Metric Denotational Semantics for PEPA

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Abstract

Stochastic process algebras, which combine the features of a process calculus with stochastic analysis, were introduced to enable compositional *performance analysis of systems. At the level of syntax, compositionality presents itself in terms of operators, which can be used to build more complex systems from simple components. Denotational semantics is a method for assigning to syntactic objects elements of a suitably chosen semantic domain. This is compositional in style, as operators are represented by certain functions on the domain, and often allows to gain additional insight by considering the properties of those functions. We consider Performance Evaluation Process Algebra (PEPA), a stochastic process algebra introduced by Hillston [9]. Based on the methodology introduced by de Bakker & Zucker, we give denotational semantics to PEPA by means of a complete metric space of suitably enriched trees. We investigate continuity properties of the PEPA operators and show that our semantic domain is fully abstract with respect to strong equivalence.*

1 Introduction

Probabilistic and stochastic phenomena are important in many areas of computing, for example, distributed systems, fault tolerance, communication protocols and performance analysis, and thus formal and automated tools for reasoning about such systems are needed.

In this paper we consider a stochastic process algebra called Performance Evaluation Process Algebra (PEPA) originally introduced by Hillston [9]. Stochastic process algebras arise through enhancing specification languages such as CCS [15] to include stochastic behaviour in the form of exponential timing, e.g. [3, 2, 8, 9], and probabilistic behaviour, e.g. [5, 14]. This is achieved by allowing basic actions (which are instantaneous in the classical process calculi) to have *duration* (exponentially distributed random variable) and replacing nondeterminism by a form of probabilistic choice. Starting with the standard operational semantics for PEPA, given in terms of a multi-transition system and strong equivalence, we aim to provide the calculus with a denotational metric space semantics, derived following the techniques introduced by de Bakker and Zucker [1], which is fully abstract with respect to strong equivalence.

The motivation behind PEPA is to introduce *compositionality* into the calculus, allowing processes to be composed from components by means of operators. However, this compositionality is present only at the level of syntax: once the model is reduced to the equilibrium state equation the compositionality sometimes breaks down and alternative methods, such as "product" solutions [10] or time scale decomposition [7], have to be applied. Denotational semantics assigns mathematical objects (here elements of metric spaces) as denotations to process terms, with operators on processes modelled in terms of operations on the denotations. This gives rise to an algebra of process denotations, which can be computed and analysed independently, and then composed. Full abstraction in our context ensures that (strong) equivalence of PEPA terms corresponds to equality of their denotations. The hope is that, through compositionality, this approach may shed some light on the handling of large systems in the workbench. Also, comparisons with the classical process algebra work are likely to be made easier, opening up the possibility for the transfer of existing methodologies such as verification and logics.

Recent research in the area has focussed mainly on the operational side, see e.g. [3, 2, 8, 9] for stochastic and [5, 14] for probabilistic behaviour. Some effort has been put into building fully abstract denotational semantics for (pure) probabilistic process algebras, of which we mention a full abstraction result for may-testing [11], where the model is rather involved and no recursion is considered; a denotational semantics for a probabilistic extension of CSP [17] in terms of conditional probability measures on the space of infinite traces; a characterisation of testing preorders for probabilistic processes which is shown to be fully abstract [19]; and our earlier work for the reactive probabilistic language RP [12]. However, to our knowledge, so far little attention has been paid to denotational models for stochastic process algebras such as PEPA above, Extended Markovian Process Algebra (EMPA) [3], Markovian Process Algebra (MPA) [2] and Markovian Timed Performance Algebra (MTIPP) [8]. We believe this area deserves more attention in the light of Edalat's work [4].

