Division of Informatics, University of Edinburgh

Laboratory for Foundations of Computer Science

Amalgamation of Transition Sequences in the PEPA Formalism

by

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Keywords :

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1 Introduction

Construction and solution of a large Markov chain corresponding to a model of a real system is not a trivial task since the number of reachable states is very large and transitions between them do not follow an easy pattern to predict. The modeller faces problems of creating an infinitesimal generator matrix $\boldsymbol{Q}$ and solving it whilst coping with limited time/space computational resources and numerical precision. However, he may apply a tool to his work which takes advantage of decompositionality of multidimensional Markov chains. He can choose one of a variety of Stochastic Process Algebras (for example: PEPA [9], TIPP [8], EMPA [2]) or the Stochastic Automata Network (SAN) method [15]. Despite the fact that none of the mentioned methods is versatile enough to allow him to deal with an arbitrary model, some classes of models may be efficiently treated with them. The advantage of the modular approach to defining a Markov chain may be taken not only when writing its description (the modeller can look at every sub-model separately) but also when solving it because possible state reductions or even calculations may be performed for processes smaller than the global one.

A class of models whose behaviour is particularly easy to express by behaviour of some unimodal Markov chains, and whose steady state solution may be calculated as a function of partial solutions of these elementary chains, is BCMP models [1]. Fluxes between states of a Markov chain underlying a BCMP model fulfil local balance equations, which are more convenient to formulate and to solve than global balance equations. A direct consequence is that the solution of the global Markov chain takes the form of a product of solutions of component chains, multiplied by a normalisation factor. A restricted subclass consists of models whose underlying Markov chains are reversible, i.e. their steady state solution remains the same when the direction of time is reversed [11]. Unfortunately, the BCMP class is not wide enough to cover more complicated models which we are interested in.

If a modeller's goal is to obtain some particular performance measures of a system he investigates, he may transform the model into another one which has a product form solution and the desired performance measure is either greater or smaller than the value for the original model. In [7], van Dijk presents his approach towards transforming an initial Markov chain into one which has product form solution and for which a chosen performance measure is bounded from above or below. It is proved in [10], for models expressed in PEPA, that a model component made of sequential reversible input-output components which are either birth-death or linear input-output or parallel input-output or symmetrical branch-join components, is also reversible under the condition that elements of a reverse pair belong to a cooperation set; hence it has a product form solution. In [16] these two approaches are applied simultaneously to some examples in order to find upper and lower bounds of an arbitrary chosen performance measure of a model which does not have a product form solution. In this paper we want to give a more formal description of transforming a Markov chain into another one with product form solution at the PEPA component level. Our goal is to remove Markov chain states for which detailed local balance equations are not satisfied and to bound interesting performance measures using a new Markov chain.

The next section contains a short description of the PEPA formalism. In Section 3 we present an existing method of reducing of sequences of activities of the undefined type τ and give conditions under which the procedure may be done. Section 4 contains descriptions of formal tools we will use to perform activity amalgamation and discusses how the amalgamation procedure affects steady state probabilities in some cases.

2 Outline of the PEPA Formalism

Entities of Performance Evaluation Process Algebra (PEPA) [9] are termed components. A component X can perform action a, $a = (\alpha, r)$, $a \in \mathcal{A}ct(X)$, $\mathcal{A}ct(X)$ is a multi-set, where α is an activity type, $\alpha \in \mathcal{A}(X)$, and r is a transition rate

according to an exponential distribution, $r \in \mathbb{R}^+$, and $\mathbb{R}^+$ is a set of positive real numbers together with the symbol $\top$ indicating unspecified transition rate. The set of types enabled in all derivatives of X, i.e. within the life cycle of X, is termed $\vec{\mathcal{A}}(X)$. Components can interact by the use of the PEPA operations:

$$(\alpha, r).X, \quad X + Y, \quad X \underset{L}{\bowtie} Y, \quad X/H, \quad \stackrel{\text{def}}{=},$$

where L is a set of types over which cooperation between some components may be performed, $H \subseteq \mathcal{A}(X)$ is a set of types which are replaced in the component X by the undefined type τ. The listed PEPA combinators are named: prefix, choice, cooperation, hiding, definitional equality, respectively. Alternatively, $X \underset{\emptyset}{\bowtie} Y$ may be written as $X||Y$.

We define some terminology which we will use later in this paper.

Definition 2.1 *The* apparent rate *of action type α in component X is denoted $r_\alpha(X)$ and is given by:*

$r_\alpha((\beta, r).X) = \begin{cases} r, & \text{if } \alpha = \beta \\ 0, & \text{if } \alpha \neq \beta \end{cases}$	$r_\alpha(X + Y) = r_\alpha(X) + r_\alpha(Y)$
$r_\alpha(X/H) = \begin{cases} r_\alpha(X), & \text{if } \alpha \notin H \\ 0, & \text{if } \alpha \in H \end{cases}$	$r_\alpha(X \underset{L}{\bowtie} Y) = \begin{cases} r_\alpha(X) + r_\alpha(Y), & \text{if } \alpha \notin L \\ \min(r_\alpha(X), r_\alpha(Y)), & \text{if } \alpha \in L \end{cases}$

For a cooperation, if α is in the cooperation set, the slowest participant determines the rate of the cooperation.

Definition 2.2 *If $X_0 \xrightarrow{(\alpha,r)} X_1$, then X_1 is a (one-step)* derivative *of X_0. More generally, if $X_0 \xrightarrow{(\alpha_0,r_0)} X_1 \cdots \xrightarrow{(\alpha_{n-1},r_{n-1})} X_n$, then X_n is a* derivative *of X_0.*

Definition 2.3 *The* derivative set *of a PEPA component X is denoted* $\mathrm{ds}(X)$ *and defined as the smallest set of components such that*

1. *if $X \stackrel{\text{def}}{=} X_0$ then $X_0 \in \mathrm{ds}(X)$,*
2. *if $X_i \in \mathrm{ds}(X)$ and there exists $a \in \mathcal{A}ct(X_i)$ such that $X_i \xrightarrow{a} X_j$ then $X_j \in \mathrm{ds}(X)$.*

From the syntactic point of view, PEPA components are divided into two groups, model components M and sequential components S:

$$\begin{array}{rcl} S & ::= & (\alpha, r).S \mid S + S \mid X \\ M & ::= & S \mid M \underset{L}{\bowtie} M \mid M/H \end{array}$$

where X denotes a constant which is a sequential component.

In [10] the authors define some structures which allow them to detect whether the Markov chain corresponding to a model is reversible. These definitions may help us to make conclusions after having performed amalgamating transformations on a model.

Definition 2.4 *A PEPA component P is said to enable* a reverse pair $(\alpha, -\alpha)$, *if* $(\alpha, r) \in \mathcal{A}ct(P)$ *and for every* (α, r)*-derivative* P' $(P \xrightarrow{(\alpha,r)} P')$, *there exists* $(-\alpha, s) \in \mathcal{A}ct(P')$ *such that* P *is an* $(-\alpha, s)$*-derivative of* P', *where* $r, s \in \mathbb{R}^+$.

Definition 2.5 *A sequential PEPA component P, with initial component P_0, is* an input-output component *if*

1. *for all* $\alpha \in \mathcal{A}(P_0)$, *such that* $P_0 \xrightarrow{(\alpha,r)} P_0'$ *for some* r, α *forms part of the reverse pair* $(\alpha, -\alpha)$,

2. *for all* $P_i \in \mathrm{ds}(P_0)$, *for all* $\alpha_i \in \mathcal{A}(P_i)$, *such that* $P_i \xrightarrow{(\alpha_i,r)} P_i'$ *for some* r, α_i *forms part of the reverse pair* $(\alpha_i, -\alpha_i)$.

Definition 2.6 *A PEPA component P_0 is* a linear input-output component *if for every* $P_i \in \mathrm{ds}(P_0)$ *and for all* Q *such that* $P_i \xrightarrow{a} Q$, P_i *and* Q *communicate exclusively via the actions* a *and* $-a$ *such that* $P_i \xrightarrow{a} Q$ *and* $Q \xrightarrow{-a} P_i$.

3 Weak Isomorphism and Insensitivity

The notion of weak isomorphism was introduced by Hillston in [9]. Two components X and Y are weakly isomorphic if there is a component C which is a compact form of both X and Y components. A compact form C of the component X is a $\mathcal{F}$-image of $\mathrm{ds}(X)$, where $\mathcal{F}$ is a surjective function, $\mathcal{F} : \mathrm{ds}(X) \longrightarrow \mathrm{ds}(Y)$, which amalgamates sequences of activities of undefined type τ into one activity of this type and sets a transition rate of the new activity in order to preserve the mean passage time through the reduced sequence (Figure 1). The stochastic process underlying the compact component C is no longer a pure Markov process because the distribution of the amalgamated sequence rate is not exponential.

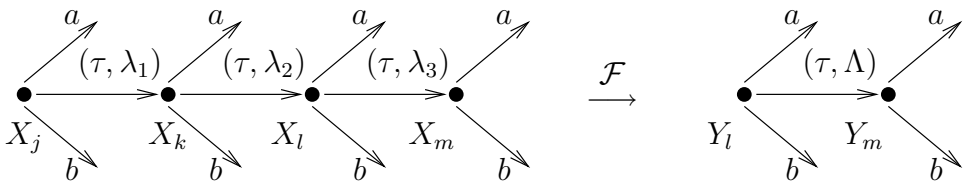

Figure 1: An example of reducible sequence in the X component and its compact form

However, it may be proven ([9]) that the corresponding process is a Generalised Semi-Markov Process (GSMP) which is defined as follows:

Definition 3.1 *A Generalised Semi-Markov Process is a stochastic process defined on a set of states S. Within each of these states s, $s \in S$ are active elements σ, $\sigma \in \Sigma$. A set of active elements is associated with a current state s. Each element of this set, σ, indicates a lifetime with decays at the state dependent rate $r(\sigma, s)$. Let $\Sigma' \cap \Sigma^* = \emptyset$, $\Sigma' \cup \Sigma^* = \Sigma$, and $\sigma \in \Sigma'$ if the element σ is an exponentially distributed lifetime, $\sigma \in \Sigma^*$ if σ is a generally distributed lifetime. When the lifetime of an active element σ expires, the process moves to a state $\overline{s}$ with probability $\pi(s, \sigma, \overline{s})$. The sojourn time in a current state is hence determined by the shortest lifetime of an active element and its termination forces other active elements to suspend their activities, and in case of elements belonging to the set Σ^*, recording residual lifetimes. A restriction which has to be introduced taking into account behaviour of elements of the set Σ^* is that no more then one active element from Σ can be either launched or killed at the same time.*

We are interested in computing a steady state probability vector of a process. We could find it basing our calculations exclusively upon mean sojourn times in the process states neglecting other moments of the general distribution if and only if the process is insensitive. Matthes in [12] formulated a theorem, stated below after [4, 9], which allows us to make the decision whether a GSMP is insensitive.

Theorem 3.1 *The following two statements are equivalent:*

1. *The process in insensitive to the element of Σ^*. That is, the distributions of the lifetimes of the elements of Σ^* may be replaced by any other distribution with the same mean, while still retaining the same equilibrium distribution.*

2. *When all elements of Σ^* are assumed to be exponentially distributed (i.e. $\Sigma = \Sigma'$), the flux out of each state due to the death of an element of Σ^* is equivalent to the flux into that state due to the birth of that element.*

In [4] Clark translates Theorem 3.1 into the PEPA formalism under the conditions that a generally distributed activity may not synchronise with any other activity and a synchronisation of sequential components may not lead to a state where more than one sequential component newly enables a generally distributed activity:

Theorem 3.2 *The following two statements are equivalent:*

1. *The PEPA model in insensitive to each generally distributed activity a,*

2. *The purely Markov process (i.e. $\Sigma = \Sigma'$), has the property that for all states s that enable transition a, the flux into s enabling a is balanced by the flux out of s due to the completion or disabling of a.*

4 Amalgamation of Sequences

In this section we describe a transformation procedure for PEPA models. Firstly, we define *local sequences* in PEPA sequential components building up a model. Secondly, we decide which activity type should be chosen as a new one to label the amalgamated transition. Thirdly, we identify a surjective function which describes the amalgamation procedure. Fourthly, we propose a classification, discuss computation of the new transition rate and give examples of local sequences according to the introduced classifying scheme. Finally, we define a *semi-input-output linear component*, analyse it to find out the impact of the amalgamation procedure on the steady state solution of the reduced input-output linear component and give an example of such a reduction in detail.

4.1 Local Sequences inside a PEPA Component

Firstly, we state the definition of an internal structure of a PEPA component which destroys its potential reversibility.

Definition 4.1 *A pair of activity types* (α_P, α_S) *is* a local sequence *in a component* X_0 *if and only if, for all* X_{j_1}, X_{j_2}, *such that* $X_{j_1}, X_{j_2} \in \mathrm{ds}(X_0)$, $X_{j_1} \not\equiv X_{j_2}$, *for which* $X_{j_1} \xrightarrow{(\alpha_P, r_{j_{12}})} X_{j_2}$ *and* X_{j_2} *is not a one-step derivative of any* X_{j_k} *except* X_{j_1} *there exists exactly one* X_{j_3}, $X_{j_3} \in \mathrm{ds}(X_0)$, $X_{j_2} \not\equiv X_{j_3}$ *such that* $X_{j_2} \xrightarrow{(\alpha_S, r_{j_{23}})} X_{j_3}$ *and* X_{j_3} *is the only one-step derivative of* X_{j_2}.

