# An application of category-theoretic semantics to the characterisation of complexity classes using higher-order function algebras

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**Abstract**

We use the category of presheaves over *PTIME*-functions in order to show that Cook and Urquhart's higher-order function algebra $PV^\omega$ defines exactly the *PTIME*-functions. As a byproduct we obtain a syntax-free generalisation of *PTIME*-computability to higher types.

By restricting to sheaves for a suitable topology we obtain a model for intuitionistic predicate logic with $\Sigma^b_1$-induction over $PV^\omega$ and use this to re-establish that the provably total functions in this system are in polynomial time computable. Finally, we apply the category-theoretic approach to a new higher-order extension of Bellantoni-Cook's system *BC* of safe recursion.

## 1 Introduction

Cook and Urquhart's system $PV^\omega$ [3] is a simply-typed lambda calculus providing constants to denote natural numbers and an operator for bounded recursion on notation like in Cobham's characterisation of polynomial-time computability.[1] Although functionals of arbitrary type can be defined in this system one can show that their presence does not increase the complexity of the definable first-order functions. Cook and Urquhart prove this by appealing to the normalisation theorem for simply-typed lambda calculus and reading off a "Cobham definition" of a function from the normal form of a first-order term of $PV^\omega$. In this paper we present an alternative method for proving such results which proceeds by exhibiting a model in which all first-order functions are in *PTIME* by definition.[2] This method provides a syntax-free generalisation of *PTIME*-computability to higher types and also seems to be more flexible with respect to extensions and variations of the syntax. We extend the method to intuitionistic predicate logic over $PV^\omega$ and apply it to a new higher-order extension of Bellantoni-Cook's system *BC* of safe recursion [1].

The central idea of our approach can be described as follows. Let $\mathbb{P}$ stand for the category of *PTIME*-functions viewed as a subcategory of the category $\mathcal{S}$ of sets and functions. The presheaf category $\hat{\mathbb{P}} \stackrel{\text{def}}{=} \mathcal{S}^{\mathbb{P}^{op}}$ contains a representable functor $O$ which has the property that the $\hat{\mathbb{P}}$-morphisms from $O$ to $O$ are in 1-1 correspondence to the *PTIME*-functions. Since $\hat{\mathbb{P}}$ is a cartesian closed category it furnishes a model for a simply-typed lambda calculus like $PV^\omega$. A term of type $o \to o$ in $PV^\omega$ gets interpreted as a morphism from $O$ to $O$ and thus yields a *PTIME*-function. Some technical work needs to be done in order to show that the $PV^\omega$-constants can be

[1] In *op. cit.* $PV^\omega$ also refers to an equational theory over this lambda calculus. We shall not be concerned with this extension here.

[2] In this paper *PTIME* denotes the class of functions on natural numbers computable in polynomial time on a Turing machine. See any textbook on complexity theory for a formal definition.

interpreted in $\hat{\mathbb{P}}$ and also to show that the function obtained via the interpretation coincides with the intended meaning of the term.

In the next two sections we describe the system $PV^\omega$ and its intended semantics. Section 4 contains the elaboration of the argument sketched above. In Section 5 we extend the method to intuitionistic predicate logic. Although, $\hat{\mathbb{P}}$ is a model for even higher-order intuitionistic logic it cannot be used directly because equality at $O$ is not decidable in $\hat{\mathbb{P}}$. In order to enforce decidability we move to the subcategory $Sh(\mathbb{P}) \subset \hat{\mathbb{P}}$ consisting of sheaves for an appropriate topology. The main result of that section is that $Sh(\mathbb{P})$ validates the scheme of bounded $\Sigma_1$-induction. Section 6 contains the material on $BC^\omega$; in Section 7 we describe several extensions and further applications which are currently under investigation.

# 2 Syntax

The system $PV^\omega$ is the simply-typed lambda calculus over one base type $o$ (for natural numbers in binary notation) and constants with types as indicated.

1. The constant zero: $0 : o$.
2. The two successor functions: $s_0, s_1 : o \to o$.
3. Integer division by two ("mixfix notation"): $\lfloor \frac{\cdot}{2} \rfloor$.
4. The (infix) functions *chop*, *pad*, and *smash* : $\div, \boxplus, \#, : o \to o \to o$.
5. The ternary conditional $Cond : o \to o \to o \to o$.
6. The bounded recursor $\mathcal{R} : o \to (o \to o \to o) \to (o \to o) \to o \to o$

Before explaining the intended meaning of these constants we first give a formal definition of the syntax. The types are defined by the grammar

$$\tau ::= o \mid \tau \to \tau'$$

We write $o^n \to \sigma$ as abbreviation for $\underbrace{o \to \ldots \to o}_{n-\text{times}} \to \sigma$. A type of the form $o^n \to o$ is called first-order; an element of such a type is also called first-order. The pseudo terms are given by

$$M ::= x \mid c \mid (MM') \mid \lambda x\colon \tau.M$$

where $x$ ranges over a countable set of variables and $c$ denotes one of the above constants. A *context* or *type assignment* is a partial function $\Gamma$ assigning types to variables. The empty such function ist mostly omitted and when $x \notin dom(\Gamma)$ we write $\Gamma, x\colon \tau$ for the extension of $\Gamma$ by $x \mapsto \tau$.

The typing judgement $\Gamma \vdash M : \tau$, read $M$ has type $\tau$ in context $\Gamma$, is now defined by the following rules.

$$(\textsc{Var}) \quad \frac{}{\Gamma \vdash x : \Gamma(x)} \quad \text{if } x \in dom(\Gamma)$$

$$(\textsc{Const}) \quad \frac{}{\Gamma \vdash c : \tau} \quad \text{if } c \text{ is a constant of type } \tau$$

$$(\textsc{App}) \quad \frac{\Gamma \vdash M : \tau \to \tau' \quad \Gamma \vdash M' : \tau}{\Gamma \vdash (MM') : \tau'}$$

$$(\textsc{Abs}) \quad \frac{\Gamma, x\colon \tau \vdash M : \tau'}{\Gamma \vdash \lambda x\colon \tau.M : \tau \to \tau'} \quad x \notin dom(\Gamma)$$

Terms are identified up to renaming of bound variables.

# 3 Set-theoretic semantics

The system $PV^\omega$ has an intended interpretation in the full type hierarchy over the set of natural numbers. To each type $\tau$ we associate a set $[\![\tau]\!]$ by

$$\begin{aligned}&[\![o]\!] = \mathbb{N}\\ &[\![\tau \to \tau]\!] = [\![\tau]\!] \to [\![\tau']\!]\end{aligned}$$

where $X \to Y$ denotes the set of all functions from $X$ to $Y$.

