our subsequent development will be modular in the sense that \( \Lambda_n \) can be replaced by any other polynomial-time computable BCK-algebra in the sense of [7]. For example, we can take Turing machines with the application defined by \( ex = \{e\}(x) \) if this result can be obtained in time at most \( \varphi(d(\ell(e) + \ell(x))) \) and 0 otherwise. Here \( \ell \) is a length function defined inductively by \( \ell(x, y) = \ell(x) + \ell(y) + 1 \), \( \ell(x) = |x| \) otherwise. The defect \( d \) is \( \ell(e) + \ell(x) - \ell(\{e\}(x)) \) and we additionally require \( \ell(\{e\}x) \leq \ell(e) + \ell(x) \). Finally, \( \varphi \) is a monotone, sublinear function satisfying \( \varphi(\ell(x)) \leq |x| \).

For example, \( \varphi(x) = Cx^{1+\epsilon} \) for arbitrary \( \epsilon > 0 \) and appropriate \( C \), see [6].

### 4.2 Realizability Sets

**Definition 4.1** Let \( X \) be a finite set of resource variables. A *realizability set over \( X \) is a pair \( A = (|A|, |\leftarrow A|) \) where \( |A| \) is a set and \( |\leftarrow A| \subseteq V(X) \times \Lambda_n \times |A| \) is a ternary relation between valuations over \( X \), affine lambda terms, and the set \( |A| \). We write \( \eta, t \rightarrow_{A} a \) for \( (\eta, t, a) \in |\leftarrow A| \).

The intuition behind \( \eta, t \rightarrow_{A} a \) is that \( a \) is an abstract semantic value, \( \eta \) measures the abstract size of \( a \), and the affine lambda term \( t \) encodes the abstract value \( a \).

**Example 4.2** The following are some useful examples of realizability sets, cf. Section 5 of [3].

(i) The realizability set \( N_x \) over \( \{x\} \) of *tally natural numbers* (*of size at most \( x \)*) is defined by: \( |N_x| = \mathbb{N} \) and

\[ \eta, t \rightarrow_{N_x} n \text{ if } t = \gamma n^\gamma \text{ and } \eta(x) \geq n \]

(ii) The realizability set \( N^2_x \) over \( \{x\} \) of *dyadic lists* (*of length at most \( x \)*) is defined by: \( |N^2_x| = \{0, 1\}^* \) and

\[ \eta, t \rightarrow_{N^2_x} w \text{ if } t = \gamma w^\gamma \text{ and } \eta(x) \geq \ell(w) \]

(iii) The realizability set \( I \) over \( \emptyset \) is defined by \( |I| = \{\ast\} \) and \( \emptyset, \lambda x. x \rightarrow_{I} \ast \)
(iv) Given a substitution \( \sigma \) from \( X \) to \( Y \), and a realizability set \( A \) over \( Y \), then a new realizability set \( A[\sigma] \) over \( X \) is defined by: \( |A[\sigma]| = |A| \) and

\[
\eta, t \models_{A[\sigma]} a \quad \text{iff} \quad \sigma(\eta), t \models_{A} a
\]

The sets \( N_x \) and \( N^2_x \) will turn out to be retracts of the denotations of the eponymous BLL formulas in our model.

In order to model the notion of positive occurrence of a resource variable in BLL formulas we introduce a corresponding concept for realizability sets.

**Definition 4.3** Let \( A \) be a realizability set over \( X \). We say that \( x \in X \) is positive (resp. negative) in \( A \), if for all \( \eta, \eta' \in \mathcal{V}(X), t \in \Lambda_a, a \in |A| \) where \( \eta \) and \( \eta' \) agree on \( X \setminus \{x\} \) and \( \eta(x) \leq \eta'(x) \) (resp. \( \eta(x) \geq \eta'(x) \)) then \( \eta, t \models_{A} a \) implies \( \eta', t \models_{A} a \).