In a local sequence (α_P, α_S) the activity type α_P is termed *the predecessor* and α_S is termed *the successor*.

Figure 2 depicts a local sequence in X, (τ, α_S). In this case the undefined type τ is the predecessor and α_S is the successor. For the state X_k we cannot write a proper local balance equation $\pi_k q_{k,m} = \pi_m q_{m,k}$ and we do not want to include this state in our calculation. Consequently we replace the component X by another one, X', in which $X'_j \xrightarrow{(\xi, R)} X'_l$. The states X'_j and X'_l are images of X_j and X_l respectively in a transforming function. However, we have to decide what type ξ is and what is its transition rate.

4.2 Choice of Activity Type

The synchronising actions in a PEPA model cannot be of undefined type. The independent actions of type with which no reward function are associated may by hidden, i.e. their types may be replaced by the undefined type. In the case when only one type of the local sequence (α_P, α_S) is a cooperating one, we pick it as the type of the new transition.

In the case when both actions are either synchronised or independent we assume that the succeeding activity type of a sequence absorbs the preceding

one. Our assumption is a direct result of an emphasis on observation. In [13] Milner proves τ-laws in which he explains the behaviour of actions labelled with undefined type τ. He states that action capabilities which are available for a component X are not changed by performing an action of type τ by this component. He also introduces a notion of weak bisimulation which leads to observational equivalence. A model from which actions labelled with undefined type τ are deleted, is weakly bisimilar to the initial one. Such an approach is a consequence of the fact that the internal actions (τ, r) cannot be observed or detected by an external agent.

Let us consider a sequence in the component X presented in Figure 2. Starting from the state X_j, the component performs its internal action (τ, λ) and the observer looking at the system in this state cannot say of what type the activity is. When the internal action is terminated, he can recognise the activity type of the consecutive action (under the condition that the next activity is of "public" type). The "private" activity of type τ takes the component into the state X_k. Because the observer may determine the activity type of the current action beginning from this state, from his point of view the state X_k is not interesting for he does not know which activity transformed the component into it. He may suppose that the time of passing from X_j to X_l has a distribution defined by a Coxian distribution equal to the convolution of the "private" and "public" exponential distributions with parameters λ and r_S, respectively [5] and the type of this transition is the same as the type of the visible transition, i.e. α_S. Reasoning as above suggests

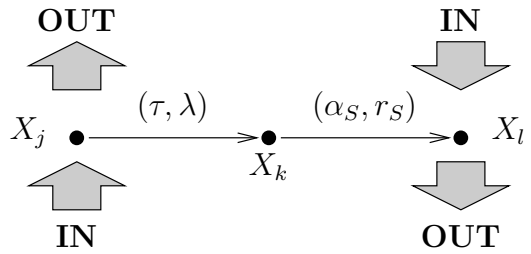

Figure 2: An example of a local sequence in the X component with internal activity type τ as a predecessor

that the activity type ξ (see Subsection 4.1) resulting from the amalgamation procedure should be replaced by the successor type α_S.

4.3 Amalgamation Procedure

To begin, we state precisely how "an internal element" (such as X_k in Figure 2), which will be removed due to the amalgamation procedure, may be identified.

Definition 4.2 *A derivative X^* is* an internal sequential derivative *of a local sequence (α_P, α_S) in X if its enabled activity multi-set is $\{\!|\, (\alpha_S, r_S) \,|\!\}$ and it is an α_P-derivative of a component whose enabled activity multi-set is $\{\!|\, (\alpha_P, r_P) \,|\!\}$.*

A function which amalgamates local sequences and removes their internal sequential derivative is defined as follows:

Definition 4.3 *A function* $f : \mathrm{ds}(X) \longrightarrow \mathrm{ds}(X')$ *is* a sequential epimorphism *from* X *to* X' *over a local sequence* (α_P, α_S) *if it is a surjective function such that*

1. *if* $X^* \in \mathrm{ds}(X)$ *and* X^* *is not an internal sequential derivative of a local sequence* (α_P, α_S)*, when* X^* *and* $f(X^*)$ *are identical,*

2. *for any* $a \in \mathcal{A}ct(X^*)$*, not part of a local sequence* (α_P, α_S)*, the set of* a*-derivatives of* $f(X^*)$ *is the same as the* f*-image of the set of* a*-derivatives of* X^**,*

3. *for any pair of activities* (α_P, r_P) *and* (α_S, r_S) *such that* $X_0^* \xrightarrow{(\alpha_P, r_P)} X_1^* \xrightarrow{(\alpha_S, r_S)} X_2^*$ *there is exactly one derivative of* $f(X_0^*)$*, which may be reached by performing* (α_S, R) *such that* $f(X_0^*) \xrightarrow{(\alpha_S, R)} f(X_2^*)$*. The expected delay between* $f(X_0^*)$ *and* $f(X_2^*)$ *is the same as the expected delay between* X_0^* *and* X_2^**.*

Sequential epimorphism is a generalisation of weak isomorphism defined in [9] (see also Section 3). However, it is much stronger since it removes states and actions and changes some transition rates.

4.4 Classification of Sequences to be Amalgamated

Let $M = \triangle(X_0^{(i)})$, $i = 0, 1, \ldots, n-1$, be a PEPA model made up of n sequential components, the $X_0^{(i)}$. The $\triangle$ operator composes the $X_0^{(i)}$ using PEPA's composition and hiding combinators. Let α be an activity type which may occur in M, i.e. $\alpha \in \vec{\mathcal{A}}(M)$. We use shorthands as stated below:

$\mathcal{M}$: the set of all the components $X_0^{(i)}$, $i = 0, 1, \ldots, n-1$,

$\mathcal{M}^{\alpha,\emptyset}$: the set of all components $X_0^{(i)}$ in which no action of type α occurs,

$\mathcal{M}^{\alpha,*}$: the set of all components $X_0^{(i)}$ in which an action of type α may occur but these components do not cooperate over α,

$\mathcal{M}^{\alpha}$: the set of all components $X_0^{(i)}$ cooperating over the activity type α.

It is obvious that $\mathcal{M} = \mathcal{M}^{\alpha,\emptyset} \cup \mathcal{M}^{\alpha,*} \cup \mathcal{M}^{\alpha}$ and $\mathcal{M}^{\alpha,\emptyset} \cap \mathcal{M}^{\alpha,*} = \mathcal{M}^{\alpha,*} \cap \mathcal{M}^{\alpha} = \mathcal{M}^{\alpha,\emptyset} \cap \mathcal{M}^{\alpha} = \emptyset$. Let a pair of activity types (α_P, α_S) be a local sequence in some sequential components $X_0^{(i)}$ of the model M. We can restrict ourselves to the cases listed in Table 1 for which $\mathcal{M}^{\alpha_P,\emptyset} = \mathcal{M}^{\alpha_S,\emptyset} = \emptyset$. If these sets were not empty our classification would be still valid as long as their elements were cyclic components. We also assume that each sequence (α_P, α_S) appears at most once in each sequential component.

To remove ambiguity, in the context of the proposed classification, we give details of the reduction of local sequences in each case. The following definitions allow us to establish the boundary conditions of the sequences we consider.

a)	$\mathcal{M}^{\alpha_P,*} = \mathcal{M}$	I	$\mathcal{M}^{\alpha_S,*} = \mathcal{M}$		
		II	$\mathcal{M}^{\alpha_S} = \mathcal{M}$		
		III	$\mathcal{M}^{\alpha_S,*} \neq \mathcal{M} \wedge \mathcal{M}^{\alpha_S} \neq \mathcal{M}$		
b)	$\mathcal{M}^{\alpha_P} = \mathcal{M}$	I	$\mathcal{M}^{\alpha_S,*} = \mathcal{M}$		
		II	$\mathcal{M}^{\alpha_S} = \mathcal{M}$		
		III	$\mathcal{M}^{\alpha_S,*} \neq \mathcal{M} \wedge \mathcal{M}^{\alpha_S} \neq \mathcal{M}$		
c)	$\mathcal{M}^{\alpha_P,*} \neq \mathcal{M}$ $\wedge$ $\mathcal{M}^{\alpha_P} \neq \mathcal{M}$	I	$\mathcal{M}^{\alpha_S,*} = \mathcal{M}$		
		II	$\mathcal{M}^{\alpha_S} = \mathcal{M}$		
		III	$\mathcal{M}^{\alpha_S,*} \neq \mathcal{M}$ $\wedge$ $\mathcal{M}^{\alpha_S} \neq \mathcal{M}$	$\star$	$\mathcal{M}^{\alpha_P} = \mathcal{M}^{\alpha_S}$
				$\star\star$	$\mathcal{M}^{\alpha_P} \cap \mathcal{M}^{\alpha_S} = \emptyset$
				$\star\star\star$	otherwise

Table 1: Possible manifestations of local sequences in sequential PEPA components of the model M

Definition 4.4 *A local sequence (α_P, α_S) in X is* a beginning sequence *in a model M built with X, if and only if, α_P is not a successor in any local sequence in X and a component whose α_P-derivative is an internal sequential derivative of (α_P, α_S) in X, is obtained only by cooperating actions of all components belonging to the set $\mathcal{M}^{\alpha_S}$.*

Definition 4.5 *A local sequence (α_P, α_S) in X is* an ending sequence *in a model M built with X, if and only if, α_S is not a predecessor in any local sequence in X and from a component which is an α_S-derivative of an internal sequential component, all outgoing actions are cooperating actions of all components belonging to the set $\mathcal{M}^{\alpha_P}$.*

Based upon these two definitions, illustrated in Figures 3, 4, and 5, we are able to define local sequences which are made of exactly two actions labelled with activity types α_P and α_S, respectively.

Definition 4.6 *A local sequence (α_P, α_S) in X included in a model M is* a strict sequence *in M if it is a beginning and ending sequence in M.*

We perform the amalgamation procedure in most investigated cases (Section 4.4.2) for beginning/ending local sequences only, because this assumption assures we will pick local sequences whose "boundary conditions" are the same in all components. Before examining these cases we explain phenomena which may occur when amalgamating local sequences.

We point out that no reward function can be associated with actions of type α_P or α_S for these actions will vanish during the amalgamation process. The choice of transition rate R will be discussed in next subsection.

Notice that the amalgamation of sequences leads to another model which is smaller in size but it does not always ensure that the reduced model has a

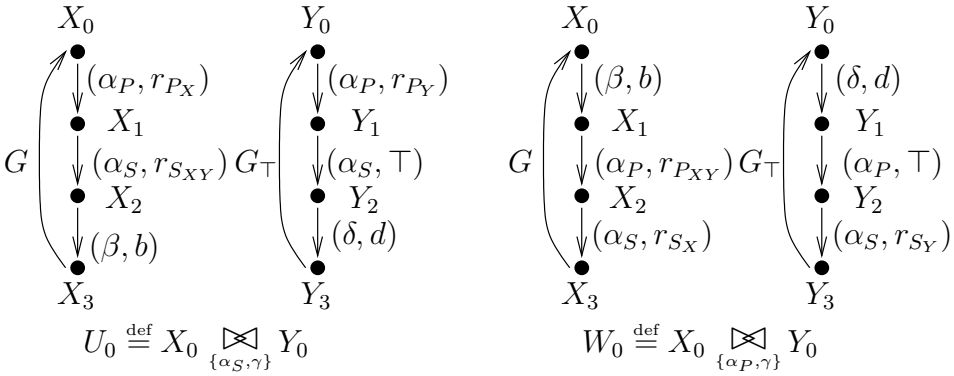

Figure 3: Examples of beginning ((α_P, α_S) in U_0) and ending ((α_P, α_S) in W_0) sequences; shorthands: $G = (\gamma, g)$, $G_\top = (\gamma, \top)$