2 Performance Evaluation Process Algebra (PEPA)

We recall basic notions of the PEPA language introduced in [9]. A *system* is described by the interaction of *components*, where components engage, singly or multiply, in *activities*. Each activity is distinguished by an *action type*, which is equivalent to the usual process algebra notion of an instantaneous action, but in contrast to the latter each activity also has associated with it a *duration* (a random variable with an exponential distribution). We denote by $\mathcal{A}$ the countable set of all possible action types. In situations where the type of the action is unknown, it may be represented by the distinguished action type, τ. Thus, an activity a is a pair (α, r), where $\alpha \in \mathcal{A}$ (the action type) and r is the activity rate, which can either be specified, i.e. a real number $r \in (0, \infty)$, or unspecified, i.e. of the form $w\top$ where $w \in \mathbb{N}$ is a weight.

Rates are used to reflect race conditions between activities in the system. If several activities are enabled at the same time, each will have its own associated timer, and the faster component succeeds. The probability of a happening within a time period t is $F_a(t) = 1 - e^{-rt}$. When activities have unspecified rates (i.e. $r = w\top$), they are understood to be *passive* (i.e. shared with another component which determines the actual rate). Comparison and certain arithmetic operations are defined for unspecified rates (see [9]), but specified and unspecified rates cannot be mixed in arithmetic expressions as the latter serve as weights inducing relative probability, while the former are (constant) parameters of exponential distribution.

The syntax for PEPA terms is defined as follows, where $L \subseteq \mathcal{A}$:

$$E ::= (\alpha, r).\, E \mid E + F \mid E \underset{L}{\bowtie} F \mid E/L \mid X \mid A$$

"+" represents choice, " $\underset{L}{\bowtie}$ " cooperation, "/" hiding, X is a member of a set of variables $\mathcal{X}$ and (assuming the existence of a countable set of constants) A is a constant. Constants are used to define recursive components.

Before the operational semantics of PEPA can be defined we first need to define the *apparent rates* of components.

Definition 2.1 *For any component E in PEPA the* apparent rate *of action type α of E, denoted $r_\alpha(E)$, is the sum of all the rates of all activites of action type α in E. This is defined inductively on the structure of components as follows.*

$$\begin{aligned}
r_\alpha((\beta, r).\, E) &= \begin{cases} r & \textit{if } \beta = \alpha \\ 0 & \textit{if } \beta \neq \alpha \end{cases} \\
r_\alpha(E + F) &= r_\alpha(E) + r_\alpha(F) \\
r_\alpha(E/L) &= \begin{cases} r_\alpha(E) & \textit{if } \alpha \notin L \\ 0 & \textit{if } \alpha \in L \end{cases} \\
r_\alpha(E \underset{L}{\bowtie} F) &= \begin{cases} \min(r_\alpha(E), r_\alpha(F)) & \textit{if } \alpha \in L \\ r_\alpha(E) + r_\alpha(F) & \textit{if } \alpha \notin L \end{cases}
\end{aligned}$$

The operational semantics of PEPA is given by the following transition rules:

Prefix

$$\frac{}{(\alpha, r).\, E \xrightarrow{(\alpha, r)} E}$$

Choice

$$\frac{E \xrightarrow{(\alpha, r)} \tilde{E}}{E + F \xrightarrow{(\alpha, r)} \tilde{E}} \qquad\qquad \frac{F \xrightarrow{(\alpha, r)} \tilde{F}}{E + F \xrightarrow{(\alpha, r)} \tilde{F}}$$

Cooperation

$$\frac{E \xrightarrow{(\alpha, r)} \tilde{E}}{E \underset{L}{\bowtie} F \xrightarrow{(\alpha, r)} \tilde{E} \underset{L}{\bowtie} F}\ (\alpha \notin L) \qquad\qquad \frac{F \xrightarrow{(\alpha, r)} \tilde{F}}{E \underset{L}{\bowtie} F \xrightarrow{(\alpha, r)} E \underset{L}{\bowtie} \tilde{F}}\ (\alpha \notin L)$$

$$\frac{E \xrightarrow{(\alpha, r_1)} \tilde{E} \quad F \xrightarrow{(\alpha, r_2)} \tilde{F}}{E \underset{L}{\bowtie} F \xrightarrow{(\alpha, R)} \tilde{E} \underset{L}{\bowtie} \tilde{F}}\ (\alpha \in L) \quad \text{where } R = \frac{r_1}{r_\alpha(E)} \frac{r_2}{r_\alpha(F)} \min(r_\alpha(E), r_\alpha(F))$$