**Definition 3.1** *Let $n$ be a nonnegative integer and let $g : \mathbb{N}^n \to \mathbb{N}$, $h : \mathbb{N}^{n+2} \to \mathbb{N}$, and $k : \mathbb{N}^{n+1} \to \mathbb{N}$ be functions. We say that $f : \mathbb{N}^{n+1} \to \mathbb{N}$ is defined from $g, h, k$ by* bounded recursion on notation*, written $f = \mathcal{R}_n(g, h, k)$, if for $\vec{x} \in \mathbb{N}^n$*

$$\begin{aligned}&f(\vec{x}, 0) = g(\vec{x})\\ &f(\vec{x}, y) = \min(h(\vec{x}, y, f(\vec{x}, \lfloor y/2 \rfloor))\ ,\ k(\vec{x}, y)),\ \textit{if } y > 0\end{aligned}$$

**Proposition 3.2 (Cobham)** *If $g, h, k$ of appropriate arities are PTIME-functions so is $\mathcal{R}_n(g, h, k)$.*

**Proof.** We have $\mathcal{R}_n(g, h, k)(\vec{x}, y) \leq k(\vec{x}, y)$ so the obvious Turing machine computing $\mathcal{R}_n(g, h, k)$ runs in polynomial time. $\square$

For $x \in \mathbb{N}$ let $|x|$ denote $\lceil \log_2(x + 1) \rceil$; the length of the binary representation of $x$. Each constant $c : \tau$ is assigned a set-theoretic meaning $[\![c]\!] \in [\![\tau]\!]$ as follows.

$$[\![0]\!] = 0, \quad [\![s_0]\!](x) = 2x, \quad [\![s_1]\!](x) = 2x + 1, \quad [\![\lfloor \tfrac{\cdot}{2} \rfloor]\!](x) = \lfloor x/2 \rfloor,$$

$$[\![\#]\!](x)(y) = 2^{|x|\cdot|y|}, \quad [\![\boxplus]\!](x)(y) = x \cdot 2^{|y|}, \quad [\![\div]\!](x)(y) = \lfloor x/2^{|y|} \rfloor,$$

$$[\![\mathcal{R}]\!](g)(h)(k) = \mathcal{R}_0(g, h, k), \quad [\![\mathit{Cond}]\!](x)(y)(z) = \begin{cases} y, & \text{if } x = 0 \\ z, & \text{otherwise} \end{cases}$$

Although this will not be required later we remark that it is an immediate consequence of Cobham's theorem [2] that every *PTIME*-function can be obtained from the above basic functions by composition and bounded recursion on notation.

An *environment* is a partial function on variables. We omit the empty environment and use the notation $\eta[x \mapsto v]$ for the environment which maps $x$ to $v$ and acts like $\eta$ otherwise.

The interpretation of a term $M$ with respect to an environment $\eta$, written $[\![M]\!]\eta$ is defined as follows.

$$\begin{aligned}&[\![x]\!]\eta = \eta(x)\\ &[\![c]\!]\eta = [\![c]\!]\\ &[\![M M']\!]\eta = [\![M]\!]\eta[\![M']\!]\eta\\ &[\![\lambda x{:}\,\tau.M]\!]\eta(v) = [\![M]\!]\eta[x \mapsto v]\end{aligned}$$

We say that an environment $\eta$ *satisfies* a context $\Gamma$, written $\eta \models \Gamma$ if $\eta(x) \in [\![\Gamma(x)]\!]$ for each $x \in \mathit{dom}(\Gamma)$. The following is immediate by induction on derivations.

**Proposition 3.3** *If $\Gamma \vdash M : \tau$ and $\eta \models \Gamma$ then $[\![M]\!]\eta \in [\![\tau]\!]$.*

The following result is an immediate consequence of Thm. 6.16 of [3]:

**Theorem 3.4** *If $\vdash M : o^n \to o$ then $[\![M]\!] : \mathbb{N}^n \to \mathbb{N}$ is a PTIME-function.*

The proof in *loc. cit.* uses a translation of a $\beta\eta$-normal form of $M$ back into a first order variant of $PV^\omega$ which is known to contain *PTIME*-functions only. Our aim in the next section will be to give an alternative semantic proof of this theorem.

# 4 Presheaf semantics of $PV^\omega$

We will now construct another model of $PV^\omega$ in which every function from $o$ to $o$ is by definition in *PTIME*. We assume some basic knowledge of category theory, see e.g. the first chapter of [9] for the required concepts.

## 4.1 Presheaves over *PTIME*-functions

The category $\mathbb{P}$ of *PTIME*-functions is defined as follows. An object of $\mathbb{P}$ is a nonnegative integer (thought of as an arity); a morphism from $m$ to $n$ is a *PTIME*-function from $\mathbb{N}^m$ to $\mathbb{N}^n$. Composition in $\mathbb{P}$ is ordinary composition of functions. So $\mathbb{P}$ is a *concrete category* in the sense that the global sections functor $\Gamma : \mathbb{P} \to \mathcal{S}$ defined by $\Gamma(n) = \mathbb{N}^n$ and $\Gamma(f) = f$ is faithful. Here and in the sequel $\mathcal{S}$ denotes the category of sets and functions.

The functor category $\hat{\mathbb{P}} = \mathcal{S}^{\mathbb{P}^{op}}$ has as objects contravariant set-valued functors (presheaves) on $\mathbb{P}$ and natural transformations as morphisms. More elementarily, an object $F \in \hat{\mathbb{P}}$ assigns a set $F_n$ to each $\mathbb{P}$-object $n \in \mathbb{N}$ and to each $f \in \mathbb{P}(m,n)$ a function $F_f : F_n \to F_m$ in such a way that $F_{\mathrm{id}}(x) = x$ and $F_{f \circ g}(x) = F_g(F_f(x))$. A $\hat{\mathbb{P}}$-morphism $\mu : F \to F'$ assigns a function $\mu_n : F_n \to F'_n$ to each $\mathbb{P}$-object $n \in \mathbb{N}$ in such a way that $\mu_n(F_f(x)) = F'_f(\mu_m(x))$ for each $f \in \mathbb{P}(m,n)$ and $x \in F_m$.

For each $n \in \mathbb{N}$ we have the *representable presheaf* $\mathcal{Y}(n) \in \hat{\mathbb{P}}$ defined by $\mathcal{Y}(n)_m = \mathbb{P}(m,n)$ and $\mathcal{Y}(n)_f(g) = g \circ f$. The assignment $\mathcal{Y}$ extends to a functor $\mathcal{Y} : \mathbb{P} \to \hat{\mathbb{P}}$—the *Yoneda embedding*—by $\mathcal{Y}(f)_k(g) = f \circ g$ whenever $f \in \mathbb{P}(m,n)$ and $g \in \mathcal{Y}(m)_k = \mathbb{P}(k,m)$. The well-known *Yoneda Lemma* says that this functor is full and faithful; indeed, if $\mu : \mathcal{Y}(m) \to \mathcal{Y}(n)$ then $f \stackrel{\mathrm{def}}{=} \mu_m(\mathrm{id}) \in \mathbb{P}(m,n)$, and we have $\mu = \mathcal{Y}(f)$. Since $\mathbb{P}$ is concrete this function $f$ can also be obtained as the function $\mu_0 : \Gamma(m) \to \Gamma(n)$.

More generally, for every presheaf $F$ and $n \in \mathbb{N}$ we have a bijection between the set $F_n$ and the set $\hat{\mathbb{P}}(\mathcal{Y}(n), F)$ of natural transformations. One direction sends $\mu : \mathcal{Y}(n) \to F$ to $\mu_n(\mathrm{id} \in F_n)$. These bijections establish an isomorphism between $F$ and the presheaf $\hat{\mathbb{P}}(\mathcal{Y}(-), F)$ which sends $n$ to $\hat{\mathbb{P}}(\mathcal{Y}(n), F)$ and $f \in \mathbb{P}(m,n)$ to $\lambda\mu.\mu \circ \mathcal{Y}(f)$.