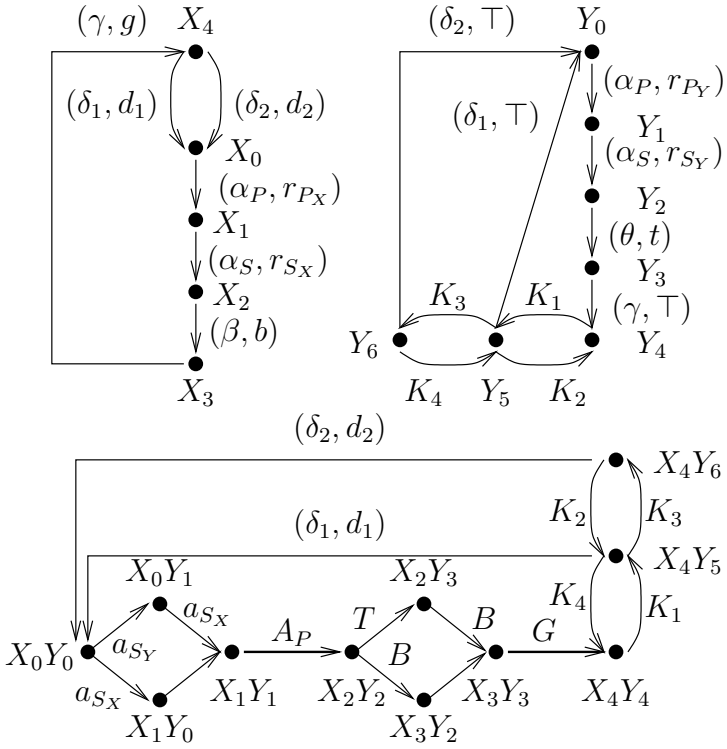

Figure 4: Example of beginning (α_P, α_S) sequence in $X_0 \underset{\{\gamma,\delta_1,\delta_2,\alpha_S\}}{\bowtie} Y_0$; shorthands: $G = (\gamma, g)$,$B = (\beta, b)$, $a_{P_X} = (\alpha_P, r_{P_X})$, $a_{P_Y} = (\alpha_P, r_{P_Y})$, $A_S = (\alpha_S, \min(r_{S_X}, r_{S_Y}))$, $T = (\theta, t)$, $K_i = (\kappa, k_i), i = 1, 2, 3, 4$

product form solution. A Markov chain in which a transition $i \xrightarrow{\lambda} (i+1)$ with exponentially distributed rate may be replaced by another one $i \xrightarrow{G(\lambda)} (i+1)$ with generally distributed rate with mean $\frac{1}{\lambda}$ and the steady state solution of the obtained stochastic process is the same, is insensitive. In general we are going

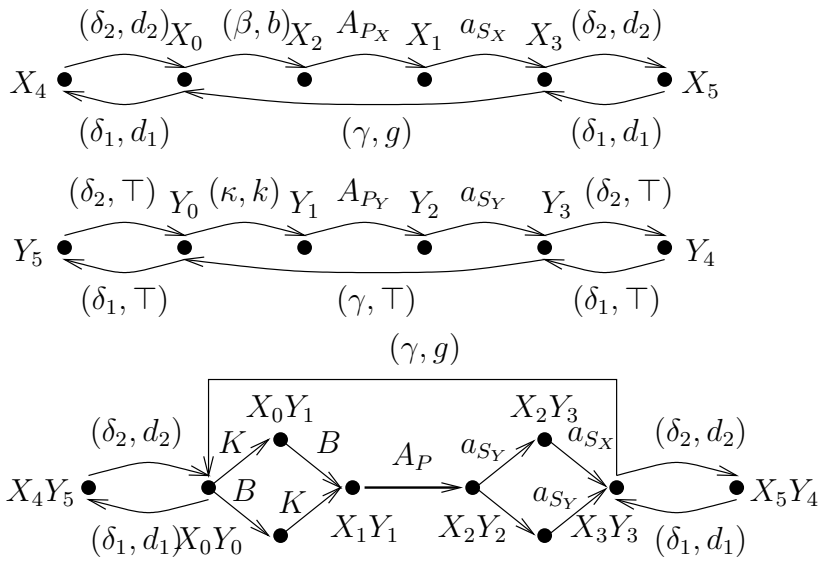

Figure 5: Example of ending (α_P, α_S) sequence in $X_0 \underset{\{\gamma,\delta_1,\delta_2,\alpha_P\}}{\bowtie} Y_0$; shorthands: $B = (\beta, b)$, $K = (\kappa, k)$ $A_{P_X} = (\alpha_P, r_{P_X})$, $A_{P_Y} = (\alpha_P, r_{P_Y})$, $A_P = (\alpha_P, \min(r_{P_X}, r_{P_Y}))$, $a_{S_X} = (\alpha_S, r_{S_X})$, $a_{S_Y} = (\alpha_S, r_{S_Y})$

to replace a sequence of transitions $i \xrightarrow{\lambda_1} (i+1) \xrightarrow{\lambda_2} \cdots \xrightarrow{\lambda_k} (i+k)$ whose behaviour is Coxian (which may be understood as an example of general distribution) by a single transition with exponentially distributed rate, $i \xrightarrow{\Lambda_k} (i+k)$. We will obtain the most satisfactory values of performance measures when the reduced chain is insensitive.

4.4.1 Phase-type Distribution

We have to consider the fact that local sequences in the components making up the investigated PEPA model interact. The synchronised actions are performed simultaneously in a group of PEPA components, the other actions are executed independently and their instances interleave with one another.

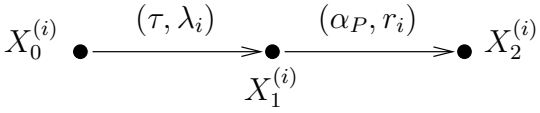

Figure 6: The $X_0^{(i)}$, $i = 0, 1, \ldots, n-1$, a sequential component of $M = \triangle(X_0^{(i)})$

We assume that a model component M is built with previously listed restrictions satisfied and it is made of $n-1$ sequential components, $M = \triangle(X_0^{(i)})$, as presented in Figure 6. Let all components $X_0^{(i)}$ cooperate over the activity type

$$
\begin{array}{lcl}
\boldsymbol{X}_{[0]} & = & (X_0^{(n-1)}, \ldots, X_0^{(1)}, X_0^{(0)}), \\
\boldsymbol{X}_{\sigma(0)} & = & (X_0^{(n-1)}, \ldots, X_1^{(\sigma(0))}, \ldots, X_0^{(0)}), \\
\boldsymbol{X}_{\sigma(1)} & = & (X_0^{(n-1)}, \ldots, X_1^{(\sigma(0))}, \ldots, X_1^{(\sigma(1))}, \ldots, X_0^{(0)}), \\
\vdots & \vdots & \qquad\qquad\qquad \vdots \\
\boldsymbol{X}_{\sigma(n-1)} & = & (X_1^{(\sigma(k))}, \ldots, X_1^{(\sigma(0))}, \ldots, X_1^{(\sigma(1))}, \ldots, X_1^{(\sigma(l))}), \\
\boldsymbol{X}_{[2]} & = & (X_2^{(n-1)}, \ldots, X_2^{(1)}, X_2^{(0)}).
\end{array}
$$

Table 2: State description for the graph in Figure 7

α_P, i.e.

$$M = X_0^{(n-1)} \underset{\{\alpha_P\}}{\bowtie} \cdots \underset{\{\alpha_P\}}{\bowtie} X_0^{(1)} \underset{\{\alpha_P\}}{\bowtie} X_0^{(0)} \tag{1}$$

A realisation of the Markov chain underlying the model defined by Eq. (1) is depicted in Figure 7, where σ is a permutation, $\sigma : \{0, 1, \ldots, n-1\} \longrightarrow \{0, 1, \ldots, n-1\}$ and the state names are given in Table 2. At each state of the chain only one coefficient can be modified and the only possible change is $X_0^{(i)} \longrightarrow X_1^{(i)}$, $i = 0, 1, \ldots, n-1$. We can assume for simplicity that $\bigwedge\limits_i X_0^{(i)} = 0$ and $\bigwedge\limits_i X_1^{(i)} = 1$, so the chain's initial state $\boldsymbol{X}_{[0]}$ is equal to $(0, 0, \ldots, 0)$ and the last one, towards which the τ transition leads, is equal to $(1, 1, \ldots, 1)$. All these states are of the form $(s^{(n-1)}, \ldots, s^{(1)}, s^{(0)})$, where $\bigwedge\limits_i (s^{(i)} = 0 \vee s^{(i)} = 1)$, so they may be ordered by numbers in binary representation, i.e. $(s^{(n-1)}, \ldots, s^{(1)}, s^{(0)}) = s^{(n-1)}2^{n-1} + \cdots + s^{(1)}2^1 + s^{(0)}2^0$. Notice that the ordering relation "$<$" for the natural numbers is also lexicographical ordering for $(s^{(n-1)}, \ldots, s^{(1)}, s^{(0)})$ treated as strings of characters. The general Markov chain diagram for the τ-type independent transitions has a "lattice-like" structure as may be seen in Figure 8 when it is 3-dimensional.

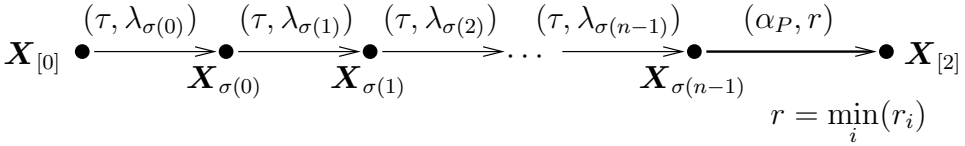

Figure 7: A realisation of the Markov chain corresponding to the model component M, $M = X_0^{(n-1)} \underset{\{\alpha_P\}}{\bowtie} \cdots \underset{\{\alpha_P\}}{\bowtie} X_0^{(1)} \underset{\{\alpha_P\}}{\bowtie} X_0^{(0)}$; state name descriptions in Table 2

Let us consider "a 3-lattice-like" Markov chain as depicted in Figure 8. The state numbered by 7 is its absorbing state and its transition matrix $\boldsymbol{Q}$ is an upper triangle matrix obtained by a tensor (Kronecker) sum [6] $\boldsymbol{Q} = \boldsymbol{Q}^{(2)} \oplus \boldsymbol{Q}^{(1)} \oplus \boldsymbol{Q}^{(0)}$,

where

$$\boldsymbol{Q}^{(i)} = \begin{bmatrix} -\lambda_i & \lambda_i \\ 0 & 0 \end{bmatrix}, \quad i = 0, 1, 2.$$

More precisely, the resulting matrix $\boldsymbol{Q} = \bigoplus_{i=0}^{N-1} \boldsymbol{Q}^{(i)}$ has a structure:

$$\boldsymbol{Q} = \left[\begin{array}{c|c} \boldsymbol{Q}^* & \boldsymbol{q} \\ \hline \boldsymbol{O} & 0 \end{array}\right],$$

where $\boldsymbol{Q}^*$ is an upper triangular matrix, $\dim(\boldsymbol{Q}^*) = n - 1$ and $\boldsymbol{q}$ is a column vector of $n - 1$ elements. The probability distribution $F(t)$ of the time until

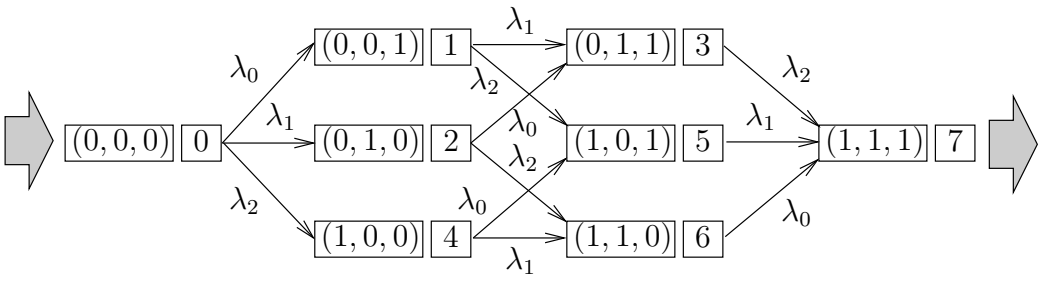

Figure 8: An example of PH-distribution, 3-lattice

absorption in the final n-lattice state (numbered by $2^n - 1$) is equal to [3, 14]:

$$F(t) = 1 - \boldsymbol{\alpha} e^{\boldsymbol{Q}^* t} \mathbf{1} \quad \text{for} \quad t \geq 0 \quad \text{and} \quad F(t) = 0 \quad \text{for} \quad t < 0,$$

where $\mathbf{1}$ is a column vector of size $n - 1$ all of whose elements are equal to 1 and $\boldsymbol{\alpha}$ is a row vector containing initial probabilities of lattice states. A pair $(\boldsymbol{\alpha}, \boldsymbol{Q}^*)$ is called a phase-type distribution (PH-distribution). Its probability density function $f(t)$ can be calculated as follows:

$$f(t) = \frac{dF(t)}{dt} = -\boldsymbol{\alpha} \boldsymbol{Q}^* e^{\boldsymbol{Q}^* t} \mathbf{1} = -\boldsymbol{\alpha} e^{\boldsymbol{Q}^* t} \boldsymbol{Q}^* \mathbf{1} = \boldsymbol{\alpha} e^{\boldsymbol{Q}^* t} \boldsymbol{q}.$$

Notice that the transformations performed above are possible since the matrices $\boldsymbol{Q}^*$ and $e^{\boldsymbol{Q}^* t}$ commute and for each i, $\sum_{k=0}^{n-2} q^*_{ik} = -q_i$, $i = 0, 1, \ldots, n-2$. The moments m_i of this distribution may be calculated by the formula:

$$m_i = (-1)^i i! \boldsymbol{\alpha} (\boldsymbol{Q}^*)^{-i} \mathbf{1}, \quad i = 1, 2, \ldots \tag{2}$$

We set the initial vector $\boldsymbol{\alpha}$ to $(1, 0, \ldots, 0)$ to indicate the fact that the investigated irreducible Markov chain always starts its performance at the state numbered by 0.

4.4.2 Review of Sequence Cases

The cases to be reviewed are listed in Table 1. Diagrams which illustrate those cases are complex and for this reason we apply some abbreviations, mostly concerning types of a local sequence. Actions which are not synchronised and are performed independently in a component X are written as a_{P_X} or a_{S_X} depending on whether they are of α_P or α_S type. For synchronised actions we introduce similar shorthand terms, for example $A_{P_{XY}}$, $A_{S_{XUZ}}$, where the lowest subscript is a list of components which cooperate over either α_P or α_S type.

