# Free-Algebra Models for the $\pi$-Calculus

Ian Stark

Laboratory for Foundations of Computer Science
School of Informatics
University of Edinburgh

# Summary

The finite $\pi$-calculus has an explicit set-theoretic functor-category model that is known to be fully-abstract for strong late bisimulation congruence [Fiore, Moggi, Sangiorgi]

We can characterise this as the initial free algebra for certain operations and equations in the setting of Power and Plotkin's enriched Lawvere theories.

This combines separate theories of nondeterminism, I/O and name creation in a modular fashion. As a bonus, we get a whole category of models, a modal logic and a computational monad. The tricky part is that everything has to happen inside the functor category $\mathcal{S}et^{\mathcal{I}}$.

# Overview

- Equational theories for different features of computation.
- Enrichment over the functor category $\mathcal{S}et^{\mathcal{I}}$.
- A theory of $\pi$.
- Free-algebra models; full abstraction; modal logic.

# Nondeterministic computation

## Operations

$$\mathsf{choice} : A^2 \longrightarrow A$$
$$\mathsf{nil} : 1 \longrightarrow A$$

## Equations

$$\mathsf{choice}(P, Q) = \mathsf{choice}(Q, P)$$
$$\mathsf{choice}(\mathsf{nil}, P) = \mathsf{choice}(P, P) = P$$
$$\mathsf{choice}(P, (\mathsf{choice}(Q, R)) = \mathsf{choice}(\mathsf{choice}(P, Q), R)$$

# Algebras for nondeterminism

For any Cartesian category $\mathcal{C}$ we can form the category $\mathcal{ND}(\mathcal{C})$ of models $(A, \mathtt{choice}, \mathtt{nil})$ for the theory. In particular, there is:

$$\mathcal{ND}(\mathcal{S}et)$$

free $\quad F \quad \dashv \quad U \quad$ forgetful

$$\mathcal{S}et$$

In fact $(U \circ F)$ is finite powerset and the adjunction is monadic: $\mathcal{ND}(\mathcal{S}et)$ is isomorphic to the category of $\mathcal{P}_{\mathrm{fin}}$-algebras.

# Computational monad for nondeterminism

$$\mathcal{ND}(Set)$$

free $\quad F \uparrow \;\dashv\; \downarrow U \quad$ forgetful

$$Set$$

The composition $T = (U \circ F) = \mathcal{P}_{fin}$ is the computational monad for finite nondeterminism. Operations $choice$ and $nil$ then induce generic effects in the Kleisli category:

from $\quad choice : A^2 \longrightarrow A^1 \qquad$ we get $\quad arb : 1 \longrightarrow T\,2$

$\qquad\quad nil : A^0 \longrightarrow A^1 \qquad\qquad deadlock : 1 \longrightarrow T\,0$

[Plotkin, Power: Algebraic Operations and Generic Effects]

# I/O computation

Operations

$$\begin{aligned} \mathrm{in} : A^V &\longrightarrow A \\ \mathrm{out} : \ A \ &\longrightarrow A^V \end{aligned}$$

Equations

none

From any Cartesian $\mathcal{C}$ we form the category $\mathcal{IO}(\mathcal{C})$ of models $(A, \mathrm{in}, \mathrm{out})$ for I/O computation over $\mathcal{C}$.

# I/O adjunction and monad

$$\mathcal{IO}(\mathit{Set})$$

free $\quad F \quad \dashv \quad U \quad$ forgetful

$$\mathit{Set}$$

The adjunction is monadic: $\mathcal{IO}(\mathit{Set}) \cong T\text{-}\mathcal{A}lg$ for the resumptions monad, the computational monad for I/O:

$$T(X) = \mu Y.(X + Y^V + Y \times V) \ .$$

The operations induce suitable effects in its Kleisli category:

from $\quad \mathit{in} : A^V \longrightarrow A^1 \qquad$ we get $\quad \mathit{read} : 1 \longrightarrow T\,V$