It is also well-known that $\hat{\mathbb{P}}$ is cartesian closed; on objects the product and exponential of two presheaves $F, G \in \hat{\mathbb{P}}$ are given by $(F \times G)_n = F_n \times G_n$ and $(F \Rightarrow G)_n = \hat{\mathbb{P}}(\mathcal{Y}(n) \times F, G)$. In particular, this means that an element of $(F \Rightarrow G)_n$ assigns to each $m \in \mathbb{N}$ and each morphism $f \in \mathbb{P}(m,n)$ a function $F_m \to G_m$. Notice here the similarity to the treatment of implication in Kripke models. A terminal object is given by $\top_n = \{()\}$.

We write $O$ for the representable presheaf $\mathcal{Y}(1)$; note that $O_n$ is the set of $n$-ary *PTIME*-functions. For presheaf $F \in \hat{\mathbb{P}}$ define $F^+ \in \hat{\mathbb{P}}$ by $F^+_n = F_{n+1}$ and $F^+_f(x) = F_{f \times \mathrm{id}}$.

**Lemma 4.1** *Let $F \in \hat{\mathbb{P}}$. The exponential $O \Rightarrow F$ is isomorphic to $F^+$.*

**Proof.** We have $(O \Rightarrow F)_n = \hat{\mathbb{P}}(\mathcal{Y}(n) \times \mathcal{Y}(1), F) \cong \hat{\mathbb{P}}(\mathcal{Y}(n+1), F) \cong F_{n+1} = F^+_n$. $\square$

## 4.2 Interpretation of $PV^\omega$ in $\hat{\mathbb{P}}$

Our aim is to interpret $PV^\omega$ types as presheaves and open terms as natural transformations. In particular, we want to interpret the base type $o$ as the representable presheaf $O$ so that by the Yoneda Lemma the first-order functions in this model are in one-to-one correspondence with the *PTIME*-functions.

A function type $\sigma \to \tau$ gets interpreted as the presheaf $[\![\sigma]\!] \Rightarrow [\![\tau]\!]$ where $[\![\sigma]\!]$ and $[\![\tau]\!]$ are the interpretations of $\sigma$ and $\tau$. Next, we want to assign a global element $[\![c]\!] : \top \to [\![\sigma]\!]$ to each constant $c{:}\,\sigma$. In $\hat{\mathbb{P}}$ like in every cartesian closed category we have the following natural isomorphisms.

$$\hat{\mathbb{P}}(\top, F \Rightarrow G) \cong \hat{\mathbb{P}}(F, G)$$
$$\hat{\mathbb{P}}(F, G \Rightarrow H) \cong \hat{\mathbb{P}}(F \times G, H)$$

To simplify the presentation we will treat these isomorphisms and the ones obtained from Lemmas 4.1as identities. The first-order constants can then be interpreted by applying the Yoneda embedding to their set-theoretic meanings. For example, we define $[\![s_0]\!] \in \hat{\mathbb{P}}(\top, O \Rightarrow O) \cong \hat{\mathbb{P}}(O, O)$ as $\mathcal{Y}(\lambda x.2x)$. That is, $[\![s_0]\!]_n(f) = \lambda\vec{x}.2f(\vec{x})$. Strictly speaking, $[\![s_0]\!]$ is defined as the transpose of this natural transformation along the isomorphism $\hat{\mathbb{P}}(\top, O \Rightarrow, O) \cong \hat{\mathbb{P}}(O, O)$.

For the recursor $\mathcal{R} : \tau$ we first note that

$$\hat{\mathbb{P}}(\top, [\![\tau]\!]_{\hat{\mathbb{P}}}) \cong \hat{\mathbb{P}}(O \times O^{++} \times O^{+} \times O, O) =: A$$

Now define $[\![\mathcal{R}]\!] \in A$ by

$$[\![\mathcal{R}]\!]_n(g, h, k, u) = \lambda\vec{x}.\mathcal{R}_n(g, h, k)(\vec{x}, u(\vec{x}))$$

where $\mathcal{R}_n$ is the first-order set-theoretic bounded recursor from Def. 3.1. It is easy to see that this is indeed a natural transformation.

In order to define an interpretation of terms in a category of presheaves a refinement of the notion of environment is needed. A C-environment $\rho$ consists of a list of distinct variables $(x_1, \ldots, x_n)$ and a list of presheaves $(X_1, \ldots, X_n)$. We write $dom(\rho)$ for the set of variables and $\rho(x)$ for $X_i$ if $x = x_i \in dom(\rho)$. Furthermore, we write $\mathrm{Dom}(\rho)$ for the cartesian product $((\ldots(\top \times X_1) \times X_2) \times \cdots \times X_n)$. If $x$ is a fresh variable and $F$ is a presheaf we write $\rho[x \mapsto F]$ for the C-environment obtained from $\rho$ by adding $x$ and $F$ to the end of the two lists. A C-environment *satisfies* a context $\Gamma$ if for each assignment $x{:}\,\sigma$ in $\Gamma$ we have $x \in dom(\rho)$ and $\rho(x) = [\![\sigma]\!]$. Now we can define an interpretation function $[\![M]\!]\rho$ on terms and C-environments in such a way that if $\Gamma \vdash M : \sigma$ and $\rho$ satisfies $\Gamma$ then $[\![M]\!]\rho$ is a morphism from $\mathrm{Dom}(\rho)$ to $[\![\sigma]\!]$.

In particular, this interpretation associates with every term $x_1{:}\,o, \ldots, x_n{:}\,o \vdash M{:}\,o$ a natural transformation $[\![M]\!]_{\hat{\mathbb{P}}}\rho : O^n \to O$ whose component at 0 is an $n$-ary *PTIME*-function.

### 4.3 Another proof of Theorem 3.4

In order to prove that the thus obtained function agrees with the intended set-theoretic meaning so as to obtain a proof of Theorem 3.4 we need to do a little bit of extra work. Let us denote by $[\![-]\!]_{\hat{\mathbb{P}}}$ the semantics in $\hat{\mathbb{P}}$ and by $[\![-]\!]_{\mathcal{S}}$ the set-theoretic semantics. For $F \in \hat{\mathbb{P}}$ we write $|F|$ for the set $F_0$ which is isomorphic to the set of global sections of $F$, i.e., morphisms from $\top$ to $F$. If $F, G$ are presheaves and $f \in |F \Rightarrow G|$ and $x \in |F|$ we write $app(f, x)$ for the element of $|G|$ obtained by transporting $f$ along the isomorphism $|F \Rightarrow G| \cong \hat{\mathbb{P}}(F, G)$ and applying it to $x \in F_0$.

Now we define a family of relations $R^\sigma \subseteq |[\![\sigma]\!]_{\hat{\mathbb{P}}}| \times [\![\sigma]\!]_{\mathcal{S}}$ by induction on $\sigma$ as follows.

$$\begin{array}{rcl} xR^o y & \iff & x = y \\ fR^{\sigma\to\tau} g & \iff & \forall x, y. xR^\sigma y \Rightarrow app(f, x) R^\tau g(y) \end{array}$$

The following slight adaptation of the "Fundamental Theorem of Logical Relations" [15] (which was stated for Henkin-models rather than cartesian closed categories) is proved by induction on terms.