We notice that \( x \) is positive in \( N_x \) and \( N^2_x \).

**Definition 4.4** Let \( A, B \) be realizability sets over some set \( X \). A morphism from \( A \) to \( B \) is a function \( f : |A| \rightarrow |B| \) satisfying the following condition:

There exists a function \( e : \mathcal{V}(X) \rightarrow \Lambda_a \) such that \( e(\eta) \) is computable in time \( q(\eta) \) for some resource polynomial \( q \) and for each \( \eta \in \mathcal{V}(X), t \in \Lambda_a, a \in |A| \), we have

\[
(1) \quad \eta, t \models_{A} a \Rightarrow \eta, e(\eta)t \models_{B} f(a)
\]

In this case we say that \( e \) witnesses \( f \) and write \( A \xrightarrow{f, e} B \) where in the notation the algorithm \( e \) is presumed to exist.

**Example 4.5** The following are some useful examples of numerical-valued morphisms:

- A morphism \( f : N_x \rightarrow N_x \) is a function \( f : \mathbb{N} \rightarrow \mathbb{N} \) that is computable in time polynomial in the input \( n \) (not in \( |n| \)). Moreover, it satisfies \( f(n) \leq n \) (by letting \( \eta(x) = n \)).

- Similarly, a morphism \( f : N^2_x \rightarrow N^2_x \) is a function \( f : \{0, 1\}^* \rightarrow \{0, 1\}^* \) that is polytime computable (in the usual sense) and moreover satisfies \( lh(f(w)) \leq lh(w) \).

- Let \( p(x) \) be a unary resource polynomial in \( x \) and let \( \sigma \) be the substitution \([\{x\}; x := p]\).

A morphism \( f : N_x \rightarrow N_x[\sigma] \) is a function \( f : \mathbb{N} \rightarrow \mathbb{N} \) that is polytime computable in the input (as above) for which \( f(n) \leq p(n) \). Composed with the above-mentioned retractions, these morphisms will be denotations of closed proofs of \( \vdash N_x \succeq N_{p(x)} \).

The following lemma illustrates how realizability sets model the syntactical iteration lemma for BLL (cf. Lemma 6.2 of [3]).

**Lemma 4.6 (Iteration Lemma)** Let \( T \) be a realizability set over \( \{x\} \) such that \( x \) is positive in \( T \). Let \( z : I \rightarrow T[x := 0] \) be a morphism (over \( \emptyset \)) and let \( s : T \rightarrow T[x := x + 1] \) be a morphism (over \( \{x\} \)). The function \( f : \mathbb{N} \rightarrow |T| \) defined by \( f(n) = s^n(z(\star)) \) is a morphism from \( N_x \) to \( T \).
Proof. The witnesses of $z$, $s$ give rise to an element $e_z \in \Lambda_a$ and to a function $e_s : \mathbb{N} \to \Lambda_a$ such that
\[
[x \mapsto 0], e_z \models_T z(*)
\]
and for each $n \in \mathbb{N}$,
\[
[x \mapsto n], t \models_T v \Rightarrow [x \mapsto n+1], e_s(n)t \models_T s(v)
\]
This is because by definition of substitution we have
\[
\emptyset, t \models_T [x:=0] v \iff [x:=0], t \models_T v \\
[x:=n], t \models_T [x:=x+1] v \iff [x:=n+1], t \models_T v
\]
We now have
\[
[x:=N], e_s(N-1)(e_s(N-2)\ldots(e_s(N-n)e_z)\ldots) \models_T f(n)
\]
whenever $n \leq N$. This follows by induction on $n$ and the fact that $x$ is positive in $T$.

We define $e : \mathbb{N} \to \Lambda_a$ recursively by
\[
e(0) = e_z \\
e(N+1) = \lambda b \in \mathbb{N} b \cdot e_z (e_s(N) (e(N) r))
\]
It follows that
\[
[x\mapsto N], e(N)^T n^2 \models_T f(n)
\]
provided that $n \leq N$.