Case a.I: None of the activities labelled with α_P or α_S is synchronised. Amalgamation of sequence may be performed entirely at the PEPA component level for components do not cooperate either over α_P or α_S and assumption about "boundary conditions" is not required. The new transition rate $R^{(i)}$ in a PEPA sequential component $X^{(i)}$ is equal to $\frac{r_P^{(i)} r_S^{(i)}}{r_P^{(i)} + r_S^{(i)}}$, $i = 0, 1, \ldots, n-1$. If both elements of a local sequence are of undefined type (τ), amalgamation has the same effect as weak isomorphism (see Section 3).

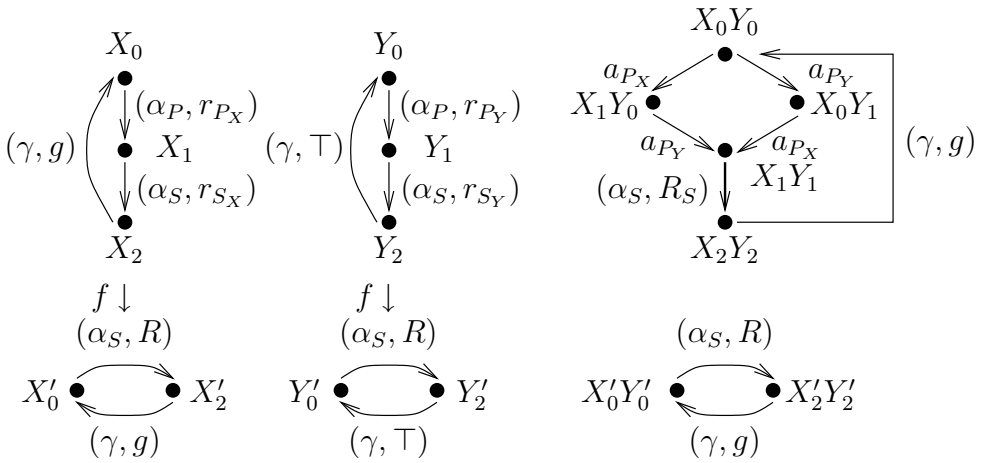

Figure 9: An example of the **a.II** case, $X_0 \underset{\{\alpha_S,\gamma\}}{\bowtie} Y_0$, $X_0' \underset{\{\alpha_S,\gamma\}}{\bowtie} Y_0'$; shorthands: $a_{P_X} = (\alpha_P, r_{P_X})$, $a_{P_Y} = (\alpha_P, r_{P_Y})$

Case a.II: All actions labelled with α_P are performed independently, all actions labelled with α_S are synchronised. Dealing with this case we are restricted to beginning sequences only and an example of a model is pictured in Figure 9. The activities of type α_P form an n-lattice in the model. The average time $\frac{1}{\Lambda_P}$ of passing through this structure is given by Eq. (2). If all but one transition rate of the activities of type α_S are equal to $\top$ then the effective cooperation rate R_S in the global chain is equal to $r_S^{(i)}$, $r_S^{(i)} \neq \top$; in other cases $R_S = \min_i(r_S^{(i)})$.

The effective rate of the replacement activity for the entire strict local sequence is equal to $R = \frac{R_S \Lambda_S}{R_S + \Lambda_S}$.

Case a.III: All actions labelled with α_P are performed independently, some of the actions labelled with α_S are synchronised. Amalgamation is possible for beginning sequences. All local sequences are replaced by actions of type α_S. For those which take place in components from $\mathcal{M}^{\alpha_S}$ the new transition rate is computed as $\frac{\min_i(r_S^{(i)})\Lambda}{\min_i(r_S^{(i)})+\Lambda}$, where $X^{(i)} \in \mathcal{M}^{\alpha_S}$, Λ is an inversion of the mean passage time through the n-lattice whose actions labelled with α_P come from components $X^{(i)}$ which belong to $\mathcal{M}^{\alpha_S}$. In components $X^{(i)}$ from $\mathcal{M} - \mathcal{M}^{\alpha_S}$ the amalgamation procedure is performed individually; the transition rates are $R^{(i)*} = \frac{r_{P_Z} r_{S_Z}}{r_{P_Z} + r_{S_Z}}$. This case is illustrated by the example pictured in Figure 10.

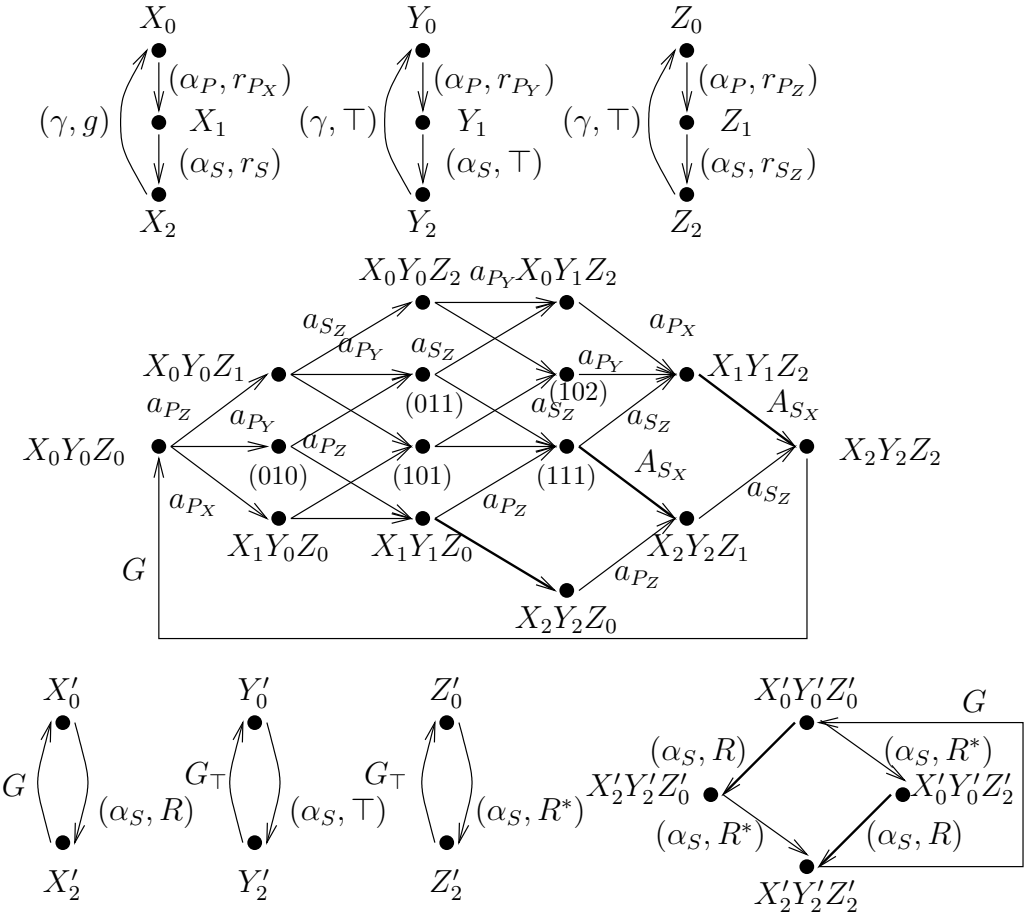

Figure 10: An example of the **a.III** case, $X_0 \underset{\{\alpha_S,\gamma\}}{\bowtie} Y_0 \underset{\{\gamma\}}{\bowtie} Z_0$, $X_0' \underset{\{\alpha_S,\gamma\}}{\bowtie} Y_0' \underset{\{\gamma\}}{\bowtie} Z_0'$; shorthands: $G = (\gamma, g)$, $a_{P_Z} = (\alpha_P, r_{P_Z})$, $a_{S_Z} = (\alpha_S, r_{S_Z})$, $a_{P_X} = (\alpha_P, r_{P_X})$, $a_{P_Y} = (\alpha_P, r_{P_Y})$, $A_{S_X} = (\alpha_S, r_S)$

Case b.I: All actions labelled with α_P are synchronised and all actions labelled with α_S are independent. This case is dual to **a.II**, so it concerns ending sequences only: firstly an action of type α_P is performed in all the sequential components, secondly the components independently execute their own actions of type α_S (Figure 11). As for **a.II** we set the transition rate of the new action, R, as an inversion of the mean of the Coxian distribution of two exponential distributions with parameters R_P and Λ_S, respectively.

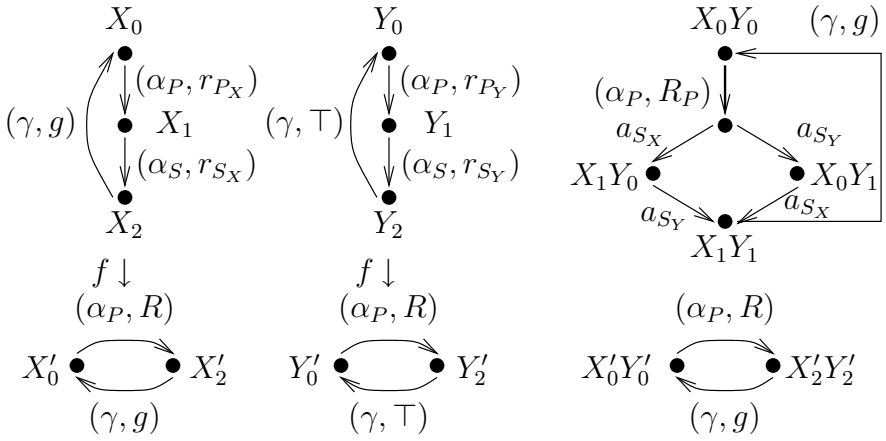

Figure 11: An example of the **b.I** case, $X_0 \underset{\{\alpha_P,\gamma\}}{\bowtie} Y_0 \underset{\{\alpha_P,\gamma\}}{\bowtie} Z_0$; shorthands: $G = (\gamma, g)$, $a_{S_X} = (\alpha_S, r_{S_X})$, $a_{S_Y} = (\alpha_S, r_{S_Y})$

Case b.II: All model components cooperate over α_P and over α_S as well. Notice that both elements of a local sequence (α_P, α_S) are synchronised and actions labelled with α_P, α_S establish automatically "boundary conditions" for launching and terminating the execution of actions of these types. The condition which has to be satisfied is that a component $X_0^{(i)}$, of which an internal sequential component $X_1^{(i)}$ is an α_P-derivative, does not contain other outgoing actions except $X_0^{(i)} \xrightarrow{(\alpha_P, r_{P_X})} X_1^{(i)}$, for all sequential components $X^{(i)}$ in the model. For the moment we assume additionally that all sequential components making up the model contain the (α_P, α_S) local sequence. These conditions ensure that the probability of choosing the (α_P, r_{P_X}) action in the state $X_0^{(i)}$ is the same as the probability of choosing (α_S, R), $R = \frac{\min_i(r_P^{(i)}) \min_i(r_S^{(i)})}{\min_i(r_P^{(i)}) + \min_i(r_S^{(i)})}$ in $X'^{(i)}_0$ although the transition rates of outgoing actions are different. Hence both these probabilities are equal to 1.0. Consequently, in the global Markov chain underlying the entire initial model, the two consecutive actions $(X_0^{(0)}, X_0^{(1)}, \ldots, X_0^{(N-1)}) \xrightarrow{(\alpha_P, \min_i(r_P^{(i)}))} (X_1^{(0)}, X_1^{(1)}, \ldots, X_1^{(N-1)}) \xrightarrow{(\alpha_S, \min_i(r_S^{(i)}))} (X_2^{(0)}, X_2^{(1)}, \ldots, X_2^{(N-1)})$ are replaced by one

$(X'^{(0)}_0, X'^{(1)}_0, \ldots, X'^{(N-1)}_0) \xrightarrow{(\alpha_S, R)} (X'^{(0)}_2, X'^{(1)}_2, \ldots, X'^{(N-1)}_2)$ and there is no other transition outgoing from the $(X^{(0)}_0, X^{(1)}_0, \ldots, X^{(N-1)}_0)$ and $(X'^{(0)}_0, X'^{(1)}_0, \ldots, X'^{(N-1)}_0)$ global states (Figure 12).