Hiding

$$\frac{E \xrightarrow{(\alpha, r)} \tilde{E}}{E/L \xrightarrow{(\alpha, r)} \tilde{E}/L}\ (\alpha \notin L) \qquad\qquad \frac{E \xrightarrow{(\alpha, r)} \tilde{E}}{E/L \xrightarrow{(\tau, r)} \tilde{E}}\ (\alpha \in L)$$

Constant

$$\frac{E \xrightarrow{(\alpha,r)} \tilde{E}}{A \xrightarrow{(\alpha,r)} \tilde{E}} \; (A \stackrel{def}{=} E)$$

As an operational equivalence we work with *strong equivalence* of PEPA terms introduced in [9]. This notion is based on (strong) bisimulation equivalence, and is similar to probabilistic bisimulation of Larsen and Skou [13]. The definition requires the notion of a *conditional transition rate* between components.

Informally, the transition rate between two components E_i,E_j via a given action type α (denoted $q(E_i,E_j,\alpha)$) is the rate at which component E_i evolves to behave as component E_j as a result of completing an activity of action type α. Formally,

$$q(E_i,E_j,\alpha) = \left\{ \sum_a r_\alpha(E_j) \mid a \in \mathcal{A}ct(E_i|E_j) \text{ and } a = (\alpha, r) \right\}$$

where $\mathcal{A}ct(E_i|E_j) = \{a \mid E_i \xrightarrow{a} E_j\}$. Moreover, if we consider a set of possible derivatives S, the *total conditional transitional rate* from E_i to S, denoted $q[E_i,S,\alpha]$, is defined to be

$$q[E_i,S,\alpha] = \sum_{E_j \in S} q(E_i,E_j,\alpha).$$

Then, informally, two PEPA components are equivalent if there is an equivalence relation between them such that, for any action type α, the total conditional transition rates from those components to any equivalence class, via activities of this type, are the same. Formally, the equivalence is defined for components $\mathcal{A}$ of PEPA as follows.

Definition 2.2 *An equivalence relation over components,* $\mathcal{R} \subseteq \mathcal{C} \times \mathcal{C}$, *is a* strong equivalence *if whenever* $(E,F) \in \mathcal{R}$ *then, for all* $\alpha \in \mathcal{A}$ *and for all* $S \in \mathcal{A}/\mathcal{C}$,

$$q[E,S,\alpha] = q[F,S,\alpha].$$

One can show by means of methods similar to those used in [15] that there exists a largest strong equivalence relation denoted $\cong$. The relation $\cong$ is a congruence for PEPA, and is sufficient to ensure that equivalent components exhibit exactly the same behaviour [9].

3 Defining Metric Denotational Semantics

In [1] de Bakker and Zucker introduce methodology based on the theory of metric spaces, by means of which given a (process) language one can derive a domain equation defining denotations for terms of that language. We illustrate their approach with the help of a simple example. Consider the language which contains prefixing $a.E$ ($a \in \mathcal{A}$), choice $E + F$ and recursion ($A = E$). We first consider *simple* processes, that is, those which are derived in the subcalculus by means of just the syntactic operator used in the inductive step of the transition rules of the calculus. As in our language these are the processes which can be derived using (successive applications of) just prefixing, the following domain equation of simple processes is reached:

$$D \cong \{p_0\} \cup (\mathcal{A} \times D)$$

where p_0 denotes the inactive process. Intuitively, the solution of this domain equation consists of p_0 and all finite sequences of the form $(a_1, (a_2 \ldots (a_n, p_0) \ldots))$ where $n \in \mathbb{N}$, together with all infinite sequences $(a_1, (a_2 \ldots))$. We can think of $(a_1, (a_2 \ldots (a_n, p_0) \ldots))$ as the process that can perform the actions $a_1, a_2 \ldots a_n$ in sequence and then terminate, and $(a_1, (a_2 \ \ldots))$ can be considered as an infinite process performing the sequence $a_1 a_2 \ldots$.