**Proposition 4.2** *Suppose that* $\Gamma = x_1{:}\sigma_1, \ldots, x_n{:}\sigma_n$ *and* $\Gamma \vdash M : \tau$. *Let* $\eta$ *be an environment satisfying* $\Gamma$*and let* $\vec{\gamma} = (\gamma_1, \ldots, \gamma_n)$ *be such that* $\gamma_i R^{\sigma_i} \eta(x_i)$ *for* $i = 1, \ldots, n$. *Then* $([\![M]\!]_{\hat{\mathbb{P}}}\rho)_0(\vec{\gamma}) R^\tau [\![M]\!]_{\mathcal{S}}\eta$ *where* $\rho = ((x_1, \ldots, x_n), ([\![\sigma_1]\!], \ldots, [\![\sigma_n]\!]))$.

It follows that if $M : o^n \to o$ then $[\![M]\!]_{\mathcal{S}}(\vec{x}) = ([\![M]\!]_{\hat{\mathbb{P}}})_0(\vec{x})$ hence $[\![M]\!]_{\mathcal{S}}$ is a *PTIME*-function.

### 4.4 Hereditarily polynomial functionals

In this section we show how $\hat{\mathbb{P}}$ together with the above logical relation yields a syntax-free generalisation of *PTIME*-computability to higher types. A functional $f \in [\![\sigma]\!]_{\mathcal{S}}$ is called *hereditarily polynomial* if there exists $f' \in |[\![\sigma]\!]_{\hat{\mathbb{P}}}|$ such that $f' R^\sigma f$. Write $H^\sigma$ for the set of hereditarily polynomial functionals.

**Proposition 4.3**

$$\begin{aligned} H^{o^n \to \sigma \to \tau} = \{f \in [\![o^n \to \sigma \to \tau]\!]_{\mathcal{S}} \mid \forall m. \forall u \in \mathbb{P}(m,n). \\ \forall g \in H^{o^m \to \sigma}. \lambda\vec{x}.f(u(\vec{x}), g(\vec{x})) \in H^{o^m \to \tau}\} \end{aligned}$$

The $\subseteq$ direction is immediate from the definition. For the other direction we use the fact that every presheaf taken on by the interpretation, in particular $[\![\tau]\!]_{\hat{\mathbb{P}}}$, is *extensional* in the sense that an element of $([\![\tau]\!]_{\hat{\mathbb{P}}})_n$ is uniquely determined by the induced function $\mathbb{N}^n \to |[\![\tau]\!]_{\hat{\mathbb{P}}}|$. This in turn is shown by induction on types.

We can take Proposition 4.3 as a *definition* of the hereditarily polynomial functionals and prove directly that the functionals taken on by the set-theoretic interpretation are hereditarily polynomial. In this way one obtains a more elementary, but conceptually less clear proof of Thm. 3.4. A partial recursive analogue to the second-order fragment of the hereditarily polynomial functionals is known as *Banach-Mazur functionals* [11]. A functional $F \in [\![(o \to o) \to o]\!]$ is Banach-Mazur if for every partial recursive function $f(x, y)$ the function $\lambda x.F(x, y)$ is partial recursive.

We remark that the sets $R^n_\sigma$ defined by

$$R^n_\sigma = \{(f_{\vec{x}})_{\vec{x} \in \mathbb{N}^n} \mid \lambda\vec{x}.f_{\vec{x}} \in H^{o^n \to \sigma}\}$$

form a *Kripke logical relation with varying arity* on the full type hierarchy over $\mathbb{N}$ in the sense of Jung and Tiuryn [8]. They point out that a such a relation corresponds to a Henkin model in a category of presheaves, but insinuate that this view could not be used in order to state properties of the full type hierarchy. Our use of an ordinary logical relation in Prop. 4.2 shows that this is not quite the case. So it is a matter of taste whether to use a functor category together with an ordinary logical relation or a Kripke logical relation. In the present situation the former approach is conceptually clearer as it starts from a "universe", namely $\hat{\mathbb{P}}$, in which the restriction to *PTIME* is built in.

## 5 Application to intuitionistic predicate logic

Indeed, $\hat{\mathbb{P}}$ supports much more linguistic constructions than merely the simply-typed lambda calculus. It is well-known, see e.g. [9], that every presheaf category forms a topos, thus a model for intuitionistic higher-order logic and even a certain variant of intuitionistic set theory. Unfortunately, in $\hat{\mathbb{P}}$ equality of natural numbers, i.e., on $O$ is not decidable as decidability amounts to the statement that two *PTIME*-functions are either equal or differ at each argument. Therefore, we cannot use $\hat{\mathbb{P}}$ directly to interpret for example Cook and Urquhart's $IPV^\omega$, an extension of $PV^\omega$ by intuitionistic predicate logic.

To repair this we replace $\hat{\mathbb{P}}$ by a subcategory of sheaves.

**Definition 5.1** *Let $n \in \mathbb{N}$ be an object of $\mathbb{P}$. A* cover *of $n$ consists of a PTIME-function $t \in \mathbb{P}(n,1)$ with range $\{1,\dots,\ell\}$ for some $\ell \in \mathbb{N}$. We write $Cov(n)$ for the set of covers of $n$. A presheaf $F \in \hat{\mathbb{P}}$ is a* sheaf *if for each $n$ and $t \in Cov(n)$ with range $1\dots\ell$ and elements $f_i \in F_n$ for $i = 1\dots\ell$ there exists a unique element $f \in F_n$ such that for every $u : \mathbb{P}(m,n)$ with $t(u(\vec{x})) = i$ we have $F_u(f) = F_u(f_i)$. We say that $f$ is obtained by pasting $f_1,\dots,f_\ell$.*

We view a cover as a finite *PTIME*-decidable partition of $\mathbb{N}^n$. A sheaf admits the definition of elements by case distinction over this partition. One can show that a sheaf in the above sense is a sheaf for a suitable Grothendieck topology on $\mathbb{P}$. Therefore, the subcategory $Sh(\mathbb{P})$ of $\hat{\mathbb{P}}$ is a topos. Products and exponentials in $Sh(\mathbb{P})$ are computed as in $\hat{\mathbb{P}}$.

**Proposition 5.2** *The representable presheaf $O = \mathcal{Y}(1)$ is a sheaf.*

**Proof.** Let $t \in \mathbb{P}(n,1)$ be a cover with range $\{1,\dots,\ell\}$ and let $f_i \in O_n (= \mathbb{P}(n,1))$ be a family of *PTIME*-functions. We define $f \in \mathbb{P}(n,1)$ by $f(\vec{x}) = f_{t(\vec{x})}(\vec{x})$. The verification is left to the reader. $\square$

This shows that the presheaves taken on by the interpretation of $PV^\omega$ types are in fact sheaves.

We now consider many-sorted first-order logic over $PV^\omega$. Formulas are built up from equations $s = t$ between $PV^\omega$-terms by boolean connectives $\vee, \wedge, \Rightarrow$ and the typed quantifiers $\exists x{:}\,\tau$ and $\forall x{:}\,\tau$. We abbreviate $\forall x{:}\,o$ and $\exists x{:}\,o$ by $\forall x$ and $\exists x$ respectively.