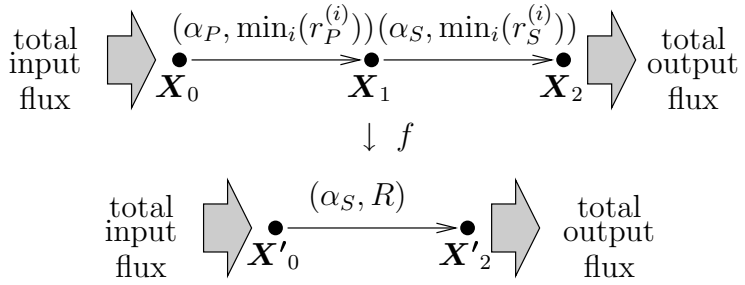

Figure 12: Amalgamation of a sequence for the **b.II** case, all components building up a model contain the local sequence (α_P, α_S); states of the global Markov chain, $\boldsymbol{X}_j = (X^{(0)}_j, X^{(1)}_j, \ldots, X^{(N-1)}_j)$, $j = 0, 1, 2$, $\boldsymbol{X'}_k = (X'^{(0)}_k, X'^{(1)}_k, \ldots, X'^{(N-1)}_k)$, $k = 0, 2$,

For the initial model, the total input flux is equal to $\pi(\boldsymbol{X}_0) \min_i(r_P^{(i)})$. The balance equation for the state $\boldsymbol{X}_1$ is

$$\pi(\boldsymbol{X}_0) \min_i(r_P^{(i)}) = \pi(\boldsymbol{X}_1) \min_i(r_S^{(i)}) \tag{3}$$

and the right hand side of this equation represents also the total output flux leaving "the channel" at the state $\boldsymbol{X}_1$. From Eq. (3) we can express $\pi(\boldsymbol{X}_1)$ as a function of $\pi(\boldsymbol{X}_0)$ and find a sum

$$\pi(\boldsymbol{X}_0) + \pi(\boldsymbol{X}_1) = \left(1 + \frac{\min_i(r_P^{(i)})}{\min_i(r_S^{(i)})}\right) \pi(\boldsymbol{X}_0).$$

In the reduced model the total flux passing through the underlying Markov chain is channelled by the transition between $\pi(\boldsymbol{X'}_0)$ and $\pi(\boldsymbol{X'}_2)$ and we may state that

$$\pi(\boldsymbol{X}_0) \min_i(r_P^{(i)}) = \pi(\boldsymbol{X'}_0) R.$$

With the transition rate R calculated as above we conclude that $\pi(\boldsymbol{X}_0) + \pi(\boldsymbol{X}_1) = \pi(\boldsymbol{X'}_0)$.

The amalgamated version of the initial model preserves throughput of activities other than suppressed α_P even if some components of the model do not contain the local sequence which is to be amalgamated. Let a model be made up of K, $K \neq N$, components containing the (α_P, α_S) local sequence, $X^{(0)}, X^{(1)} \ldots X^{(K-1)}$, with synchronisations for $X^{(i)}_0 \xrightarrow{(\alpha_P, r_P^{(i)})} X^{(i)}_1 \xrightarrow{(\alpha_S, r_S^{(i)})} X^{(i)}_2$, $i = 0, 1, \ldots, K-1$, and $N - K$ components, $X^{(K)}, X^{(K+1)} \ldots X^{(N-1)}$, in which this local sequence does

not appear. All actions taking place in $X^{(K)}$, $X^{(K+1)} \ldots X^{(N-1)}$ components do not change the states of the first K components whilst in these first components states are fixed to $(X_j^{(0)}, X_j^{(1)}, \ldots, X_j^{(K-1)})$, $j = 0, 1, 2$ because their transition rates do not depend upon the first part of the chain's state. We may examine separately the Markov sub-chains constituted by the $N-K$ last components only. Notice, that those sub-chains are identical because functional transition rates are not allowed in PEPA models. Moreover they may not be irreducible because a subset of the components may require cooperation with any of the first K components and these components are blocked waiting for the sequence of actions of α_P and α_S to be accomplished. A sum of all incoming side fluxes for all global states with the first K entries fixed and the last $N-K$ coefficients changeable is equal to a sum of all outgoing side fluxes for those global states (Figure 13). For this reason we may consider groups of states $(X_j^{(0)}, \ldots, X_j^{(K-1)}, X_*^{(K)}, \ldots, X_*^{(N-1)})$, $j = 0, 1, 2$, $*$ — any state, and actions between them labelled with the α_P, α_S types, as "a channel" capable of carrying the total flux passing through the global Markov chain.

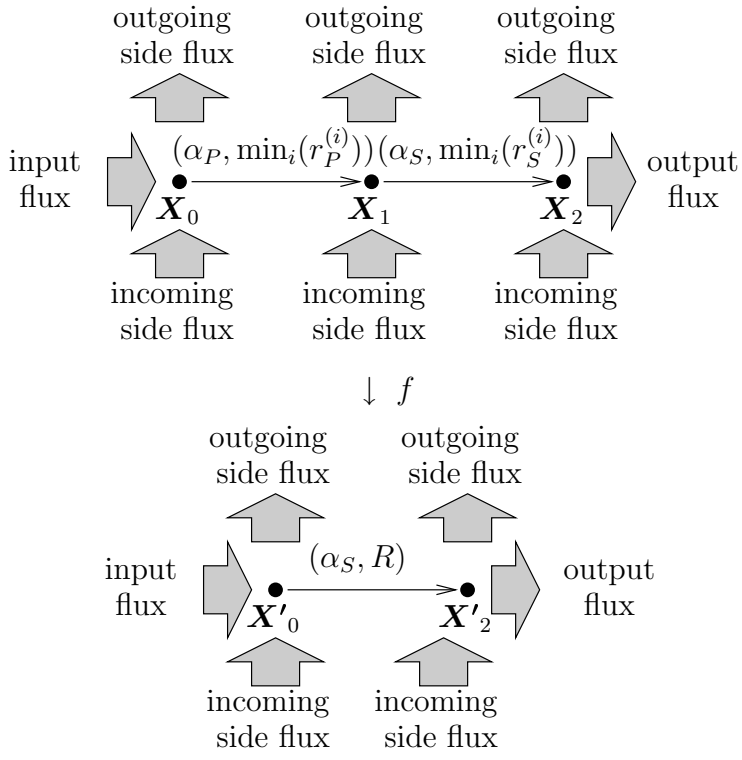

Figure 13: Amalgamation of sequence for the **b.II** case, some components building up a model do not contain the local sequence (α_P, α_S) the global Markov chain, $\boldsymbol{X}_j = (X_j^{(0)}, \ldots, X_j^{(K-1)}, X_*^{(K)}, \ldots, X_*^{(N-1)})$, $j = 0, 1, 2$, $\boldsymbol{X'}_k = (X'^{(0)}_k, \ldots, X'^{(K-1)}_j, X'^{(K)}_*, \ldots, X'^{(N-1)}_*)$, $k = 0, 2$, $*$ — any state

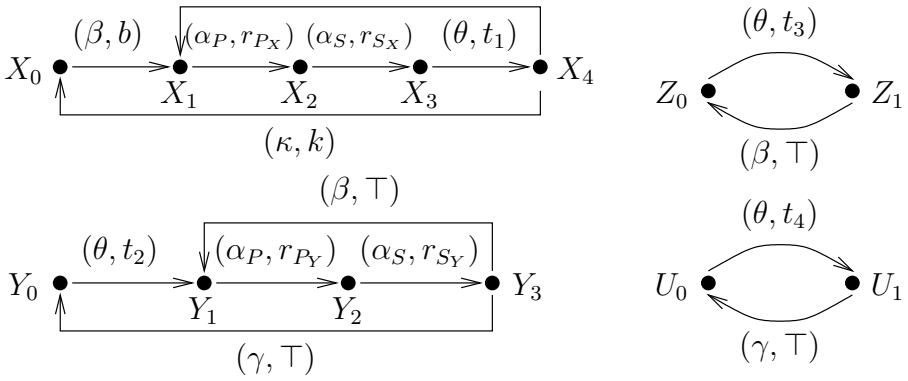

Figure 14: An example of the **b.II** model, $((X_1 \underset{\{\alpha_P,\alpha_S,\beta,\gamma\}}{\bowtie} Y_1) \underset{\{\beta\}}{\bowtie} Z_0) \underset{\{\gamma\}}{\bowtie} U_0$

$X_0 = (\beta, b).X_1;$	$X'_0 = (\beta, b).X'_1;$
$X_1 = (\alpha_P, r_{P_X}).X_2;$	$X'_1 = (\alpha_P, R).X'_3;$
$X_2 = (\alpha_S, r_{S_X}).X_3;$	
$X_3 = (\theta, t_1).X_4;$	$X'_3 = (\theta, t_1).X'_4;$
$X_4 = (\kappa, k).X_0 + (\gamma, g).X_1;$	$X'_4 = (\kappa, k).X'_0 + (\gamma, g).X'_1;$
$Y_0 = (\theta, t_2).Y_1;$	$Y'_0 = (\theta, t_2).Y'_1;$
$Y_1 = (\alpha_P, r_{P_Y}).Y_2;$	$Y'_1 = (\alpha_P, \top).Y'_3;$
$Y_2 = (\alpha_S, r_{S_Y}).Y_3;$	
$Y_3 = (\gamma, \top).Y_0 + (\beta, \top).Y_1;$	$Y'_3 = (\gamma, \top).Y'_0 + (\beta, \top).Y'_1;$
$Z_0 = (\theta, t_3).Z_1;$	$Z'_0 = (\theta, t_3).Z'_1$
$Z_1 = (\beta, \top).Z_0;$	$Z'_1 = (\beta, \top).Z'_0;$
$U_0 = (\theta, t_4).U_1;$	$U'_0 = (\theta, t_4).U'_1;$
$U_1 = (\gamma, \top).U_0;$	$U'_1 = (\gamma, \top).U'_0;$
$((X_1 \underset{\{\alpha_P,\alpha_S,\beta,\gamma\}}{\bowtie} Y_1) \underset{\{\beta\}}{\bowtie} Z_0) \underset{\{\gamma\}}{\bowtie} U_0$	$((X_1 \underset{\{\alpha_P,\alpha_S,\beta,\gamma\}}{\bowtie} Y_1) \underset{\{\beta\}}{\bowtie} Z_0) \underset{\{\gamma\}}{\bowtie} U_0$

Table 3: Representation of the model illustrated in Figure 14 with the PEPA formalism, $R = \frac{\min(r_{P_X} r_{P_Y}) \min(r_{S_X} r_{S_Y})}{\min(r_{P_X} r_{P_Y}) + \min(r_{S_X} r_{S_Y})}$

We solve a model of this case using the PEPA Workbench software.

As an example, diagrams of the model are illustrated in Figure 14 and its description with the PEPA formalism in Table 3. The initial model consists of 24 states, the amalgamated one, 20. Throughput of γ activities, ϑ_γ, is calculated as

$$(\pi(X_4, Y_3, Z_1, U_1) + \pi(X_4, Y_3, Z_0, U_1)) \cdot g$$

for the initial and reduced models. Throughput of α_S activities, ϑ_{α_S}, is found

using the formula

$$\sum_{z\in\{Z_0,Z_1\},u\in\{U_0,U_1\}} \pi(X_2,Y_2,z,u) * \min(r_{S_X}, r_{S_Y})$$

for the initial model and

$$\sum_{z'\in\{Z'_0,Z'_1\},u'\in\{U'_0,U'_1\}} \pi(X'_1,Y'_1,z',u') \cdot R$$

for the reduced one where $R = \frac{\min(r_{P_X},r_{P_Y})\min(r_{S_X},r_{S_Y})}{\min(r_{P_X},r_{P_Y})+\min(r_{S_X},r_{S_Y})}$. Assuming that all the initial model's transition parameters are equal to 1.0 we get $\vartheta_\gamma = 0.11$ and $\vartheta_{\alpha_S} = 0.22$ in both cases. The sum of probabilities

$$\sum_{(z,u)\in\{Z_0,Z_1\}\times\{U_0,U_1\}} (\pi(X_1,Y_1,z,u) + \pi(X_2,Y_2,z,u))$$

is equal to the sum derived from the amalgamated model

$$\sum_{(z',u')\in\{Z'_0,Z'_1\}\times\{U'_0,U'_1\}} \pi(X'_1,Y'_1,z',u') = 0.44$$

and for the final local sequence states

$$\sum_{(z,u)\in\{Z_0,Z_1\}\times\{U_0,U_1\}} \pi(X_3,Y_3,z,u) = \sum_{(z',u')\in\{Z'_0,Z'_1\}\times\{U'_0,U'_1\}} \pi(X'_3,Y'_3,z',u') = 0.22.$$

Case b.III: All actions labelled with α_P are synchronised and some actions labelled with α_S are synchronised. Amalgamation in this case may be performed for ending sequences only. Let $\mathcal{M}^{\alpha_P/\alpha_S}$, $\mathcal{M}^{\alpha_P/\alpha_S} \subset \mathcal{M}^{\alpha_P}$, be the set of $X_0^{(i)}$ components which belong to the $\mathcal{M}^{\alpha_P}$ set and whose indices are the same as those of elements of the $\mathcal{M}^{\alpha_S}$ set. The sub-model consisting of these sequential components fulfils the conditions of the **b.II** case. The other sub-model, the components of which belong to $\mathcal{M}/\mathcal{M}^{\alpha_P/\alpha_S}$ (or — equivalently — to $\mathcal{M}^{\alpha_S,*}$) satisfies the conditions of the **b.I** case. The apparent transition rate of the synchronised action of type α_P is calculated from rate values for all the components $X^{(i)}$, $R_P = \min_i(r_P^{(i)})$. The transition rate of the action of α_S type executed simultaneously by elements of $\mathcal{M}^{\alpha_S}$ is equal to $R_S = \min_{i:X_0^{(i)}\in\mathcal{M}^{\alpha_S}}(r_S^{(i)})$. The total transition rate of the action of type α_S we obtain with the use of Eq. (2) for transition rates of α_S activities of $X_0^{(i)}$, $X_0^{(i)} \in \mathcal{M}^{\alpha_S,*}$ together with "cooperation rate" R_S.