Furthermore, it is easy to see that $e$ is polytime given that we have $e(N+1) = P e(N) e_s(N) e_z$ for some fixed $P \in \Lambda_a$ (independent of $N$).

Therefore, $\eta \mapsto e(\eta(x))$ witnesses the function $f(n) = s^n(z(*))$. \qed

The following analogous version for dyadic lists (cf. Lemma 6.2 of [3]) is proved similarly.

**Lemma 4.7 (Iteration lemma for dyadic lists)** Let $T$ be a realizability set over $\{x\}$ such that $x$ is positive in $T$. Let $z : I \to T[x := 0]$ be a morphism (over $\emptyset$) and let $s_0, s_1 : T \to T[x := x+1]$ be morphisms (over $\{x\}$). The function $f : \{0,1\}^* \to |T|$ defined by $f(\epsilon) = z$ and $f(iw) = s_i(f(w))$ is a morphism from $\mathbb{N}_2^\omega$ to $T$.

We remark that one can also prove more general versions of the preceding iteration lemmas allowing for extra resource variables in $T$ as parameters.

**Proposition 4.8** Let $X$ be a finite set (of resource variables). Realizability sets over $X$ and morphisms between them form a category $\mathcal{B}(X)$ such that the mapping $A = (|A|, \models_A)$ \mapsto $|A|$ extends to a functor from $\mathcal{B}(X)$ to the category $\text{Set}$ of sets. This means that composition in $\mathcal{B}(X)$ is given by ordinary set-theoretic composition of functions.

**Proof.** The identity function $id : |A| \to |A|$ is witnessed by the algorithm $e(\eta) = \lambda x.x$ which is clearly polytime computable.

If $A_0 \xrightarrow{f} A_1$ and $A_1 \xrightarrow{g} A_2$ then the composition $g \circ f : |A_0| \to |A_2|$ can be witnessed by
\[
e(\eta) = \lambda z.e_1(\eta)(e_0(\eta)z) = B e_1(\eta) e_0(\eta)
\]
where $B = \lambda yz.x(yz)$. Now $e$ is polytime using the fact that application in $\Lambda_a$ is polytime. \qed
Recall the definition in Example 4.2 of the realizability set $A[\sigma]$ over $X$ when $A$ is a realizability set over $Y$ and $\sigma$ is a substitution from $X$ to $Y$.

**Proposition 4.9** Let $\sigma$ be a substitution from $X$ to $Y$. The assignment $A \mapsto A[\sigma]$ extends to a functor $-[\sigma] : \mathcal{B}(Y) \to \mathcal{B}(X)$ with $f[\sigma] \overset{def}{=} f$.

**Proof.** We have to show that if $A \xrightarrow{f} B$ then we can find $e'$ so that $A[\sigma] \xrightarrow{f[e']} B[\sigma]$. Unfolding the definitions reveals that $e'(\eta) = e(\sigma(\eta))$ does the job. \qed

This allows us to consider morphisms between realizability sets over different sets of resource variables. Namely, if $X \subseteq Z$ we have a “weakening substitution” $weak_{X,Z}$ from $Z$ to $X$ given by $weak_{X,Z}(x) = x$. Thus, if $A$ is a realizability set over $X$ and $B$ is a realizability set over $Y$ we can consider morphisms from $A[weak_{X,Z}]$ to $B[weak_{X,Z}]$. Such a morphism is a function $f : |A| \to |B|$ such that there exists an algorithm $e : \mathcal{Y}(X \cup Y) \to A_\eta$ such that $e(\eta)$ is computable in time $q(\eta)$ for some resource polynomial $q$ and

$$\eta|_X, t \mid\frac{\vdash_A a}{\mid B \mid b}$$

where $\eta|_X$ denotes the restriction of $\eta$ to $X$. We shall sloppily refer to such morphisms as being morphisms from $A$ to $B$. In this sense, the only morphism from $N_x$ to $N_y$ where $x \neq y$ is the constant zero function.