If $\varphi$ is a formula with free variables recorded in a context $\Gamma$ and $\rho \models \Gamma$ then we can associate a subsheaf $[\![phi]\!] \hookrightarrow \mathrm{Dom}(\rho)$ by induction on the structure of $\varphi$ using the internal Heyting algebra structure of the subobject classifier in $Sh(\mathbb{P})$. Such a subsheaf consists of a family of subsets $[\![\varphi]\!]_n \subseteq \mathrm{Dom}(\rho)_n$ closed under the reindexing functions $\mathrm{Dom}(\rho)_u : \mathrm{Dom}(\rho)_n \to \mathrm{Dom}(\rho)_n$ for $u \in \mathbb{P}(m,n)$ and also closed under pasting.

Fortunately, these subsets admit an explicit description known as *Kripke-Joyal semantics* [12, Ch. VI] which we are going to describe for the present case. Let $\rho, \varphi$ be as above. If $\gamma \in \mathrm{Dom}(\rho)_n$ then we write $n, \gamma \Vdash_\rho \varphi$ for $\gamma \in ([\![\varphi]\!]\rho)_n$ and say $\varphi$ is valid at $n$ under $\gamma, \rho$. We say that $n, \gamma \Vdash_\rho \varphi$ is well-formed to mean that $\gamma \in \mathrm{Dom}(\rho)_n$ and $\rho$ satisfies a context in which $\varphi$ is well-formed. A formula $\varphi$ is valid in $Sh(\mathbb{P})$ if $n, \gamma \Vdash_\rho \varphi$ holds whenever it is well-formed. If $\gamma \in \mathrm{Dom}(\rho)_n$ and $u \in \mathbb{P}(m,n)$ then we write $\gamma\lceil u$ for $\mathrm{Dom}(\rho)_u(\gamma)$.

**Proposition 5.3** *The following equivalences of validities hold if they are well-formed.*

$$
\begin{array}{lcl}
n,\gamma \Vdash_\rho s = t & \iff & ([\![s]\!]\rho)_n(\gamma) = ([\![t]\!]\rho)_n(\gamma) \\
n,\gamma \Vdash_\rho \varphi \Rightarrow \psi & \iff & \forall m\,.\,\forall u \in \mathbb{P}(m,n)\,.\,m,\gamma\lceil u \Vdash_\rho \varphi \Rightarrow m,\gamma\lceil u \Vdash_\rho \psi \\
n,\gamma \Vdash_\rho \varphi \wedge \psi & \iff & n,\gamma \Vdash_\rho \varphi \ \wedge\ n,\gamma \Vdash_\rho \psi \\
n,\gamma \Vdash_\rho \varphi \vee \psi & \iff & \exists t \in Cov(n)\,.\,\forall m\,.\,\forall u \in \mathbb{P}(m,n)\,.\,t \circ u \textit{ constant } \Rightarrow \\
 & & \quad m,\gamma\lceil u \Vdash_\rho \varphi \ \vee\ m,\gamma\lceil u \Vdash_\rho \psi \\
n,\gamma \Vdash_\rho \neg\varphi & \iff & \forall u \in \mathbb{P}(m,n)\,.\,m,\gamma\lceil u \nVdash_\rho \varphi \\
n,\gamma \Vdash_\rho \forall x{:}\,\tau\,.\,\varphi & \iff & \forall m\,.\,\forall u \in \mathbb{P}(m,n)\,.\,\forall v \in [\![\tau]\!]_m\,.\,m,(\gamma\lceil u, v) \Vdash_{\rho[x \mapsto [\![\tau]\!]]} \varphi \\
n,\gamma \Vdash_\rho \exists x{:}\,\tau\,.\,\varphi & \iff & \exists v \in [\![\tau]\!]_n\,.\,n,(\gamma, v) \Vdash_{\rho[x \mapsto [\![\tau]\!]]} \varphi \\
n,\gamma \Vdash_\rho \forall x\,.\,\varphi & \iff & n+1,(\gamma,\pi) \Vdash_{\rho[x \mapsto O]} \varphi
\end{array}
$$

**Proof.** See *loc. cit.* for all but the last two clauses. The last clause (in which $\pi$ stands for the last projection in $O_{n+1}$) is an easy consequence of the general clause for universal quantifiers. The clause for the existential quantifier is slightly simpler than the one in *loc. cit.* because in the present case covers are always disjoint. $\square$

Of course, we could now forget about $Sh(\mathbb{P})$ and take the above as an (ad-hoc) definition of a forcing relation. The advantage of using the topos is again a conceptual one. It provides a more general and different point of view. An immediate technical advantage is that we get for free that validity is compatible with intuitionistic logic in the sense that if $n, \gamma \Vdash_\rho \varphi$ and $\psi$ follows from $\varphi$ in intuitionistic predicate logic then $n, \gamma \Vdash_\rho \psi$. In particular, $n, \gamma \Vdash_\rho \varphi$ for all $n, \gamma, \rho$ if $\varphi$ is intuitionistically valid. It is also clear from the interpretation of atomic formulas that all the defining equations for $PV^\omega$ terms are valid.

**Proposition 5.4** *Suppose that $x_1{:}\, o, \ldots, x_n{:}\, o, y{:}\, o \vdash s : o$ is a $PV^\omega$-term and that $\exists y.s(\vec{x}, y)$ is valid in $Sh(\mathbb{P})$. Then there exists a PTIME-function $t \in \mathbb{P}(n, 1)$ such that $[\![s]\!]_{\mathcal{S}}(\vec{x}, t(\vec{x})) = 0$.*

**Proof.** This follows immediately from Prop. 4.2 and Prop. 5.3. $\square$

**Proposition 5.5** *Equality at type $o$ is decidable in $Sh(\mathbb{P})$, i.e., the formula $\varphi$ defined as $x = y \vee \neg x = y$ is valid.*

**Proof.** It suffices to show that $2, \gamma \Vdash_\rho \varphi$ where $\mathrm{Dom}(\rho)$ is $O \times O$ and $\gamma$ consists of the two projections in $O_2$. To see this, we consider the cover $t \in \mathbb{P}(2, 1)$ defined by $t(x, y) = 1$ if $x = y$ and $t(x, y) = 2$ otherwise. If $u = (u_x, u_y) \in \mathbb{P}(m, 2)$ has the property that $t \circ u$ is constant then either $u_x = u_y$ or $u_x$ and $u_y$ differ at all arguments. In the first case we have $m, \gamma\lceil u \Vdash_\rho x = y$, in the second case we have $m, \gamma\lceil u \Vdash_\rho \neg x = y$ hence the result. $\square$

Let $Lessequ(x, y)$ stand for a $PV^\omega$-term such that $[\![Lessequ]\!]_{\mathcal{S}}(x, y) \Leftrightarrow x \leq y$. An $\Sigma^b_1$ formula is one of the form $\exists y.Lessequ(y, k) = 0 \wedge q = 0$ where $k, q$ are terms of $PV^\omega$ and $y$ is not free in $k$.

A formula of the form

$$PIND(\varphi) \stackrel{def}{\Longleftrightarrow} \varphi(0) \wedge (\forall x.\varphi(\lfloor \tfrac{x}{2} \rfloor) \Rightarrow \varphi(x)) \Rightarrow \forall x.\varphi(x)$$

where $\varphi$ is $\Sigma^b_1$ is called an instance of NP-induction.