In the second step, to model the whole calculus de Bakker and Zucker "lift" the denotations of simple processes to *sets* using the induced Hausdorff distance between the sets, and add semantic operators to model the remaining syntactic operators of the calculus. This corresponds to the introduction of an appropriate powerset operator $\mathcal{P}$ into the equation. Assuming deterministic choice, for the above simple language we obtain:

$$D \cong \{p_0\} \cup \mathcal{P}(\mathcal{A} \times D)$$

where $\mathcal{P}$ denotes the non-empty closed sets. Now set-theoretic union on D corresponds to the syntactic choice: the denotation $\{(a, p_0)\} \cup \{(b, p_0)\}$ can intuitively be thought of as the process that can either perform the action a and then terminate, or perform the action b and then terminate.

Upon initial analysis, there appears to be a clear similarity between terms in PEPA and those of the probabilistic process algebra RP of [12], in that in both languages there are actions a process can perform, and *values* associated with the possible successor processes. The difference is that, for PEPA terms, the values are *rates*, whereas for RP terms, the values are *values of a probabilistic distribution*. It might seem that by mapping rates to (relative) probabilities (in the sense of a discrete time Markov chain) PEPA can be reduced to a probabilistic language, and that the semantics for the latter, e.g. as developed in [12], will also be adequate for the former. Unfortunately, this approach fails as it allows to identify denotations of non-equivalent components. To see this, consider the processes below where $r_1, r_2 \in \mathbb{R}^{>0}$ such that $r_1 \neq r_2$:

Then, for any such r_1, r_2, the probabilities of p and q performing the action α and then terminating are given by:

$$\frac{r_1}{r_\alpha(p)} \quad \text{and} \quad \frac{r_2}{r_\alpha(q)}$$

respectively. Moreover, since (α, r_1) and (α, r_2) are the only activities that p and q can perform, the apparent rates are $r_\alpha(p) = r_1$ and $r_\alpha(q) = r_2$. Thus, the probabilities of p and q performing the action α and terminating are both 1, and therefore p and q would be mapped on to the same denotation. However, since $r_1 \neq r_2$, they are distinguished by strong equivalence.

So, there are important differences between the two languages, and hence the construction of [12] cannot directly be applied to PEPA. To give another example, in RP the probability of a path can be computed by multiplying the probabilities of each step; similar calculations, for example summing up (or multiplication) of the rates along paths are meaningless for PEPA, as we demonstrate below.

Then it follows that the total rate (= the sum of rates along the path) of both $\tilde{p}$ and $\tilde{q}$ of performing the action α and then the action β is $r_1 + r_2$, and yet, since $r_1 \neq r_2$, they are distinguished by strong equivalence.

We omit the discussion of unspecified rates to simplify the presentation; unspecified rates can be added at a cost of an extra level of complexity which is not essential in understanding the basic model construction. As our guiding principle for the construction we shall adopt the property that small differences in rates result in small differences in the distance between denotations, and vice versa.

Furthermore, we shall assume that the rates are bounded, and that there also exists a lower bound on rates. These two conditions hold for any finite PEPA term and are respectively required to prevent unboundedness of the metric and the rate 0 entering the model through limits of Cauchy sequences.

Similarly to the metric model for RP [12], we construct the denotations for processes as enriched trees (arcs have rates as well as actions). The model is derived in two steps following the techniques introduced in [1]: first, we define the metric on simple components (those defined in terms of the prefix), and then generalise the metric construction to suitably defined trees.