Case c.I: Some actions labelled with α_P are synchronised, all actions of α_S type are performed independently. The amalgamation procedure may be performed for all components in which local sequences are ending ones. In the components

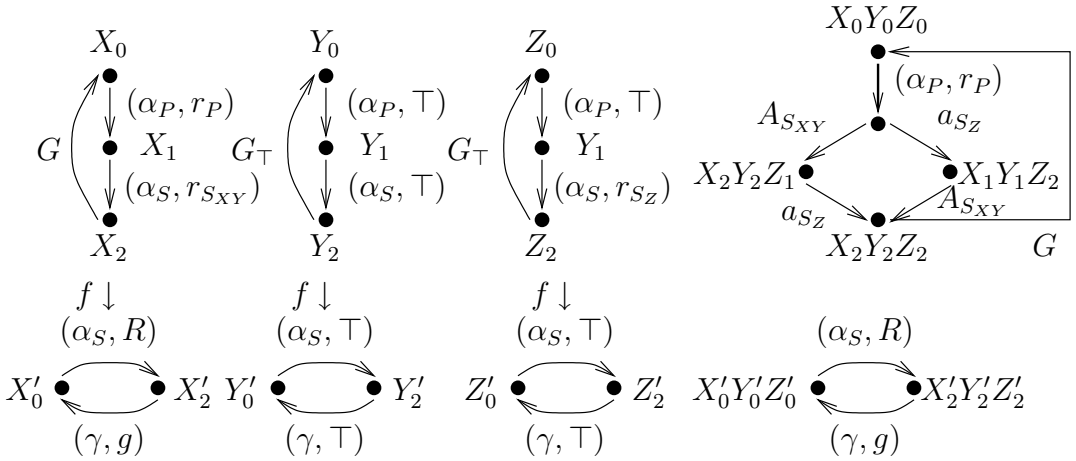

Figure 15: An example of the **b.III** case, $X_0 \underset{\{\alpha_P,\alpha_S,\gamma\}}{\bowtie} Y_0 \underset{\{\alpha_P,\gamma\}}{\bowtie} Z_0$, $X'_0 \underset{\{\alpha_S,\gamma\}}{\bowtie} Y'_0 \underset{\{\alpha_S,\gamma\}}{\bowtie} Z'_0$; shorthands: $G = (\gamma, g)$, $G_\top = (\gamma, \top)$, $A_{S_{XY}} = (\alpha_S, r_{S_{XY}})$, $a_{S_Z} = (\alpha_S, r_{S_Z})$

whose indices belong to $\mathcal{M}^{\alpha_P}$ the new activity type is set as α_P, in the others as α_S. For the first group of components $X^{(i)}$, the transition rate of the introduced action is equal to $R = \frac{\min_i(r_P^{(i)})\Lambda}{\min_i(r_P^{(i)})+\Lambda}$, $X^{(i)} \in \mathcal{M}^{\alpha_P}$, where Λ is the inversion of the mean passage time through the n-lattice computed according to formula (2). For the latter the transition rates are evaluated for each component $X^{(i)}$ separately, $R^{(i)*} = \frac{r_P^{(i)} r_S^{(i)}}{r_P^{(i)} + r_S^{(i)}}$, $X^{(i)} \in \mathcal{M} - \mathcal{M}^{\alpha_P}$. The amalgamation procedure cannot lead to a model whose Markov chain is reversible (Figure 16).

Case c.II: Some actions labelled with α_P are synchronised, all actions of α_S type are synchronised. This case consists of instances of the **a.II** and **b.I** cases so only beginning sequences may be subjects of amalgamation. The total transition rate resulting from the simultaneous execution of activities of type α_S is equal to $R_S = \min_i(r_S^{(i)})$. The transition rate of actions of type α_P performed in all components belonging to $\mathcal{M}^{\alpha_P}$ is equal to $R_P = \min_i(r_P^{(i)})$ and this value is included in the computation of total transition rate of passing through an n-lattice, according to Eq. (2).

Case c.III: Some components of the model cooperate over α_P, some of them over α_S. This case has to be considered more carefully, by three sub-cases:

- ⋆ The same components cooperate over α_P and α_S. If $\mathcal{M}^{\alpha_P} = \mathcal{M}^{\alpha_S}$ we deal with cases **a.I** and **b.II** applied to two disjoint sets of sequential components and amalgamation is possible when done independently, see Figure 17. The transition rate R_{XY} is equal to the inversion of the mean passage

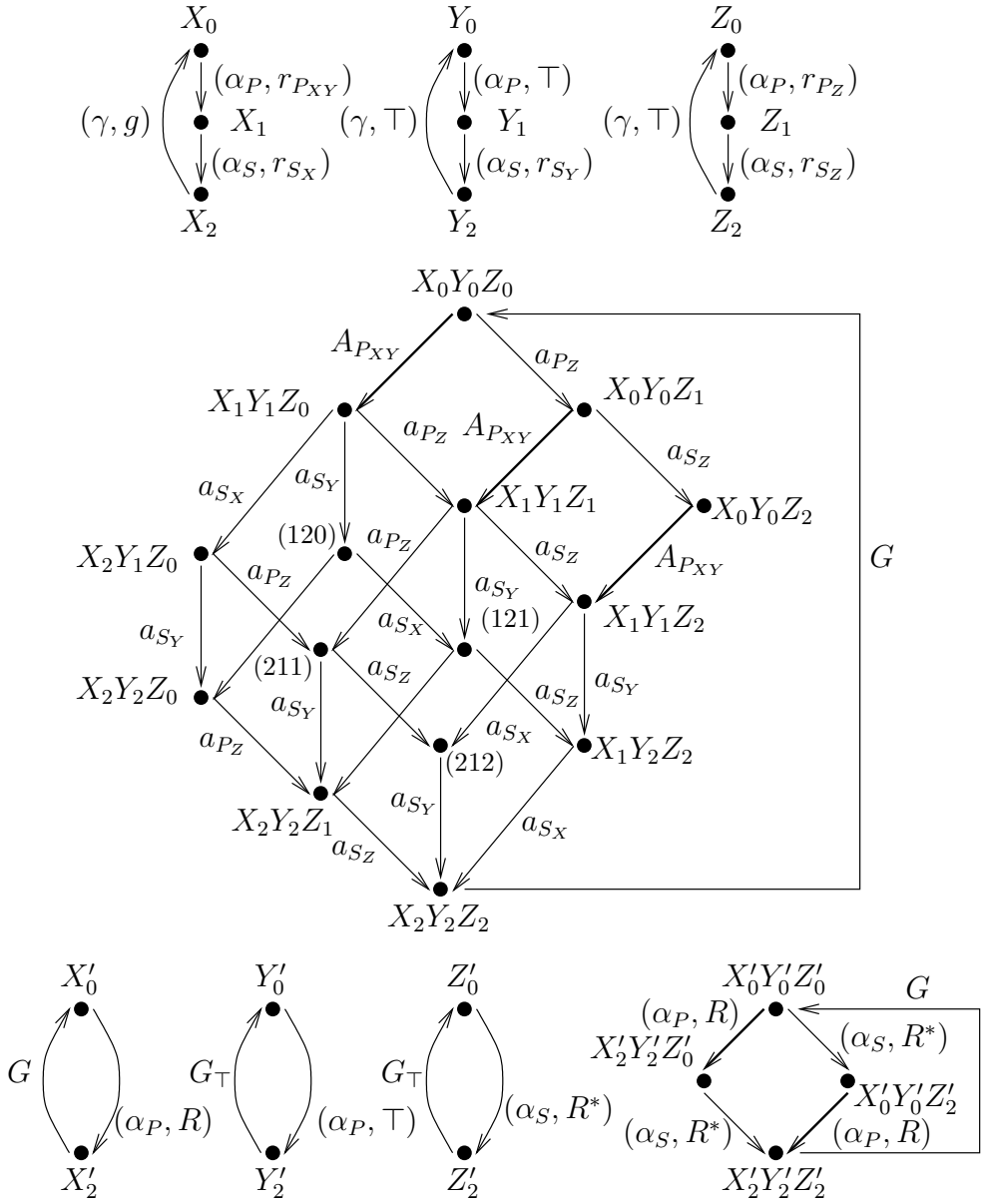

Figure 16: An example of the **c.I** case, $(X_0 \underset{\{\alpha_P,\gamma\}}{\bowtie} Y_0) \underset{\{\gamma\}}{\bowtie} Z_0$, $(X'_0 \underset{\{\alpha_P,\gamma\}}{\bowtie} Y'_0) \underset{\{\gamma\}}{\bowtie} Z'_0$; shorthands: $G = (\gamma, g)$, $A_{P_{XY}} = (\alpha_P, r_{P_{XY}})$, $a_{P_Z} = (\alpha_P, r_{P_Z})$, $a_{S_Z} = (\alpha_S, r_{S_Z})$, $a_{S_X} = (\alpha_S, r_{S_X})$, $a_{S_Y} = (\alpha_S, r_{S_Y})$

time through the synchronised transition α_P in X_0 and Y_0 and through the 2-lattice with parameters R_{P_Z} and R_{P_U}. The rate R_{ZU} is computed analogously: upon the synchronised transition α_S in Z_0 and U_0 and the confluence with R_{S_X} and R_{S_Y}. Notice that the resulting model component $X'_0 \underset{\{\alpha_S,\gamma\}}{\bowtie} Y'_0 \underset{\{\alpha_S,\gamma\}}{\bowtie} Z'_0$ does not have product form solution.

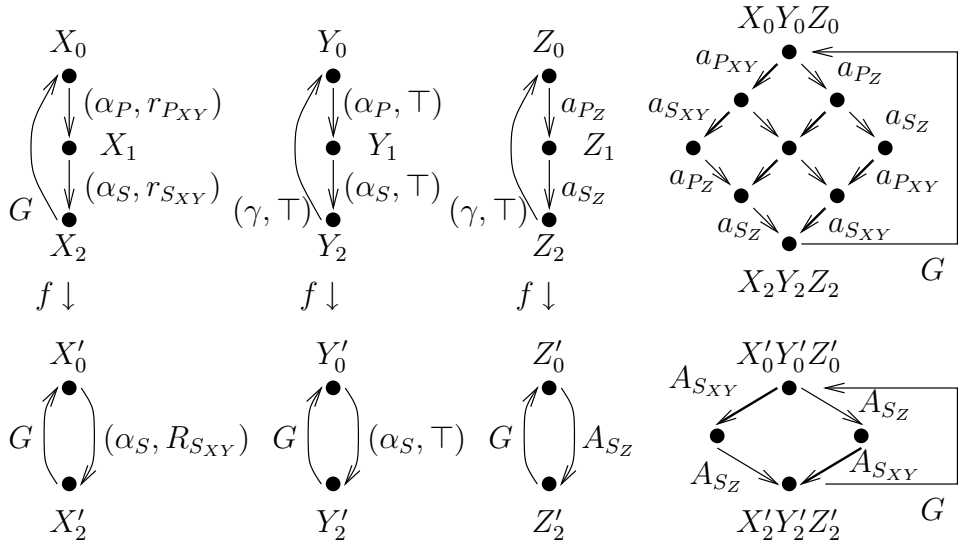

Figure 17: An example of the **c.III.**$\star$ case, $(X_0 \underset{\{\alpha_P,\alpha_S,\gamma\}}{\bowtie} Y_0) \underset{\{\gamma\}}{\bowtie} Z_0$, $X'_0 \underset{\{\alpha_S,\gamma\}}{\bowtie} Y'_0 \underset{\{\alpha_S,\gamma\}}{\bowtie} Z'_0$; shorthands: $G = (\gamma, g)$, $a_{P_Z} = (\alpha_P, r_{P_Z})$, $a_{S_Z} = (\alpha_S, r_{S_Z})$, $a_{P_{XY}} = (\alpha_P, r_{P_{XY}})$, $a_{S_{XY}} = (\alpha_S, r_{S_{XY}})$, $A_{S_{XY}} = (\alpha_S, R_{S_{XY}})$, $A_{S_Z} = (\alpha_S, R_{S_Z})$

- $\star\star$ All model components cooperate over α_P or over α_S. Those which cooperate over α_P do not cooperate over α_S and vice versa. A model splits automatically into two sub-models depending on the activity type, α_P or α_S, over which synchronisation is carried out. One of them fulfils case **a.II** conditions, the other those of **b.I**. An example where amalgamation can be performed for both sub-cases is pictured in Figure 18. However, the reduced model cannot have a product form solution.

- $\star\star\star$ All model components cooperate over α_P or over α_S. Some of them cooperate only over α_P, some only over α_S, some over both the types. Amalgamation is not possible as presented in Figure 19 by an example in which $\mathcal{M}^{\alpha_P} = \{Y_0, Z_0\}$, $\mathcal{M}^{\alpha_P,*} = \{X_0\}$, $\mathcal{M}^{\alpha_S} = \{X_0, Y_0\}$, $\mathcal{M}^{\alpha_S,*} = \{Z_0\}$.

4.4.3 Performance Values Results for the Reviewed Cases

Models of each case investigated in the previous subsection were computed in order to find throughput of transitions not involved in the amalgamation procedure. Depending on whether the underlying Markov chain was insensitive or not, the results were exact or approximate. A summary for examples we have presented in Table 4.