The following is immediate:

**Lemma 4.10** Suppose that $A \in \mathcal{B}(Y)$ is a realizability set and $\sigma, \sigma' : X \to Y$ are substitutions. Suppose furthermore that $\sigma(y) \leq \sigma'(y)$ if $y$ occurs positively in $A$, that $\sigma(y) \geq \sigma'(y)$ if $y$ occurs negatively in $A$, and that $\sigma(y) = \sigma'(y)$ otherwise. Then the identity function is a morphism from $A[\sigma]$ to $A[\sigma']$.

### 4.3 The Category of Realizability Sets

We will now show that the categories $\mathcal{B}(X)$ have the appropriate categorical structure to model the BLL connectives.

**Definition 4.11** We define the following monoidal structure on $\mathcal{B}(X)$:

- $I = (|I|, \mid I \mid_\lambda x.x)$, where $|I| = \{*\}$ and $\eta, t \mid\frac{\vdash_I *}{\mid I \mid *}$ for $t = \lambda x.x$ and $\eta$ arbitrary.

- If $A_1, A_2$ are realizability sets over $X$ we define $A_1 \otimes A_2$ by $|A_1 \otimes A_2| = |A_1| \times |A_2|$ and $\eta, t \mid\frac{\vdash_{A_1 \otimes A_2} (a_1, a_2)}{\mid A_1 \mid a_i}$ if $t = \langle t_1, t_2 \rangle$, where $\eta, t_i \mid\frac{\vdash_{A_i} a_i}{\mid A_i \mid a_i}$ for $i = 1, 2$.

**Proposition 4.12** Let $f : A \to B$ be a $\mathcal{B}(X)$ morphism, $C \in \mathcal{B}(X)$. Then

i) The function $f \otimes C : |A \otimes C| \to |B \otimes C|$ defined by $(f \otimes C)(a, c) = (f(a), c)$ is a morphism from $A \otimes C$ to $B \otimes C$. 

10
ii) The canonical set-theoretic maps $|A \otimes (B \otimes C)| \rightarrow |(A \otimes B) \otimes C|$, $|A \otimes B| \rightarrow |B \otimes A|$, and $|A \otimes I| \rightarrow |A|$ induce isomorphisms between the associated objects.

iii) For appropriately typed substitution $I[\sigma] = I$ and $(A_1 \otimes A_2)[\sigma] = A_1[\sigma] \otimes A_2[\sigma]$.

This says in particular that $\mathcal{B}(X)$ is a symmetric monoidal category, and the forgetful functor $\mathcal{B}(X) \rightarrow \text{Set}$ is a monoidal functor. Clause (iii) states that substitution is a monoidal functor. This says that the collection of the categories $\mathcal{B}(X)$ forms a fibred symmetric monoidal category in the sense of [14].

**Proof.** Ad i). If $e$ witnesses $f$ then we define $e'(\eta) = \lambda x \otimes y.e(\eta)x \otimes y$. Obviously, $e'$ witnesses $f \otimes C$ and, since $e'(\eta) = P\ e(\eta)$ for some $P \in \Lambda_a$, the function $e'$ is polytime. The other cases are analogous.

**Proposition 4.13** For any two objects $A, B \in \mathcal{B}(X)$, there is a linear function space object $A \rightarrow B \in \mathcal{B}(X)$, where

(i) $|A \rightarrow B| = |A| \Rightarrow |B|$  
(ii) $\eta, t \models_{A \rightarrow B} f$ iff whenever $\eta, t' \models_{A} a$ then $\eta, t' \models_{B} f(a)$.

This structure makes $\mathcal{B}(X)$ a symmetric monoidal closed category, i.e. there is a natural bijection $\mathcal{B}(X)(C \otimes A, B) \cong \mathcal{B}(X)(C, A \rightarrow B)$. Moreover, $(A \rightarrow B)[\sigma] = A[\sigma] \rightarrow B[\sigma]$ so that $-[\sigma]$ is a monoidal closed functor.