4 A Metric for Simple Components of PEPA

We first turn our attention to *simple components* of PEPA, that is, those which can be defined in terms of the prefix operator $(\alpha, r).\, E$, cf. [1].

Intuitively, a simple component will be represented by a pair consisting of an activity and another simple component. Assuming D denotes the set of such simple components, and letting p_0 denote the process which performs no activities (this is needed as a basis for induction) we reach the domain equation:

$$D \cong \{p_0\} \cup \left(\mathcal{A} \times \mathbb{R}^{>0}\right) \times D$$

where $\mathbb{R}^{>0} \stackrel{\mathit{def}}{=} \{r \mid r \in \mathbb{R} \text{ and } 0 < \mathbf{r} < r < \mathbf{R} < \infty\}$; here $\mathbf{r}$ denotes the lower bound on rates and $\mathbf{R}$ the upper bound.

Thus, any (non-trivial) simple component p is of the form $p = (\alpha, r)\tilde{p}$, where $\alpha \in \mathcal{A}$, $r \in \mathbb{R}^{>0}$ and $\tilde{p} \in D$, and we can think of p as the component that can perform the action α with rate r and then become $\tilde{p}$. When defining the distance function on simple components we rely on the following intuition: roughly speaking, the distance will depend on how *similar* the initial activities are, and how *close* the successor components are. More specifically, the distance is set to 1 if the simple components differ on their initial actions, and otherwise we calculate the (normalised absolute value) of the difference between the initial rates and combine this with the distance of the successor components.

Formally, following the techniques of [1], we inductively define a collection of metric spaces $(D_n, d_n)_n$, $n = 0,1,\ldots$, where the elements of the spaces model *finite* simple components. Informally, $D_0 \subset D_1 \subset \ldots \subset D_n \ldots$ form a sequence of sets, with D_n modelling the components capable of performing up to n steps.

Definition 4.1 *Let (D_n, d_n), $n = \mathbb{N}$, be a collection of metric spaces defined inductively by putting $D_0 = \{p_0\}$, $d_0(p,q) = 0$ (since $p, q \in D_0 \iff p = q = p_0$), and*

$$D_{n+1} = \{p_0\} \cup \left(\mathcal{A} \times \mathbb{R}^{>0}\right) \times D_n,$$

with d_{n+1} given by: $d_{n+1}(p,q) = 0$ if $p = q = p_0$ and 1 if $p = p_0$, $q \neq p_0$ or $p \neq p_0$, $q = p_0$. Otherwise, $p = (\alpha_1, r_1)\tilde{p}$ and $q = (\alpha_2, r_2)\tilde{q}$ for some (α_1, r_1),$(\alpha_2, r_2) \in \mathcal{A} \times \mathbb{R}^{>0}$, $\tilde{p}, \tilde{q} \in D_n$, and we put

$$d_{n+1}(p,q) = \begin{cases} 1 & \text{if } \alpha_1 \neq \alpha_2 \\ \frac{1}{2}\left(\frac{1}{\mathbf{R}}|r_1 - r_2| + d_n(\tilde{p},\tilde{q})\right) & \text{if } \alpha_1 = \alpha_2. \end{cases}$$

Lemma 4.1 *Let (D_n, d_n) $n = 0,1,\ldots$ be defined as above. Then $0 \leq d_n(p,q) \leq 1$ for all $p, q \in D_n$.*