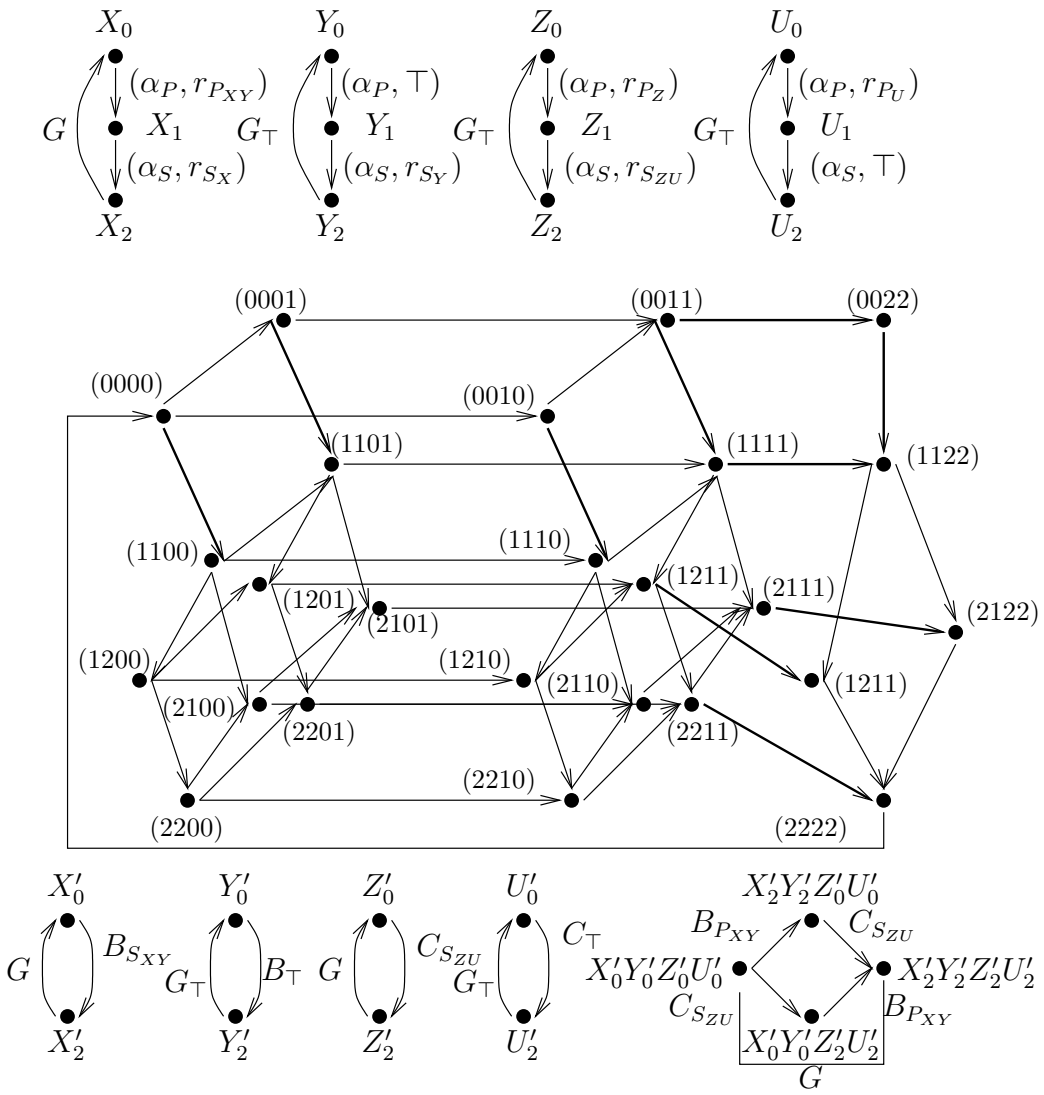

Figure 18: An example of the **c.III.⋆⋆** case, $(X_0 \underset{\{\alpha_P,\gamma\}}{\bowtie} Y_0) \underset{\{\gamma\}}{\bowtie} (Z_0 \underset{\{\alpha_S,\gamma\}}{\bowtie} U_0)$, shorthands: $G = (\gamma, g)$, $G_\top = (\gamma, \top)$, $A_{P_{XY}} = (\alpha_P, r_{P_{XY}})$, $A_{S_{ZU}} = (\alpha_S, r_{S_{ZU}})$, $a_{P_Z} = (\alpha_P, r_{P_Z})$, $a_{P_U} = (\alpha_P, r_{P_U})$, $a_{S_X} = (\alpha_S, r_{P_X})$, $a_{P_Y} = (\alpha_S, r_{P_Y})$, $B_{P_{XY}} = (\alpha_P, R_{XY})$, $B_\top = (\alpha_P, \top)$, $C_{S_{ZU}} = (\alpha_S, R_{ZU})$, $C_\top = (\alpha_S, \top)$

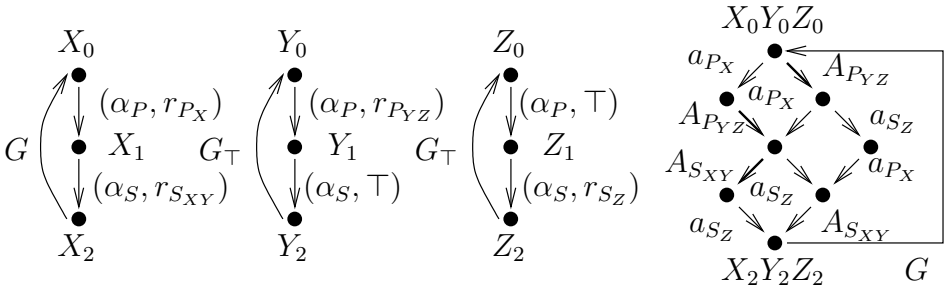

Figure 19: An example of the **c.III.**$\star\star\star$ case, $(X_0 \underset{\{\alpha_S,\gamma\}}{\bowtie} Y_0) \underset{\{\alpha_P,\gamma\}}{\bowtie} Z_0$, shorthands: $G = (\gamma, g)$, $G_\top = (\gamma, \top)$, $a_{P_X} = (\alpha_P, r_{P_X})$, $A_{P_{YZ}} = (\alpha_P, r_{P_{YZ}})$, $a_{S_Z} = (\alpha_S, r_{S_Z})$, $A_{S_{XY}} = (\alpha_S, r_{S_{XY}})$

Case:	**Markov Chain**	**Throughput Results**
a.I	not insensitive	approximate
a.II	insensitive	exact
a.III	not insensitive	approximate
b.I	insensitive	exact
b.II	not insensitive	approximate
b.III	insensitive	exact
c.I	not insensitive	approximate
c.II	insensitive	exact
c.III.$\star$	not insensitive	approximate
c.III.$\star\star$	not insensitive	approximate
c.III.$\star\star\star$	*not applicable*	

Table 4: Throughput values for models of each case

4.5 Semi-Input-Output Linear Components

The amalgamation procedure may be performed "sequence by sequence". A reduced model M' is obtained by amalgamating suitable sequences in the components $X_0^{(i)}$ of M, $(\alpha_{P_1}, \alpha_{S_1}), \ldots, (\alpha_{P_k}, \alpha_{S_k})$. It may happen that we can proceed further with reduction of sequences $(\beta_{P_1}, \beta_{S_1}), \ldots, (\beta_{P_k}, \beta_{S_k})$ in the components $X'^{(i)}_0$ of M' and obtain another model M''. Iteration can be carried out until there is no sequence in the components $X_0^{N\times'}$ of $M^{N\times'}$ suitable for amalgamation.

Definition 4.7 *A PEPA sequential component X of a model M is* a semi-input-output linear component *when it can be transformed by the iterative amalgamation procedure into a sequential input-output linear component Y.*

The number of states in the modified chain is smaller than in the initial one and the probability mass is dispersed according to activity transition rates and states which have been deleted. We will try, however, to reason about what performance measures we are able to derive from the new chain (in terms of the example in Figure 20).

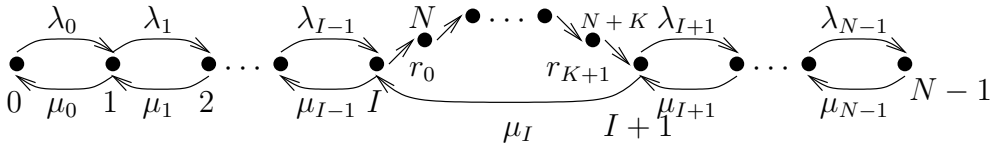

Figure 20: The Markov chain underlying a semi-input-output linear component with one potentially amalgamated sequence

The transition between the states I and $I+1$ in the Markov chain pictured in Figure 20 can be considered as performed according to a Coxian distribution with all probabilities equal to 1. The steady state probabilities of the Markov chain states l, $l = 1, 2, \ldots, I$ may be stated as:

$$\pi_l = \varrho_0 \varrho_1 \cdots \varrho_{l-1} \pi_0, \tag{4}$$

where $\varrho_l = \frac{\lambda_l}{\mu_l}$. Probabilities of states lying upon the edge between I and $I+1$ are:

$$\begin{array}{lcllcl}
\pi_I r_0 & = & \pi_N r_1 & \Longrightarrow \quad \pi_N & = & \frac{r_0}{r_1}\pi_I \\
\pi_N r_1 & = & \pi_{N+1} r_2 & \Longrightarrow \quad \pi_{N+1} = \frac{r_1}{r_2}\pi_N = \frac{r_1}{r_2}\frac{r_0}{r_1}\pi_I & = & \frac{r_0}{r_2}\pi_I \\
 & \vdots & & \vdots & \vdots & \\
\pi_{N+K} r_{K+1} & = & \pi_{N+K-1} r_K & \Longrightarrow \quad \pi_{N+K} = \frac{r_K}{r_{K+1}}\pi_{N+K-1} & = & \frac{r_0}{r_{K+1}}\pi_I.
\end{array} \tag{5}$$

For the states I and $I+1$ we can write global balance equations which allow us to calculate π_{I+1} and π_{I+2}:

$$\pi_{I+1}\mu_I \;=\; \pi_I(\mu_{I-1} + r_0) - \pi_{I-1}\lambda_{I-1} \quad \text{which rewritten:}$$

$$
\begin{aligned}
\pi_{I+1}\mu_I &= \varrho_0\varrho_1\cdots\varrho_{I-2}(\varrho_{I-1}\mu_{I-1} + \varrho_{I-1}r_0 - \lambda_{I-1})\pi_0 \quad \text{yields:} \\
\pi_{I+1} &= \varrho_0\varrho_1\cdots\varrho_{I-1}\frac{r_0}{\mu_I}\pi_0 \\
\pi_{I+1}(\lambda_{I+1}+\mu_I) &= r_{K+1}\pi_{N+K} + \mu_{I+1}\pi_{I+2} \quad \text{replacing } \pi_{N+K} \text{ as in Eq. (5)} \\
\mu_{I+1}\pi_{I+2} &= \pi_{I+1}(\lambda_{I+1}+\mu_I) - r_0\pi_I \\
\mu_{I+1}\pi_{I+2} &= \frac{r_0}{\mu_I}(\lambda_{I+1}+\mu_I)\pi_I - r_0\pi_I \\
\pi_{I+2} &= \varrho_0\varrho_1\cdots\varrho_{I-1}\frac{r_0}{\mu_I}\varrho_{I+1}\pi_0.
\end{aligned}
$$

States l located on the right hand side of a cycle with Coxian distribution i.e. chain states numbered by $l = I+2, I+3, \ldots, N-1$, have probabilities equal to

$$\pi_l = \varrho_0\varrho_1\cdots\varrho_{I-1}\frac{r_0}{\mu_I}\varrho_{I+1}\cdots\varrho_{l-1}\pi_0.$$

A sum of probabilities of "Coxian states" (i.e. Markov chain states numbered by $l = N, N+1, \ldots, N+K$) is given by:

$$\sum_{i=N}^{N+K}\pi_i = \pi_I r_0\sum_{j=1}^{K+1}\frac{1}{r_j} = \varrho_0\varrho_1\cdots\varrho_{I-1}\pi_0\sum_{j=1}^{K+1}\frac{1}{r_j},$$

where π_i are substituted by expression from Eq. (5). The normalisation equation of the investigated chain depending upon π_0 is formulated as:

$$
\begin{aligned}
&\pi_0(1+\varrho_0+\varrho_0\varrho_1+\cdots+\varrho_0\varrho_1\cdots\varrho_{I-1}+\varrho_0\varrho_1\cdots\varrho_{I-1}\textstyle\sum_{j=1}^{K+1}\frac{1}{r_j}+ \\
&\varrho_0\varrho_1\cdots\varrho_{I-1}\tfrac{r_0}{\mu_I}+\varrho_0\varrho_1\cdots\varrho_{I-1}\tfrac{r_0}{\mu_I}\varrho_{I+1}+\cdots+\varrho_0\varrho_1\cdots\varrho_{I-1}\tfrac{r_0}{\mu_I}\varrho_{I+1}\cdots\varrho_{N-2}) = 1
\end{aligned}
\tag{6}
$$