**Proof.** The evaluation map $|A \otimes (A \rightarrow B)| \rightarrow |B|$ given by $(a, f) \mapsto f(a)$ is witnessed by $e(\eta) = \lambda x \otimes y.x$. If $C \otimes A \xrightarrow{f} B$ then $\lambda(f) : C \rightarrow A \rightarrow B$ given by $\lambda(f)(c)(a) = f(c, a)$ is witnessed by $e'(\eta) = \lambda x y.e(\eta)(x \otimes y)$. Just as in Prop. 4.12, i) we have $e'(\eta) = P\ e(\eta)$ for some $P \in \Lambda_a$, which establishes that $e'$ is polytime.

We notice that the forgetful functor $\mathcal{B}(X) \rightarrow \text{Set}$ is also monoidal closed, i.e. sends $\otimes$ to $\times$ and $\rightarrow$ to $\Rightarrow$.

**Definition 4.14** Given a polynomial $p \in \mathcal{P}(X)$ and a realizability set $A$ over $X \cup \{x\}$ where $x \notin X$ then we define a realizability set $!_{x < p} A$ over $X$ (i.e. $x$ is “bound” by $!_{x < p}$) by

- $|!_{x < p} A| = |A|$,  
- $\eta, t \models_{!_{x < p} A} a$ if
  - $t = \otimes_{i < p(\eta)}^n t_i$ for some family $(t_i)_{i < p(\eta)}$,  
  - $\eta[x \mapsto i], t_i \models_{A} a$ for each $i < p(\eta)$.

Whenever we write $!_{x < p} A$ in the sequel we implicitly assume that $x$ does not occur in $p$.

**Proposition 4.15** If $f : A \rightarrow B \in \mathcal{B}(X \cup \{x\})$ is a morphism then $!_{x < p}(f) \overset{\text{def}}{=} f$ is a morphism $!_{x < p} A \rightarrow !_{x < p} B$. This says that $!_{x < p}$ extends to a functor from $\mathcal{B}(X \cup \{x\})$ to $\mathcal{B}(X)$ which is mapped to the identity by the forgetful functor to $\text{Set}$. 

11
Proof. If $A \xrightarrow{f} B$ then we can witness $f : \neg x < p A \to \neg x < p B$ by

$$e'(\eta) = \lambda \bigotimes_{i < p(\eta)} x_i. \bigotimes_{i < p(\eta)} e(\eta)x_i$$

We notice that

$$e'(\eta) = P e(\eta) \ldots e(\eta)$$

for some $P \in \Lambda$, so that $e'(\eta)$ is computable in time $O((p(\eta)q(\eta))^2)$ if $e(\eta)$ is computable in $O(q(\eta))$. Note that this is a rather generous estimate. \qed

We now show that we have the appropriate categorical structure to interpret the rules of BLL.

**Proposition 4.16** The following are morphisms:

- $\epsilon_A : \neg x < 1 A \to A[x := 0]$ where $\epsilon_A(a) = a$
- $\epsilon_A : \neg x < 1 A \to I$ where $\epsilon_A(a) = *$
- $d_A : \neg x < p + q A \to \neg x < p A \otimes \neg y < q A[x := p + y]$
  where $d_A(a) = (a, a)$
- $\delta_A : \neg y < \sum_{z < q} q(z)x \to \neg x < p \neg z < q(x) A[y := z + \sum_{z < q} q(z)]$
  where $\delta_A(a) = a$ and $p \in P(X), q \in P(X \cup \{x\})$
  and $x, y$ are fresh
- $\tau_{A, B} : \neg x < p A \otimes \neg x < p B \to \neg x < p (A \otimes B)$
  where $\tau_{A, B}(a, b) = (a, b)$
- $w_A : \neg x < p A \to \neg x < q A$ where $w_A(a) = a$ and $q \leq p$