**Theorem 5.6** *All instances of NP-induction are valid in $Sh(\mathbb{P})$.*

**Proof.** Let $\varphi(x) \equiv \exists y.Lessequ(y, k(x)) \wedge q(x, y)$ be a $\Sigma^b_1$-formula. Since $PV^\omega$ contains all $PTIME$-functions and equality is decidable in $Sh(\mathbb{P})$ we can find a term $s$ such that $PIND(\varphi)$ is equivalent (in $Sh(\mathbb{P})$) to

$$\varphi(0) \wedge (\forall x \forall y.\exists z.s(x, y, z) = 0) \Rightarrow \forall x.\varphi(x)$$

where $s$ is such that (in $Sh(\mathbb{P})$)

$$s(x, y, z) = 0 \Longleftrightarrow A(\lfloor \tfrac{x}{2} \rfloor, y) \Rightarrow A(x, z)$$

and $A$ is the decidable matrix of $\varphi$.

Now suppose that the antecedent is valid at $n$ under $\gamma, \rho$.

Validity of $\varphi(0)$ gives a function $g \in \mathbb{P}(n,1)$ such that $n, (\gamma, g) \Vdash_{\rho[x\mapsto O]} A(0,x)$. Define $k' \in \mathbb{P}(n+1,1)$ by $k'(\vec{x},x) = [\![k]\!]\rho[x \mapsto O]_0(\gamma{\restriction}\vec{x},x)$. Naturality of $[\![k]\!]$ yields $([\![k]\!]\rho[x \mapsto O])_n(\gamma,u)(\vec{x}) = k'(\vec{x},u(\vec{x}))$ for all $u \in \mathbb{P}(n,1)$ and $\vec{x} \in \mathbb{N}^n$. Analogously, define $q' \in \mathbb{P}(n+2,1)$. We conclude that $g(\vec{x}) \leq k'(\vec{x},0)$ and $q'(\vec{x},0,g(\vec{x})) = 0$.

Similarly, validity of $\forall x \forall y.\exists z.s(x,y,z) = 0)$ at $n$ gives a function $h \in \mathbb{P}(n+2,1)$ such that $y \leq k'(\vec{x},\lfloor\frac{x}{2}\rfloor) \wedge q'(\vec{x},\lfloor\frac{x}{2}\rfloor,y) = 0$ implies $h(\vec{x},x,y) \leq k'(\vec{x},x)$ and $q'(\vec{x},x,h(\vec{x},x,y)) = 0$. Now let $f \in \mathbb{P}(n+1,1)$ be $\mathcal{R}_n(g,h,k)$. It follows by meta-level induction on $|x|$ that $f(\vec{x},x) \leq k'(x)$ and $q'(\vec{x},x,f(\vec{x},x)) = 0$ hence $n,\gamma \Vdash \forall x.\varphi(x)$. □

This together with Prop. 5.4 yields another proof of Cook and Urquhart's result that the Skolem functions of intuitionistic predicate logic over $PV^\omega$ with NP-induction are the *PTIME*-functions.

# 6 Higher-order extension of Cook-Bellantoni's *BC*

In this section we apply the semantic method to a higher-order extension of Bellantoni-Cook's function algebra *BC* which also captures the complexity class *PTIME*, but which does not contain explicit size restrictions like the function $k$ in the recursor of $PV^\omega$. Unlike $PV^\omega$ the higher-order extension of *BC* we are going to describe is new.

We begin with a brief description of the first-order system *BC*. For more detailed account we refer to [1].

In *BC* the variables of a function $f$ are split into two zones separated by a semicolon: $f(\vec{x};\vec{y})$. The $\vec{x}$-variables are called *normal*; the $\vec{y}$-variables are called *safe*. Both range over binary natural numbers like in $PV^\omega$. We may substitute any term for a safe variable, but may substitute a term for a normal variable only if it does not depend on safe variables.

The system provides unbounded recursion on notation; however the recursive argument of a function must be normal, whereas recursive calls are allowed via safe variables only. In other words, if $g(\vec{x};\vec{y})$ and $h(x,\vec{x};y,\vec{y})$ have already been defined then we may define $f(x,\vec{x};\vec{y})$ by

$$\begin{aligned} &f(0,\vec{x};\vec{y}) = g(\vec{x};\vec{y}) \\ &f(x,\vec{x};\vec{y}) = h(x,\vec{x};f(\lfloor x/2 \rfloor,\vec{x};\vec{y}),\vec{y}), \text{if } x > 0 \end{aligned}$$

The main result of [1] is that the *PTIME*-functions can be obtained as the functions $f(\vec{x};)$ from certain basic functions via the above processes of "safe composition" and "safe recursion on notation". One part of the proof proceeds by showing inductively that whenever a function $f(\vec{;}\vec{y})$ is definable then $|f(\vec{x};\vec{y})| \leq p(|\vec{x}|) + \max(|\vec{y}|)$. That is, the size of $f(\vec{x};\vec{y})$ is polynomial in the size of the normal variables (on which we can recur) and constant in the size of the safe variables (which we can only carry through as parameters).

In order to obtain a higher-order generalisation of this system we will study a presheaf topos based on the *PTIME*-computable functions which satisfy such a growth restriction.

**Definition 6.1** *A function $f : \mathbb{N}^m \times \mathbb{N}^n \to \mathbb{N}$ is $(m,n)$-polymax if it is in PTIME and there exists an $m$-variate polynomial $p$ such that*

$$|f(\vec{x},\vec{y})| \leq p(|\vec{x}|) + \max(|\vec{y}|)$$

*for each $\vec{x} \in \mathbb{N}^m$ and $\vec{y} \in \mathbb{N}^n$.*

*The category $\mathcal{B}$ has pairs $(m,n)$ of natural numbers as objects; a $\mathcal{B}$-morphism from $(m,n)$ to $(1,0)$ consists of an $m$-ary PTIME-function; a $\mathcal{B}$-morphism from*

*$(m,n)$ to $(0,1)$ consists of an $(m,n)$-polymax function. A morphism from $(m,n)$ to $(m',n')$ consists of $m'$ morphisms from $(m,n)$ to $(1,0)$ and $n'$ morphisms from $(m,n)$ to $(0,1)$.*

It follows by an easy calculation that this is indeed a category, i.e., that the set-theoretic componentwise composition of two $\mathcal{B}$-morphisms is a $\mathcal{B}$-morphism again.

Our aim is to interpret a higher-order version of $BC$ in the presheaf category $\hat{\mathcal{B}}$. We first notice that if $F \in \hat{\mathbb{P}}$ then we can define $L(F) \in \hat{\mathcal{B}}$ by $L(F)_{(m,n)} \stackrel{\text{def}}{=} F_m$ and $L(F)_{\vec{u},\vec{v}} = F_{\vec{u}}$. Conversely, if $G \in \hat{\mathcal{B}}$ then we can define $R(G) \in \hat{\mathbb{P}}$ by $R(G)_m \stackrel{\text{def}}{=} G_{(m,0)}$. As indicated, $L$ is left adjoint to $R$ and the composition $L \circ R$ induces a comonad $\Box : \hat{\mathcal{B}} \to \hat{\mathcal{B}}$ on $\hat{\mathcal{B}}$ which is explicitly described by $\Box(F)_{(m,n)} \stackrel{\text{def}}{=} F_{(m,0)}$ and $\Box(F)_{(\vec{u},\vec{v})} \stackrel{\text{def}}{=} F_{(\vec{u},())}$.