Proof. The proof follows by induction on $n \in \mathbb{N}$. For $n = 0$ there is nothing to prove. Now suppose $0 \leq d_k(p,q) \leq 1$ for all $p, q \in D_k$ and consider $d_{k+1}(p,q)$ for any $p, q \in D_{k+1}$. Then, by definition, we have $0 \leq d_{k+1}(p,q) \leq 1$ in all cases except when $p = (\alpha, r_1)\tilde{p}$ and $q = (\alpha, r_2)\tilde{q}$ for some (α, r_1),$(\alpha, r_2) \in \mathcal{A} \times \mathbb{R}^{>0}$ and $\tilde{p}, \tilde{q} \in D_k$. In the latter case we have by definition:

$$\begin{aligned} d_{k+1}(p,q) &= \frac{1}{2}\left(\frac{1}{\mathbf{R}}|r_1 - r_2| + d_k(\tilde{p},\tilde{q})\right) \\ &= \frac{1}{2}\frac{1}{\mathbf{R}}|r_1 - r_2| + \frac{1}{2}d_k(\tilde{p},\tilde{q}) \\ &\leq \frac{1}{2} + \frac{1}{2} = 1 \quad \text{by induction and since } \mathbf{R} \text{ is a maximum rate.} \end{aligned}$$

□

Proposition 4.1 *(D_n, d_n) is a metric space for all $n \in \mathbb{N}$.*

Proof.

1. We show $d_{k+1}(p,q) = 0$ if and only if $p = q$. By definition, $d_{k+1}(p,p) = 0$ for all $p \in D_{k+1}$; now suppose $d_{k+1}(p,q) = 0$, then again by definition it must be the case that $p = (\alpha, r_1)\tilde{p}$ and $q = (\tilde{\alpha}, r_2)\tilde{q}$ such that $\alpha = \tilde{\alpha}$, $r_1 = r_2$ and $d_k(\tilde{p},\tilde{q}) = 0$. By the induction hypothesis $d_k(\tilde{p},\tilde{q}) = 0$ if and only if $\tilde{p} = \tilde{q}$, and hence $p = q$ as required.
2. $d_{k+1}(p,q) = d_{k+1}(q,p)$ for all $p, q \in D_{k+1}$ follows by the commutativity of $+$ and induction.
3. The inequality $d_{k+1}(p,q) + d_{k+1}(q,s) \geq d_{k+1}(p,s)$ for all p, q and $r \in D_{k+1}$ follows from Lemma 4.1 in all cases except $p = (\alpha, r_1)\tilde{p}$, $q = (\alpha, r_2)\tilde{q}$ and $s = (\alpha, r_3)\tilde{s}$. In

the latter case, by definition of d_{k+1} and rearranging the terms, we have:

$$\begin{aligned}
&d_{k+1}(p,q) + d_{k+1}(q,s) - d_{k+1}(p,s) \\
&= \frac{1}{2\mathbf{R}}\left(|r_1 - r_2| + |r_2 - r_3| - |r_1 - r_3|\right) + \frac{1}{2}\left(d_k(\tilde{p},\tilde{q}) + d_k(\tilde{q},\tilde{s}) - d_k(\tilde{p},\tilde{s})\right) \\
&\geq 0 + \frac{1}{2}\left(d_k(\tilde{p},\tilde{q}) + d_k(\tilde{q},\tilde{s}) - d_k(\tilde{p},\tilde{s})\right) \quad \text{from properties of the euclidean metric} \\
&\geq 0 \quad \text{by the induction hypothesis.}
\end{aligned}$$

□

Let $D_\omega = \bigcup_n D_n$ and $d_\omega = \bigcup_n d_n$. (D_ω, d_ω) is the metric space of all *finite* simple components, and will need to be completed to include the infinite components as well. Before we do this, however, we would like to point out that D_ω can be endowed with the classical metric $(\tilde{d}_n)$ of [1]. The definition is similar to the definition of d_n above except in the following case, in which it is given by:

$$\tilde{d}_{n+1}((\alpha_1,r_1)\tilde{p},(\alpha_2,r_2)\tilde{q}) = \begin{cases} 1 & \text{if } (\alpha_1,r_1) \neq (\alpha_2,r_2) \\ \frac{1}{2}\tilde{d}_n(\tilde{p},\tilde{q}) & \text{if } (\alpha_1,r_1) = (\alpha_2,r_2). \end{cases}$$

The main difference is that the euclidean distance between rates is not taken into account. Let us now consider the simple components $q_1, q_2 \in D_\omega$ shown in Figure 1.