**Proof.** The map $\epsilon_A$ can be witnessed by $e(\eta) = \lambda \bigotimes_{i < 1} x_i. x_0$. The map $\epsilon_A$ may be witnessed by $e(\eta) = \lambda y. \lambda x. x$. The map $d_A$ may be witnessed by

$$e(\eta) = \lambda \bigotimes_{i < (p+q)(\eta)} x_i. \bigotimes_{i < p(\eta)} (\bigotimes_{i < p(\eta)} x_i \otimes (\bigotimes_{j < q(\eta)} x_{j+p(\eta)}))$$

To see this, assume $\eta, t \models A[x := i] t_i \models A a$. Then for each $i < p(\eta)$ we have $\eta, t \models A \otimes t_i \models A a$, so $\eta, \bigotimes_{i < p(\eta)} t_i \models A \otimes A a$. Moreover, for each $j < q(\eta)$ we have $\eta,\bigotimes_{i < q(\eta)} \bigotimes_{j < q(\eta)} t_i \models A \otimes A a$, hence $\eta,\bigotimes_{i < q(\eta)} t_{j+p(\eta)} \models A \otimes A a$. Therefore, $\eta, \bigotimes_{j < q(\eta)} t_{j+p(\eta)} \models A \otimes A a$, so the result follows.

Next, $\delta_A$ may be witnessed by

$$e(\eta) = \lambda \bigotimes_{j < q(\eta)} \bigotimes_{i < p(\eta)} x_j. \bigotimes_{j < q(\eta)} \bigotimes_{i < p(\eta)} \bigotimes_{j < q(\eta)} x_j \otimes \bigotimes_{k < i} x_{j+p(\eta)}$$

The verification is similar to the one of $d_A$.

Finally, $\tau_{A, B}$ may be witnessed by

$$e(\eta) = \lambda u \otimes v. u(\lambda x_1 x_2 \ldots x_{p(\eta)-1} v(\lambda y_1 y_2 \ldots y_{p(\eta)-1}. \bigotimes_{i < p(\eta)} x_i \otimes \bigotimes_{i < p(\eta)} y_i))$$

We omit the “waste” morphism $w_A$, which is obvious. \qed
4.4 Interpreting the Syntax in \( B \)

We shall now give the details of the interpretation of the syntax of BLL in terms of realizability sets and morphisms between them.

**Definition 4.17** Let \( X \) be a set of resource variables. A *second-order environment* over \( X \) is a partial function \( \rho \) which assigns to a second-order variable \( \alpha \) of arity \( n \) a pair \((l, C)\) such that:

- \( l = (y_0, \ldots, y_{n-1}) \) is a list of \( n \) pairwise different resource variables not occurring in \( X \),
- \( C \) is a realizability set over \( X \cup \{y_0, \ldots, y_{n-1}\} \) in which the \( y_i \) are positive.

For second-order environment \( \rho \) we write \( |\rho| \) for the mapping \( \alpha \rightarrow |C| \) when \( \rho(\alpha) = (l, C) \).

If \( \sigma : X \rightarrow Y \) is a substitution and \( \rho \) is a second-order environment over \( Y \) we define a second-order environment \( \rho[\sigma] \) over \( X \) by \( \rho[\sigma](\alpha) = (l, C[\sigma]) \) when \( \rho(\alpha) = (l, C) \). We assume here that the variables in \( l \) are not contained in \( Y \). Otherwise, the substitution cannot be defined.

Recall the set-theoretic semantics \([-]\) defined in Section 3. From now on, we will write \([-]_\text{Set} \) for this set-theoretic semantics to distinguish it from a realizability semantics \([-]_R \) which we now define.