$\qquad \mathit{out} : A^1 \longrightarrow A^V \qquad\qquad \mathit{write} : V \longrightarrow T\,1$

# Notions of computation determine monads

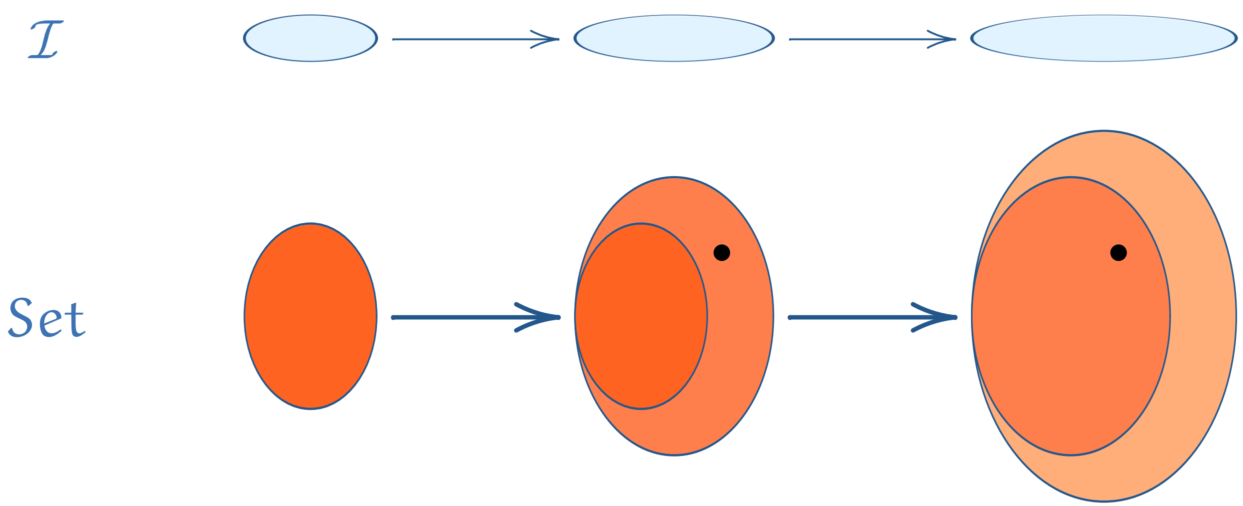
Operations $+$ Equations $\longrightarrow$ Free-algebra models of computational features

$\longrightarrow$ Monads $+$ generic effects

- Characterise known computational monads *and* effects.
- Simple and flexible combination of theories.
- Enriched models and arities: countably infinite, posets, $\omega$Cpo.

# The functor category $\mathcal{S}et^{\mathcal{I}}$

To account for names, we work with structures that vary according to the names available.




An object $B \in \mathcal{S}et^{\mathcal{I}}$ is a varying set: it specifies for any finite set of names $s$ the set $B(s)$ of values using names from $s$, together with information about how these values change with renaming.

# Structure within $\mathcal{S}et^{\mathcal{I}}$

We use $\mathcal{S}et^{\mathcal{I}}$ both as the arena for building name-aware algebras and monads, and as the source of arities for operations.

Relevant structure includes:

- Pairs $A \times B$ and function space $A \to B$;
- Separated pairs $A \otimes B$ and fresh function space $A \multimap B$;
- The object of names $N$;
- The shift endofunctor $\delta A = A(\_ + 1)$, with $\delta A = N \multimap A$.

In particular, the object $N$ serves as a varying arity.