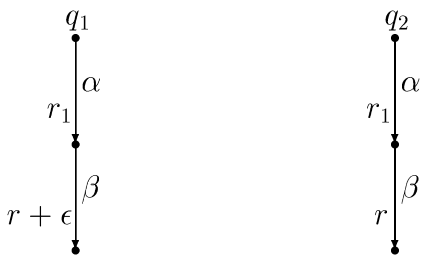

Figure 1 Metric d_ω vs metric of [1]

Then, by definition of our distance, we have:

$$\begin{aligned}
d_2(q_1,q_2) &= \frac{1}{2}\left(\frac{1}{\mathbf{R}}|r_1 - r_1| + \frac{1}{2}\Big(\frac{1}{\mathbf{R}}|r+\epsilon - r| + 0\Big)\right) \\
&= \frac{\epsilon}{4\mathbf{R}}.
\end{aligned}$$

Thus, as $\epsilon \to 0$, we have $d(q_1,q_2) \to 0$, which reflects our guiding principle that small differences in rates induce small differences in the distance. On the other hand, according to the ultra-metric $\tilde{d}$ we have $\tilde{d}_2(q_1,q_2) = \frac{1}{2}$ for all $\epsilon > 0$.

It can be shown that our metric, although not an ultra-metric in general, nevertheless specialises to the metric of [1]. To see this, consider a restriction of the set of rates $\mathbb{R}^{>0}$ to just a singleton set $\mathbf{1}$ (meaning all simple components perform at the same rate or instantaneously). Then any $p\,(\neq p_0) \in D_{n+1}$ in this restricted setting will be of the form

$(\alpha, \mathbf{1})\tilde{p}$ for some $\alpha \in \mathcal{A}$ and $\tilde{p} \in D_n$, and so our metric reduces to:

$$d_{n+1}(p,q) = \begin{cases} 1 & \text{if } \alpha_1 \neq \alpha_2 \\ \frac{1}{2} d_n(\tilde{p},\tilde{q}) & \text{if } \alpha_1 = \alpha_2. \end{cases}$$

since clearly $|\mathbf{1} - \mathbf{1}| = 0$, and so coincides with the metric of de Bakker and Zucker [1].

The space (D_ω, d_ω) still lacks *infinite* simple components needed to model recursive computations. The infinite PEPA component $A = (\alpha, r).A$ is modelled by the Cauchy sequence $\langle p_n \rangle_n$, where $p_n \in D_n$, with respect to d_ω, as shown in Figure 2.

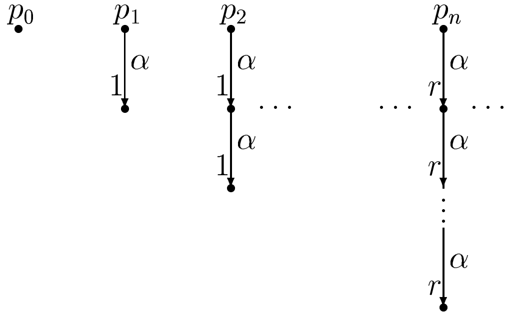

Figure 2 Example of recursive simple computations

Then it is straightforward to show (by induction) that $d_\omega(p_n, p_m) = 2^{-\min\{m,n\}}$, and so the sequence is indeed Cauchy, and its limit is the infinite component we require.

As in [1], we now apply the standard metric completion technique to derive the metric space D of (finite and infinite) simple components consisting of D_ω together with all limit points $p = \lim_{n\to\infty} p_n$, with $\langle p_n \rangle_n$ a Cauchy sequence in D_ω, such that $p_n \in D_n$ for all $n \in \mathbb{N}$.

Definition 4.2 *(D, d) is the completion of (D_ω, d_ω).*

By a straightforward application of the techniques of [1] it follows that D satisfies the required domain equation (up to isometry).