We have a natural transformation $\varepsilon(F) : \Box(F) \to F$ defined by $\varepsilon(F)_{(m,n)}(x) = F_\pi(x)$ where $\pi \in \mathcal{B}((m,n),(m,0))$ is the projection. Furthermore, if $f : \Box(F) \to G$ then we can obtain a morphism $f^! : \Box(F) \to \Box(G)$ by $f^!_{(m,n)}(x) = f_{(m,0)}(x)$. The functor $\Box$ preserves products and the terminal object in $\hat{\mathcal{B}}$. We remark that such a product-preserving comonad can be seen as a counterpart to an S4-modality under the Curry-Howard isomorphism.

We define the presheaf $O \in \hat{\mathcal{B}}$ by $O_{(m,n)} = \mathcal{B}((m,n)\,,\,(0,1))$, i.e., as the presheaf of polymax functions. We will use the presheaf $O$ to model safe natural numbers and $\Box O$ to model the normal ones. More generally, the comonad $\Box$ yields a normal variant of every type. In $\hat{\mathcal{B}}$ we have the following generalisation of Lemma 4.1.

**Lemma 6.2** *For $F \in \hat{\mathcal{B}}$ we have the following natural isomorphisms:*

$$\begin{array}{l}(O \Rightarrow F)_{(m,n)} \cong F_{(m,n+1)}\\ (\Box O \Rightarrow F)_{(m,n)} \cong F_{(m+1,n)}\end{array}$$

In particular, the $\hat{\mathcal{B}}$-morphisms from $\Box O^m \times O^n$ to $O$ are in 1-1 correspondence with the $(m,n)$-polymax functions.

Now, Lemma 6.2 allows us to lift first-order safe recursion on notation to a global element in $\hat{\mathcal{B}}$ of the following presheaf:

$$\Box O \Rightarrow O \Rightarrow (\Box O \Rightarrow O \Rightarrow O) \Rightarrow O$$

In a similar way we can define appropriately typed constants corresponding to the other basic functions and constructions in the first-order system $BC$. Interestingly, the syntactic formulation of a simply-typed lambda calculus which supports a comonad and thus could be interpreted in $\hat{\mathcal{B}}$ is not straightforward. There exists one formulation due to Pfenning [4] in which the passage from normal to safe values (the transformation $\varepsilon$) as well as lifting of functions depending on normal variables only (the $f \mapsto f^!$) operation are both witnessed by special term formers. In [7] the author has presented a lambda calculus with subtyping in which (like in first-order $BC$) this does not happen. We will describe the relevant fragment of this system, to be called $BC^\omega$.

## 6.1 The system $BC^\omega$

The syntax of $BC^\omega$ is defined as follows.

- $o$ is a type. If $\sigma, \tau$ are types so are $\sigma \to \tau$ and $\Box\sigma \to \tau$. Note that $\Box\tau$ is not a type.
- The subtyping relation between types is the partial order generated by the axiom $\sigma \to \tau \leq \Box\sigma \to \tau$ and the rule that $\sigma \to \tau \leq \sigma' \to \tau'$ whenever $\sigma' \leq \sigma$ and $\tau \leq \tau'$.

- The constants of $BC^\omega$ with their types are $0{:}\,o$, $s_0, s_1{:}\,o \to o$, the safe recursor $\mathcal{R}{:}\,\Box o \to o \to (\Box o \to o \to o) \to o$, and definition by cases $\mathcal{C} : o \to o \to (o \to o) \to (o \to o) \to o$.
- A context of $BC^\omega$ is a sequence of bindings of the form $x{:}\,\sigma$ or $x{:}\,\Box\sigma$. If $x{:}\,\sigma$ or $x{:}\,\Box\sigma$ appears in $\Gamma$ then $x \in dom(\Gamma)$ and $\Gamma(x) = \sigma$. A context is called *modal* if all its bindings are of the second kind.
- The terms of $BC^\omega$ together with their types are defined by the following rules.

$$(\textsc{Var}) \quad \frac{x \in dom(\Gamma)}{\Gamma \vdash x : \Gamma(x)} \quad \text{if } x \in dom(\Gamma)$$

$$(\textsc{Const}) \quad \frac{c : \tau \text{ a constant}}{\Gamma \vdash c : \tau}$$

$$(\textsc{Sub}) \quad \frac{\Gamma \vdash M : \tau \qquad \tau \leq \tau'}{\Gamma \vdash M : \tau'}$$

$$(\textsc{App}) \quad \frac{\Gamma \vdash M : \tau \to \tau' \quad M' : \tau}{\Gamma \vdash (MM') : \tau'}$$

$$(\textsc{App-}\Box) \quad \frac{\Gamma, \Delta \vdash M : \Box\tau \to \tau' \quad \Gamma \vdash M' : \tau}{\Gamma, \Delta \vdash (MM') : \tau'} \quad \Gamma \text{ modal}$$

$$(\textsc{Abs}) \quad \frac{\Gamma, x{:}\,\tau \vdash M : \tau'}{\Gamma \vdash \lambda x{:}\,\tau.M : \tau \to \tau'} \quad x \notin dom(\Gamma)$$

$$(\textsc{Abs-}\Box) \quad \frac{\Gamma, x{:}\,\Box\tau \vdash M : \tau'}{\Gamma \vdash \lambda x{:}\,\tau.M : \Box\tau \to \tau'} \quad x \notin dom(\Gamma)$$

The set-theoretic interpretation of $BC^\omega$-types is defined by $[\![o]\!] = \mathbb{N}$ and $[\![\sigma \to \tau]\!] = [\![\Box\sigma \to \tau]\!] = [\![\sigma]\!] \to [\![\tau]\!]$. Variables, application, and abstractions are modelled as in the case of $PV^\omega$. The intended set-theoretic meaning of the constants is given by

$$[\![0]\!] = 0, \quad [\![s_0]\!](x) = 2x, \quad [\![s_1]\!](x) = 2x+1,$$

$$[\![\mathcal{C}]\!](0, g, h_0, h_1) = g \quad [\![\mathcal{C}]\!](2x, g, h_0, h_1) = h_0(x) \quad [\![\mathcal{C}]\!](2x+1, g, h_0, h_1) = h_1(x)$$

$$[\![\mathcal{R}]\!](0, g, h) = g$$

$$[\![\mathcal{R}]\!](x, g, h) = h(x, [\![\mathcal{R}]\!](\lfloor x/2 \rfloor, g, h))$$

## 6.2 Presheaf interpretation of $BC^\omega$

Our aim is to show that every closed term of type $\Box o^n \to o$ denotes an $n$-ary *PTIME*-function. To do this, we interpret $BC^\omega$ in the presheaf category $\hat{\mathcal{B}}$ by decreeing that

$$\begin{aligned} [\![o]\!] &= O \\ [\![\sigma \to \tau]\!] &= [\![\sigma]\!] \Rightarrow [\![\tau]\!] \\ [\![\Box\sigma \to \tau]\!] &= \Box[\![\sigma]\!] \Rightarrow [\![\tau]\!] \end{aligned}$$