# Theory of $\pi$: operations

## Nondeterminism

| | | |
|---|---|---|
| $\mathit{nil} : 1 \longrightarrow A$ | inactive process | $0$ |
| $\mathit{choice} : A^2 \longrightarrow A$ | process sum | $P + Q$ |

## I/O

| | | |
|---|---|---|
| $\mathit{out} : A \longrightarrow A^{N \times N}$ | output prefix | $\bar{x}y.P$ |
| $\mathit{in} : A^N \longrightarrow A^N$ | input prefix | $x(y).P$ |
| $\mathit{tau} : A \longrightarrow A$ | silent prefix | $\tau.P$ |

## Dynamic name creation

| | | |
|---|---|---|
| $\mathit{new} : \delta A \longrightarrow A$ | restriction | $\nu x.P$ |

# Theory of $\pi$: interlude

Each operation induces a corresponding effect:

$$\begin{aligned} \mathsf{send} : N \times N &\longrightarrow T\,1 & \mathsf{deadlock} : 1 &\longrightarrow T\,0 \\ \mathsf{receive} : N &\longrightarrow T\,N & \mathsf{arb} : 1 &\longrightarrow T2 \\ \mathsf{skip} : 1 &\longrightarrow T\,1 & \mathsf{fresh} : 1 &\longrightarrow T\,N \end{aligned}$$

Other possible operations:

- $\mathsf{par}$ is not algebraic (because $(P \mid Q); R \neq (P; R) \mid (Q; R)$ )
- $\mathsf{eq}, \mathsf{neq} : A \longrightarrow A^{N \times N}$ definable from $N \times N \cong N \otimes N + N$
- $\mathsf{bout} : \delta A \longrightarrow A^N$ can be defined from $\mathit{new}$ and $\mathsf{out}$

# Theory of $\pi$: operations

## Nondeterminism

| | | |
|---|---|---|
| $nil : 1 \longrightarrow A$ | inactive process | $0$ |
| $choice : A^2 \longrightarrow A$ | process sum | $P + Q$ |

## I/O

| | | |
|---|---|---|
| $out : A \longrightarrow A^{N \times N}$ | output prefix | $\bar{x}y.P$ |
| $in : A^N \longrightarrow A^N$ | input prefix | $x(y).P$ |
| $tau : A \longrightarrow A$ | silent prefix | $\tau.P$ |

## Dynamic name creation

| | | |
|---|---|---|
| $new : \delta A \longrightarrow A$ | restriction | $\nu x.P$ |

# Theory of $\pi$: component equations

## Nondeterminism

$choice$ is associative, commutative and idempotent, with identity $nil$.

## I/O

None.

## Dynamic name creation

$$new(x.p) = p$$

$$new(x.new(y.p)) = new(y.new(x.p))$$

# Theory of $\pi$: combining equations

## Commuting

$$new(x.choice(p, q)) = choice(new(x.p), new(x.q))$$

$$new(z.out_{x,y}(p)) = out_{x,y}(new(z.p)) \qquad z \notin \{x, y\}$$

$$new(z.in_x(p_y)) = in_x(new(z.p_y)) \qquad z \notin \{x, y\}$$

$$new(z.tau(p)) = tau(new(z.p))$$

## Interaction

$$new(x.out_{x,y}(p)) = nil$$

$$new(x.in_x(p_y)) = nil$$

# Models of the theory of $\pi$

The category $\mathcal{PI}(\mathcal{S}et^{\mathcal{I}})$ of $\pi$-algebras has objects of the form $(A \in \mathcal{S}et^{\mathcal{I}}; in, out, \dots, new)$ satisfying the equations given.

In any $\pi$-algebra $A$, each finite $\pi$-calculus process $P$ has interpretation $[\![P]\!]_A$ defined by induction over the structure of $P$, using the operations of the theory (and the expansion law for parallel composition).

**Thm:** Every such $\pi$-algebra interpretation respects strong late bisimulation congruence:

$$P \approx Q \quad \Longrightarrow \quad [\![P]\!]_A = [\![Q]\!]_A \, .$$

Of course, this doesn't yet give us any actual $\pi$-algebras to work with.

# Models of the theory of $\pi$

The category of $\pi$-algebras has a forgetful functor to $\mathcal{S}et^{\mathcal{I}}$, taking each algebra to its underlying (varying) set:

$$\begin{array}{c} \mathcal{PI}(\mathcal{S}et^{\mathcal{I}}) \\ \downarrow \; U \quad \text{forgetful} \\ \mathcal{S}et^{\mathcal{I}} \end{array}$$

Naturally, we now look for a free functor left adjoint to $U$, and its accompanying monad.