Let \( A \) be a BLL formula with free resource variables in \( X \) and \( \rho \) be an environment over \( X \) defined on all free second-order variables occurring in \( A \). We assume without loss of generality that all bound variables in \( A \) and in \( \rho \) are distinct from each other.

By induction on \( A \) we define a realizability set \([A]_\rho \) such that

\[
[A]_\rho = [A]_\rho^{\text{Set}}
\]

The defining clauses are as follows:

\[
\begin{align*}
\alpha(p) & = C[\sigma] \\
\text{where } \rho(\alpha) = ((y_0, \ldots, y_{n-1}), C) \text{ and } p = (p_0, \ldots, p_{n-1}) \text{ and } \sigma(y_i) = p_i \\
A \otimes B & = [A]_\rho \otimes [B]_\rho \\
A \circ B & = [A]_\rho \circ [B]_\rho \\
\models_{x < p} A & = \models_{x < \rho} [A]_\rho^{\text{weak}_{X,\{x\}}} \\
\forall \alpha A & = (\bigcup_{C \in D} [A]_\rho^{\text{Set}}\left[\rho[\alpha = C]\right] = [A]_\rho^{\text{Set}}\left[\rho[\alpha = C]\right])
\end{align*}
\]

where

\[
\eta, t \models [\forall \alpha A]_\rho \iff \eta, t \models [A]_\rho^{\text{Set}} \text{ for all } l, C \text{ as in Def. 4.17}
\]

The following lemmas are immediate by structural induction.

**Lemma 4.18** Let \( \sigma \) be a substitution from \( X \) to \( Y = \{y_0, \ldots, y_{n-1}\} \) and let \( A \) be a BLL formula with free resource variables from \( Y \) and let \( \rho \) be an environment over \( Y \) such that \( \rho[\sigma] \) is defined. Then

\[
[A[y_0 := \sigma(y_0), \ldots, y_{n-1} := \sigma(y_{n-1})]]_{\rho[\sigma]} = [A]_\rho[\sigma]
\]

13
Lemma 4.19 Let $A$ be a BLL formula possibly containing the second-order variable $\alpha$ of
arity $n$. Let $B$ be a BLL formula containing free resource variables \{\(y_0, \ldots, y_{n-1}\)\} which
do not occur in $A$. Then
\[
\left[ A[\alpha := \lambda y_0 \ldots \lambda y_{n-1} B] \right]_{\rho}^B = \left[ A \right]_{\rho(y_0 \ldots y_{n-1})}^B,
\]

Lemma 4.20 Let $A$ be a BLL formula in which resource variable $x$ occurs positively (resp.
 negatively) and $\rho$ be an appropriate second-order environment. Then $x$ is positive (resp.
 negatively) in $\left[ A \right]_{\rho}^B$.

Theorem 4.21 (Soundness) Let $\pi$ be a proof of a sequent $A_1, \ldots, A_n \vdash B$ involving re-
source variables from $X$. Let $\rho$ be a second-order environment binding all the second-order
variables occurring in the sequent. Then the set-theoretic function
\[
\left[ \pi \right]_{\rho}^{Set} : \left[ A_1 \otimes \ldots \otimes A_n \right]_{\rho}^{Set} \longrightarrow \left[ B \right]_{\rho}^{Set}
\]
is a morphism of realizability sets from $\left[ A_1 \otimes \ldots \otimes A_n \right]_{\rho}^{Set}$ to $\left[ B \right]_{\rho}^{Set}$. Recall that $\left[ A_1 \otimes \ldots \otimes A_n \right]_{\rho}^{Set} = \left[ A_1 \otimes \ldots \otimes A_n \right]_{\rho}^B$ and $\left[ B \right]_{\rho}^{Set} = \left[ B \right]_{\rho}^B$.