The interpretation of the constants follows the pattern set out in Section 4 and the informal description in Section 6. To interpret the subsumption rule we define a coercion morphism $c_{\sigma,\tau} : [\![\sigma]\!] \to [\![\tau]\!]$ whenever $\sigma \leq \tau$ by induction on the definition of subtyping. In particular, we define $c_{\sigma\to\tau,\Box\sigma\to\tau}$ as

$$[\![\sigma]\!] \Rightarrow [\![\tau]\!] \xrightarrow{\varepsilon([\![\sigma]\!])\Rightarrow[\![\tau]\!]} \Box[\![\sigma]\!] \Rightarrow [\![\tau]\!]$$

In order to formally interpret terms we use two C-environments one accounting for the modal variables in a context $\Gamma$ the other one for the non-modal ones. If $\rho, \psi$ are C-environments we say that $\rho, \psi$ satisfies $\Gamma$ if $dom(\varphi) \cap dom(\psi) = \emptyset$ and whenever $x{:}\Box\sigma$ occurs in $\Gamma$ then $\rho(x) = [\![\sigma]\!]$ and if $x{:}\sigma$ occurs in $\Gamma$ then $\psi(x) = [\![\sigma]\!]$. The semantics of a term $\Gamma \vdash M : \tau$ w.r.t. two C-environments $\rho, \psi$ satisfying $\tau$ is then a morphism $[\![M]\!](\rho,\psi) : \Box\mathrm{Dom}(\rho) \times \mathrm{Dom}(\psi) \longrightarrow [\![\tau]\!]$. We define this semantics by induction on typing derivations and then show that it is independent of the chosen derivation, which is not unique due to the presence of subtyping. This coherence proof amounts to checking that application and abstraction commute with the coercions in the appropriate sense which in turn is obvious from the definition.

Apart from the constants and subsumption the only rules differing substantially from the situation in $PV^\omega$ are VAR and APP-$\Box$. For the former we use a projection followed by an instance of $\varepsilon$ making use of the fact that $\Box$ preserves products. For APP-$\Box$ we employ the lifting operation $f \mapsto f^!$.

Now, for presheaf $F \in \hat{\mathcal{B}}$ we put $|F| \stackrel{\mathrm{def}}{=} F_{(0,0)}$ and define a family of relations $R^\sigma \subseteq |[\![\sigma]\!]_{\mathcal{B}}| \times [\![\sigma]\!]_{\mathcal{S}}$ exactly as in Section 4. Note that $|\Box F| = |F|$ so that the case $\Box\sigma \to \tau$ can be treated just as $\sigma \to \tau$. Again, we can show that the respective meanings of closed terms are related and conclude

**Theorem 6.3** *If $M : \Box o^m \times o^n \to o$ is a closed term of $BC^\omega$ then $[\![M]\!]_{\mathcal{S}}$ is a $(m,n)$-polymax function.*

# 7 Further work

The work reported here is raw material for the author's forthcoming habilitation thesis. Several strands of further development are currently under investigation in this context.

Important from the point of view of programming languages is the study of the strength of safe recursion with result type other than $o$. In [7] we have introduced a *linear* recursor of type

$$\Box o \to \sigma \to (\Box o \to \sigma \multimap \sigma) \to \sigma$$

where $\sigma = o^n \to o$ and $f : \sigma \multimap \sigma$ means intuitively that the functional argument to $f$ is applied at most once. The semantics of the linear recursor is the same as the one of $\mathcal{R}$. In *loc. cit.* we sketch a category-theoretic proof that this linear recursor does not increase the strength of $BC^\omega$. Its advantage is that it allows for shorter and more direct definitions of functions. The main contribution of [7] consists of the definition of a formal system in which such linear function types can be expressed.

Without the linearity constraint safe recursion with higher result type allows us to define exponentiation. We conjecture that safe recursion with result type $o \to o$ defines precisely the Kalmar elementary functions. Leivant and Marion [10] have given a characterisation of polynomial space using tiered recursion with higher result type. A similar result for safe recursion which uses a somewhat adhoc modification of safe recursion with result type $\Box o \to o$ is also announced in [7]. In order to prove it one uses presheaves over a certain category of second-order functionals computable in polynomial space. We conjecture that safe recursion with first-order result types other than $o$, e.g., product types $o \times o$, list types, or trees does not increase the strength, but of course provides a more comfortable style of programming.

Another line of further work consists of applications to logical systems as opposed to function algebras sketched in Section 5. The construction of the category $Sh(\mathbb{P})$ as in Def. 5.1 makes sense for $\mathbb{P}$ replaced by any category of functions admitting definition by cases. Therefore, it is possible to extend the development to richer systems such as the intuitionistic bounded arithmetic with $\Sigma^b_i$-induction (in this case one would consider sheaves over Buss' function class $\Box^p_i$ which consists of the polynomial time closure of the characteristic functions of predicates in the $i$-th level of the polynomial hierarchy.). This would provide a new proof of Harnik's results [6]. We have also made an attempt at generalising the approach to second order intuitionistic bounded arithmetic which according to Buss [2, Ch. 10] captures polynomial space.

Concerning the realm of safe recursion it might be worthwhile to study the logical structure of the topos $\hat{\mathcal{B}}$ or a suitable subcategory of sheaves so as to obtain a logical system corresponding to $BC^\omega$.

Unfortunately, the sheaf-theoretic approach does not seem to be applicable to classical logic. In particular, the double negation topology on $\hat{\mathbb{P}}$ is trivial because if $F$ is a $\neg\neg$-sheaf then $F_n \cong \mathbb{N}^n \to F_0$ as the sieve consisting of all constant maps is a $\neg\neg$-cover. Therefore, the category of $\neg\neg$-sheaves is equivalent to the category of sets. Of course, we can translate syntactic methods such as Cook and Urquhart's version of the *Dialectica* interpretation into category-theoretic language, but apparently no further insight can be drawn from such an exercise.

On the other hand, we might draw profit of the intuitionistic nature of $Sh(\mathbb{P})$ and similar categories and look for principles which are incompatible with classical logic. A candidate for such a principle is the following polynomial version of Church's thesis, where $S(e, x, t, y)$ means that Turing machine $e$ on input $x$ halts after $t$ steps with result $y$.

$$\forall f\colon o \to o \exists k, C, e. \forall x. \exists t \leq |x|^k + C. S(e, x, t, f(x))$$

We conjecture that this principle is valid in $Sh(\mathbb{P})$. Together with the axiom of choice for base type which is valid in $Sh(\mathbb{P})$ this contradicts the principle of excluded middle.

# 8 Related work

Closely related to the present work is Phil Mulry's thesis [13] where a topos of sheaves—the *recursive topos*—over the category of partial recursive functions is investigated. It is noted that the morphisms from $O \Rightarrow O$ to $O$ in this topos correspond to the Banach-Mazur functionals [11]. However, no formal correspondence like Prop. 4.3 above is provided. Also, the recursive topos is studied *per se*; applications to logical systems appear only in so far as the recursive topos provides a model for certain systems of recursive analysis. One of Mulry's central results is a characterisation of the canonical topology on the topos of presheaves over the partial recursive functions. His characterisation makes essential use of recursion theory and does not obviously carry over to the case of polynomial time. We do not know whether the topology on $\mathbb{P}$ used in Section 5 is the canonical one.

We further report that a construction similar to the hereditarily polynomial functionals has been used by Ehrhard and Colson [5] to extend another first-order notion (Vuillemin-Milner sequentiality) to higher types.

The present work should be distinguished from Otto's work [14] where category-theoretic methods are used to analyse the syntax rather than the semantics of function algebras for complexity classes.

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