As it happens, using both closed structures at the same time means that general results engaged earlier don't immediately apply :-(

# Free models for $\pi$

Each component theory has a standard monad:

| | |
|---|---|
| **Nondeterminism** | $\mathcal{P}_{\mathrm{fin}}(X)$ |
| **I/O** | $\mu Y.(X + N \times N \times Y + N \times Y^N + Y)$ |
| **Name creation** | $\mathrm{Dyn}(X) = \int^k X(\_ + k)$ |

Weaving these together as monad transformers gives

$$\mu Y.\mathcal{P}_{\mathrm{fin}}(\mathrm{Dyn}(X + N \times N \times Y + N \times Y^N + Y))\ldots$$

# Free models for $\pi$

Each component theory has a standard monad:

**Nondeterminism** $\mathcal{P}_{\mathrm{fin}}(X)$

**I/O** $\mu Y.(X + N \times N \times Y + N \times Y^N + Y)$

**Name creation** $\mathrm{Dyn}(X) = \int^k X(\_ + k)$

Weaving these together as monad transformers gives

$$\mu Y.\mathcal{P}_{\mathrm{fin}}(\mathrm{Dyn}(X + N \times N \times Y + N \times Y^N + Y))$$

. . . but the algebras for this do not satisfy the interaction equations between $\mathit{new}$ and $\mathrm{in}/\mathrm{out}$.

## Free models for $\pi$

Each component theory has a standard monad:

| | |
|---|---|
| **Nondeterminism** | $\mathcal{P}_{\mathrm{fin}}(X)$ |
| **I/O** | $\mu Y.(X + N \times N \times Y + N \times Y^N + Y)$ |
| **Name creation** | $Dyn(X) = \int^k X(\_ + k)$ |

The correct monad for the combined theory is

$$Pi(X) = \mu Y.\mathcal{P}_{\mathrm{fin}}(Dyn(X) + N \times N \times Y + N \times \delta Y + N \times Y^N + Y)$$

which adds bound output but otherwise does little with name creation.

# Results

**Thm:** There is an adjunction making the category of $\pi$-algebras monadic over $\mathit{Set}^{\mathcal{I}}$.

$$\mathcal{PI}(\mathit{Set}^{\mathcal{I}})$$

free $\quad \mathsf{Pi} \uparrow \;\dashv\; \downarrow \mathsf{U} \quad$ forgetful

$$\mathit{Set}^{\mathcal{I}}$$

The composition $\mathsf{T}_\pi = (\mathsf{U} \circ \mathsf{Pi})$ is a computational monad for concurrent name-passing programs, with effects $\mathit{send}$, $\mathit{receive}$, $\mathit{arb}$, $\mathit{deadlock}$, $\mathit{skip}$ and $\mathit{fresh}$.

# Results

We have the following:

- A category $\mathcal{PI}(\mathcal{S}et^{\mathcal{I}})$ of $\pi$-algebras, all sound models of $\pi$-calculus bisimulation.

$$P \approx Q \quad \Longrightarrow \quad [\![P]\!]_A = [\![Q]\!]_A$$

- An explicit free-algebra construction $\mathsf{Pi} : \mathcal{S}et^{\mathcal{I}} \longrightarrow \mathcal{PI}(\mathcal{S}et^{\mathcal{I}})$ such that all $\mathsf{Pi}(X)$ are fully-abstract models of $\pi$.

$$P \approx Q \quad \Longleftrightarrow \quad [\![P]\!]_{\mathsf{Pi}(X)} = [\![Q]\!]_{\mathsf{Pi}(X)}$$

- The initial free algebra $\mathsf{Pi}(0)$ is in fact the previously known fully-abstract model.