Proof. By induction on BLL derivations using using Lemma 4.10 for the axiom; using
Propositions 4.12, 4.13, 4.15, 4.16 for the term formers associated with $\otimes$, $\neg$, $\lambda x. \ll x<\rho$, and using
the above three lemmas for universal application and abstraction. We also make use of
obvious translations of syntactic constructs into categorical combinators, e.g. application
into evaluation and composition or storage into a combination of $\delta$, $\tau$, and functoriality of
$\ll x<\rho$. \qed

Proof of Theorem 3.1 Applying the iteration principle (Theorem 4.6) to the denotations
of 0 and $S$ shows that the function $\phi : \mathbb{N} \rightarrow \left[ \mathbb{N}_x \right]_{\rho}^{Set}$ is a morphism in $\mathcal{B}(\{x\})$ from $\mathbb{N}_x$ to
$\left[ \mathbb{N}_x \right]_{\rho}^B$. Similarly, the function $\psi : \left[ \mathbb{N}_{\rho(x)} \right]_{\rho}^{Set} \rightarrow \mathbb{N}$ is a morphism in the other direction (by polymorphic application).
Thus the function $f = \psi \circ \left[ \pi \right]_{\rho}^{Set} \circ \phi$ in the theorem is a morphism.
The result follows by the analysis in Example 4.5. We proceed analogously in the case of
dyadic lists, using the corresponding iteration principle. \qed

5 Additional Remarks

We notice that for any realizability set $A$ over $X$ the unique function $|A| \rightarrow \{\star\}$ is a morphism $A \rightarrow I$ of realizability sets witnessed by $e(\eta) = \lambda y. \lambda x. x$. In particular, this gives projections for $\otimes$. This shows that we can model an affine variant of BLL which has the following
additional rule Weak:
\[
\Gamma \vdash B \quad \Gamma, A \vdash B
\]
In particular, we see that this rule does not add computational strength.

Recall that we assumed the existence of a universe $\mathcal{U}$ in our ambient set theory which
is closed under $\mathcal{U}$-indexed products. As is well-known, no such universe exists in classical ZF
set theory but it is consistent with constructive set theories [9, 13]. We found it convenient to
assume the existence of such a $U$ because it allows the use of informal set-theoretic arguments
(provided they are constructive). For the reader who feels uneasy about such sleight-of-hand,
we offer the following ways of making this rigorous (all of which, however, complicate the
argument):

- Formalize the entire discussion in the Calculus of Inductive Constructions [2].
- Formalize the entire discussion in a realizability topos [6, 13].
- Making the previous point explicit, we can stipulate that the carrier sets of realizability
sets must be a subquotient (by a per) of the set of untyped lambda terms (not
necessarily linear!). And furthermore, a morphism between realizability sets must be
uniformly tracked by an untyped lambda term, in the obvious sense. This would allow
one to interpret polymorphic quantification as intersection of pers.

References


[5] A. Goerdt, Characterizing complexity classes by higher-type primitive recursive defini-

in Computer Science(LICS) 1999, pp. 464-476.

[7] M. Hofmann, Type Systems for Polynomial-time Computation, Habilitationsschrift,
Darmstadt University of Technology, 1999 (appears as Technical Report ECS-LFCS-99-406, Dept. of Computer Science, U. Edinburgh.) A revised and abridged version to
appear in Ann. Pure and Applied Logic under the title “Safe recursion with higher types
and BCK algebra ”.

[8] M. Hofmann, Programming Languages capturing complexity classes, SIGACT News
(Logic Column 9), ftp.research.bell-labs.com/dist/riecke/hofmann.ps.gz

135-165.

IEEE Symp. on Principles of Programming Languages (POPL’93), 1993.


Martin Hofmann
Div. of Informatics (LFCS)
University of Edinburgh
King’s Buildings
Edinburgh EH9 3JZ
U.K.
mxh@dcs.ed.ac.uk

Philip J. Scott
Dept. of Mathematics & Statistics
Univ. of Ottawa
585 King Edward
Ottawa, Ont.
Canada K1N6N5

phil@mathstat.uottawa.ca, or
phil@scie.uottawa.ca