A possible reduction will replace the series of transitions between I and $I+1$ by one transition with rate R equal to the mean of the Coxian distribution. The resulting Markov chain has a normalisation equation as follows:

$$
\begin{aligned}
&\pi'_0(1+\varrho_0+\varrho_0\varrho_1+\cdots+\varrho_0\varrho_1\cdots\varrho_{I-1}+\varrho_0\varrho_1\cdots\varrho_{I-1}\tfrac{R}{\mu_I}+ \\
&\varrho_0\varrho_1\cdots\varrho_{I-1}\tfrac{R}{\mu_I}\varrho_{I+1}+\cdots++\varrho_0\varrho_1\cdots\varrho_{I-1}\tfrac{R}{\mu_I}\varrho_{I+1}\cdots\varrho_{N-2}) = 1.
\end{aligned}
\tag{7}
$$

The expressions in brackets in Eq. (6) and (7) are replaced by shorthand terms d and D, respectively, i.e. $\pi_0 = \frac{1}{d}$ and $\pi'_0 = \frac{1}{D}$. Since r_0 is always greater than R, factors $\varrho_0\cdots\varrho_{I-1}\frac{r_0}{\mu_I}\varrho_{I+1}\cdots\varrho_l$ are always greater than corresponding ones with R. As they appear in denominators, we get $\pi_0 < \pi'_0$. From (4) and the fact that π'_l, $l = 1, 2, \ldots, I$, have similar structure we conclude that

$$\pi_l < \pi'_l, \qquad l = 1, 2, \ldots, I.$$

In case of the chain depicted in Figure 21 sequential transitions between I and $I+1$ take place in both directions. As before, state probabilities up to the

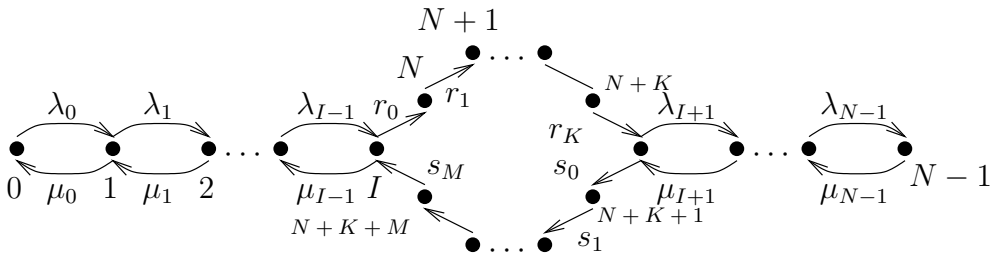

Figure 21: The Markov chain underlying a semi-input-output linear component with two potentially amalgamated sequences

state I may be computed depending upon π_0 as in Eq. (4). Analogous reasoning for the segmented branches gives us:

$$\begin{array}{lcll} \pi_{N+i} & = & \frac{r_0}{r_i}\pi_I, & i = 0, 1, \ldots, K \\ \pi_{N+K+j} & = & \frac{s_0}{s_j}\pi_{I+1}, & j = 1, 2, \ldots, M \end{array}$$

The global balance equations for chain states I and $I+1$ using which we compute the probabilities π_{I+1} and π_{I+2} are:

$$\begin{array}{lcl@{\quad}c@{\quad}lcl} \pi_I(r_0 + \mu_{I-1}) & = & \lambda_{I-1}\pi_{I-1} + s_M\pi_{N+K+M} & \Longrightarrow & \pi_{I+1} & = & \frac{r_0}{s_0}\pi_I \\ \pi_{I+1}(s_0 + \lambda_{I+1}) & = & \mu_{I+1}\pi_{I+2} + r_{K+1}\pi_{N+K} & \Longrightarrow & \pi_{I+2} & = & \frac{r_0}{s_0}\varrho_{I+1}\pi_I \end{array}$$

and on the right hand side of the state $I + 1$, i.e. for the chain states numbered by l, $l = I + 2, I + 3, \ldots, N$, the probabilities are equal to:

$$\pi_l = \varrho_0\varrho_1 \cdots \varrho_{I-1}\frac{r_0}{s_0}\varrho_{I+1} \cdots \varrho_{l-1}\pi_0.$$

From the normalisation equations we obtain, for the initial chain, $\pi_0 = \frac{1}{d}$ and for the reduced one: $\pi'_0 = \frac{1}{D}$,

$$\begin{array}{rcl} d & = & 1 + \varrho_0 + \cdots + \varrho_0\varrho_1 \cdots \varrho_{I-1} + \varrho_0 \cdots \varrho_{I-1}\frac{r_0}{s_0} + \varrho_0 \cdots \varrho_{I-1}\frac{r_0}{s_0}\varrho_{I+1} + \\ & & \varrho_0 \cdots \varrho_{I-1}\frac{r_0}{s_0}\varrho_{I+1} \cdots \varrho_{N-2} + \Sigma \end{array} \tag{8}$$

and

$$\begin{array}{rcl} D & = & 1 + \varrho_0 + \cdots + \varrho_0\varrho_1 \cdots \varrho_{I-1} + \varrho_0 \cdots \varrho_{I-1}\frac{r_0}{s_0} + \varrho_0 \cdots \varrho_{I-1}\frac{R}{S}\varrho_{I+1} + \\ & & \varrho_0 \cdots \varrho_{I-1}\frac{R}{S}\varrho_{I+1} \cdots \varrho_{N-2} \end{array} \tag{9}$$

where Σ is a shorthand for a sum of probabilities of states lying on the segmented branches. The parameters R and S are means of the Coxian distributions, i.e.

$$R = \frac{\sum_{i=0}^{K+1}\prod_{j=1,j\neq i}^{K+1} r_j}{\prod_{i=0}^{K+1} r_i}, \qquad S = \frac{\sum_{i=0}^{M}\prod_{j=1,j\neq i}^{M} s_j}{\prod_{i=0}^{M} s_i}.$$

The fraction $\frac{r_0}{s_0}$ is greater than or equal to $\frac{R}{S}$ if and only if

$$\frac{\prod_{i=0}^{M} s_i \sum_{i=0}^{K+1} \prod_{j=1,j\neq i}^{K+1} r_j}{\prod_{i=1}^{K+1} r_i \sum_{i=0}^{M} \prod_{j=1,j\neq i}^{M} s_j} \leq 1 \tag{10}$$

If the condition (10) is satisfied, the denominator d from Eq. (8) is greater than the denominator D derived from Eq. (9) and $\pi_0 < \pi'_0$ which may be extended up to I, $\pi_i < \pi'_i$, $i = 1, 2, \ldots, I$.

A direct conclusion is that if the initial state 0 is a starting point of the segmented edge (or, dually, the final state $N-1$ is a starting point of such a series) and this is the only segmented sequence in the investigated chain, $\pi_0 + \sum_{i=0}^{K} \pi_{N+i} = \pi'_0$ ($\pi_{N-1} + \sum_{i=0}^{M} \pi_{N+i} = \pi'_{N-1}$ dually) holds.

4.6 An Example of Amalgamation in Semi-Input-Output Linear Components of a PEPA Model

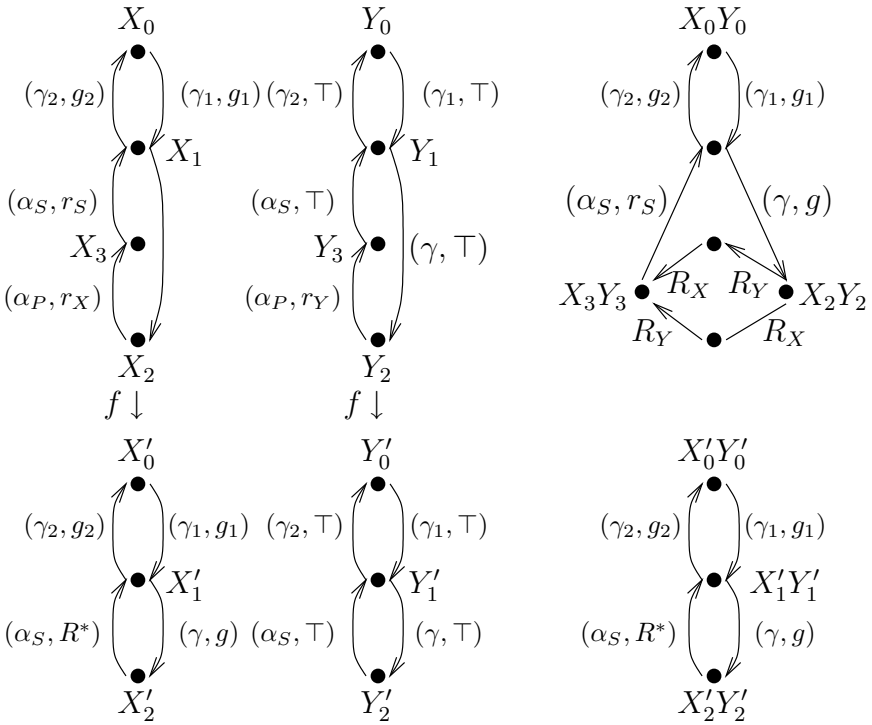

Figure 22: A PEPA model to be reduced, $X_0 \underset{\{\gamma_1,\gamma_2,\alpha_S,\gamma\}}{\bowtie} Y_0$ and its compact form $X'_0 \underset{\{\gamma_1,\gamma_2,\alpha_S,\gamma\}}{\bowtie} Y'_0$; shorthands $R_X = (\alpha_P, r_X)$, $R_Y = (\alpha_P, r_Y)$

After solving global balance equations of the Markov chain underlying the

PEPA model $X_0 \underset{\{\gamma_1,\gamma_2,\alpha_S,\gamma\}}{\bowtie} Y_0$ presented Figure 22 we compute a sum of probabilities:

$$\pi_{X_2Y_2} + \pi_{X_3Y_2} + \pi_{X_2Y_3} + \pi_{X_3Y_3} = \frac{g_1 g\big(r_s(r_X^2+r_Xr_Y+r_Y^2)+r_Xr_Y(r_X+r_Y)\big)}{(g_1+g_2)r_Sr_Xr_Y(r_X+r_Y)+g_1g\big(r_s(r_X^2+r_Xr_Y+r_Y^2)+r_Xr_Y(r_X+r_Y)\big)} = \frac{g_1 g\big(r_s(r_X^2+r_Xr_Y+r_Y^2)+r_Xr_Y(r_X+r_Y)\big)}{d}. \quad (11)$$

The local sequence (α_P, α_S) is a beginning sequence in the model $X_0 \underset{\{\gamma_1,\gamma_2,\alpha_S,\gamma\}}{\bowtie} Y_0$ and it may be amalgamated according to the algorithm provided for the case **a.II** in Section 4.4.2. The mean transition rate R of passing through the 2-lattice confluence labelled with (α_P, r_X) and (α_P, r_Y) is calculated by the formula (2). The value of R^* is found as the mean of the transition rate of the Coxian distribution which is the convolution of two exponential distributions with parameters R and r_S, respectively,

$$R^* = \frac{r_S r_X r_Y (r_X + r_Y)}{r_X r_Y (r_X + r_Y) + r_S(r_X^2 + r_X r_Y + r_Y^2)}.$$

The local balance equations written for the Markov chain underlying the reduced model $X_0' \underset{\{\gamma_1,\gamma_2,\alpha_S\gamma\}}{\bowtie} Y_0'$ are:

$$\pi_{X_1'Y_1'} = \frac{g_1}{g_2}\pi_{X_0'Y_0'}, \qquad \pi_{X_2'Y_2'} = \frac{g_1}{g_2}\frac{g}{R^*}\pi_{X_0'Y_0'}$$

which give us: $\pi_{X_2'Y_2'} = \frac{g_1 g}{R^*(g_1+g_2)+g_1g_2}$ and this expression is the same as the one in Eq. (11). Notice that the throughput of the activity labelled with α_S is also preserved in the reduced chain; hence $r_S\pi_{X_3Y_3} = \frac{g_1 g r_X r_Y r_S(r_X+r_Y)}{d}$ is equal to $R^*\pi_2' = \frac{g_1 g R^*}{R^*(g_1+g_2)+g_1g_2}$.

5 Conclusions and Further Work

In this paper we have investigated which sequences in sequential PEPA components may be potentially amalgamated in order to obtain a smaller model preserving some features of the initial one. We make a decision, based upon whether an action is synchronised or not and upon the observational assumption, as to which action type should be chosen for the amalgamated transition. We have proposed definitions of distinguishing sequences of actions to be amalgamated and a classification of them depending on the model topology. For each case of possible sequence amalgamation we gave an explanation of how to calculate a transition rate for the introduced action. We defined and discussed in detail PEPA sequential components termed semi-input-output linear components. Finally, we showed the impact of the amalgamation procedure upon these components, including also an illustrative example.

Amalgamation of sequences may lead to models whose underlying Markov chains are reversible. We are interested in further investigations in which cases the reduced model has this feature. The crucial point is to investigate for which listed local sequence cases an underlying Markov chain to be reduced is insensitive. Such a chain may be used for computations of values of some performance measures. Not insensitive chains may be used for computations of bounds of these measures. For the other hand, we also want to study in the future upper and lower bounds and to research models obtained from the initial one by adding extra actions in order to get another Markov chain.

Acknowledgements
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