# Parallel composition

Parallel composition of $\pi$-calculus processes is not algebraic, but we can nevertheless handle it in the following ways:

- All $\pi$-algebras can support $(P \mid Q)$ externally by expansion.
- All free $\pi$-algebras have an internally-defined map

$$\mathsf{par}_{X,Y} : \mathsf{Pi}(X) \times \mathsf{Pi}(Y) \longrightarrow \mathsf{Pi}(X \times Y) \ .$$

- Any multiplication $\mu : X \times X \to X$ then gives us

$$\mathsf{par}_{\mu} : \mathsf{Pi}(X) \times \mathsf{Pi}(X) \longrightarrow \mathsf{Pi}(X) \ .$$

- For $X = 0$, this is standard parallel composition; for $X = 1$ we get the same with an extra success process $\checkmark$.

# Modal logic

Any theory gives rise to a modal logic over its algebras, with possibility and necessity modalities for every operation.

$$P \models \Diamond out_{x,y}(\phi) \iff \exists Q.\ P \sim \bar{x}y.Q \ \wedge\ Q \models \phi$$

$$P \models \Box out_{x,y}(\phi) \iff \forall Q.\ P \sim \bar{x}y.Q \ \Rightarrow\ Q \models \phi$$

$$P \models \Diamond choice(\phi, \psi) \iff \exists Q, R.\ P \sim Q + R \ \wedge\ Q \models \phi \ \wedge\ R \models \psi$$

HML is definable:

$$\langle \bar{x}y \rangle \phi \ = \ \Diamond choice(\Diamond out_{x,y}(\phi), true)$$

We could also take other algebraic operations and define modalities. However, in no case is there a $(\phi \mid \psi)$ modality.

# Review

Operations and equations with enriched arities can give algebraic models for features of computation.

Taking $\mathcal{S}et^{\mathcal{I}}$ for both arities and algebras, we can give a modular theory for the $\pi$-calculus:

$$\pi = (\text{Nondeterminism} + \text{I/O} + \text{Name creation}) \ / \ \mathit{new} \leftrightarrow \mathsf{i/o}$$

We have an explicit formulation of free algebras for this theory; all of these are fully abstract for bisimulation congruence.

The induced computational monad is almost, but not quite, the combination of its three components.

# What next?

- Use $\mathrm{Cpo}^{\mathcal{I}}$ for the full $\pi$-calculus. (OK, FM-$\mathrm{Cpo}$)
- Partial order arities for testing equivalences. [Hennessy]
- Modify equations for early/open/weak bisimulation.
- Try $\mathrm{Pi}(X)$ for applied $\pi$.
- Investigate algebraic $\mathrm{par}$. (with effect $\mathrm{fork} : 1 \to T2$?)

- Build a proper theory of arities over two closed structures.

OR

- Exhibit $\mathcal{S}et^{\mathcal{I}}$ as the category of algebras for a theory of equality testing in $\mathcal{S}et^{\mathcal{F}}$, and then redo everything in the single Cartesian closed structure of $\mathcal{S}et^{\mathcal{F}}$.

# Constructions in $\mathcal{S}et^{\mathcal{I}}$

## Cartesian closed

$$(A \times B)(k) = A(k) \times B(k)$$

$$B^{A}(k) = [A(k + \_), B(k + \_)]$$

## Monoidal closed

$$(A \otimes B)(k) = \int^{k' + k'' \hookrightarrow k} A(k') \times B(k'')$$

$$(A \multimap B)(k) = [A(\_), B(k + \_)]$$

# More constructions in $\mathcal{S}et^{\mathcal{I}}$

Object of names, shift operator

$$N(k) = k$$
$$\delta A(k) = A(k+1)$$

Connections

$$A \otimes B \longrightarrow A \times B \qquad \delta A \cong N \multimap A$$
$$(A \to B) \longrightarrow (A \multimap B) \qquad \delta N \cong N + 1$$

When $A$ and $B$ are pullback-preserving, the two maps are injective and surjective respectively.