Theorem 4.1 $D \cong \{p_0\} \cup (\mathcal{A} \times \mathbb{R}^{>0}) \times D.$

5 A Domain Equation for PEPA

In the previous section we have defined denotations of simple components of PEPA (the elements of D) by representing them either as p_0 (termination), or as limits $\lim_{n\to\infty} p_n$ of Cauchy sequences of (finite) simple components $p_n \in D_n$. To allow choice it is necessary to use *sets* of elements of D as denotations for all components of PEPA. In the non-stochastic setting, e.g. [1], this is achieved by introducing an appropriate powerset (e.g. the non-empty closed subsets) into the domain equation, and considering the *Hausdorff distance* between the sets, which for a complete metric space yields a complete metric space of subsets.

Definition 5.1 *Let (M,d) be a metric space and let X,Y be subsets of M. The* Hausdorff distance *between sets X,Y is given by:*

(a) $d(x,Y) = \inf\{d(x,y) \mid y \in Y\}$
(b) $d(X,Y) = \max\{\sup\{d(x,Y) \mid x \in X\}, \sup\{d(y,X) \mid y \in Y\}\}$

where by convention $\inf \emptyset = 1$ *and* $\sup \emptyset = 0$.

Once the metric space of subsets has been derived, the operations of union or disjoint union (for the branching time case) are commonly used to model choice in the process algebra. Unfortunately, in PEPA the situation is more complex due to the assumed interpretation of the rates. For example, the PEPA components:

$$(\alpha, r_1).\,E + (\alpha, r_2).\,E \quad \text{and} \quad (\alpha, r_1 + r_2).\,E$$

are identified (i.e. are strongly equivalent), and thus should correspond to *the same* denotation in the the metric model, or else full abstraction with respect to equality could not be shown. One possible solution is to map components such as $(\alpha, r_1).\,E + (\alpha, r_2).\,E$ to denotations of the form $(\alpha, r_1 + r_2)\mathcal{M}(E)$, and prevent sets of the form $(\alpha, r_1)\mathcal{M}(E) \cup (\alpha, r_2)\mathcal{M}(E)$ from occurring in the metric model.

The above restriction can be achieved by imposing the following reactiveness condition on sets similar to [12].

Definition 5.2 *Let $X \subseteq (\mathcal{A} \times \mathbb{R}^{>0}) \times P$, where P is a set. Then X is said to satisfy the* reactiveness condition *if, for any $p, q \in (\mathcal{A} \times \mathbb{R}^{>0}) \times P$ where $p = (\alpha, r)\tilde{p}$ and $q = (\alpha, \tilde{r})\tilde{q}$, if $p, q \in X$ then either $\tilde{p} \neq \tilde{q}$ or $p = q$.*

We can now introduce a series of metric spaces (P_n, d_n), $n \in \mathbb{N}$, where we let $\mathcal{P}_{fr}$ denote the powerset operator restricted to the finite subsets satisfying the reactiveness condition.

Definition 5.3 *Let (P_n, d_n) $n = 0,1\ldots$ be a collection of metric spaces defined inductively by $P_0 = \{p_0\}$, $d_0(p,q) = 0$, and*

$$P_{n+1} = \{p_0\} \cup \mathcal{P}_{fr}\left((\mathcal{A} \times \mathbb{R}^{>0}) \times P_n\right),$$

where $d_{n+1}(p, p_0) = d_{n+1}(p_0, p) = 1$ for $p \neq p_0$, and otherwise $d_{n+1}(p,q)$ is the Hausdorff distance between sets $p = X \subseteq (\mathcal{A} \times \mathbb{R}^{>0}) \times P_n$ and $q = Y \subseteq (\mathcal{A} \times \mathbb{R}^{>0}) \times P_n$ induced by the distance $d_{n+1}(x,y)$ between points $x \in X, y \in Y$.

Again, it follows by induction that (P_n, d_n) is a metric space for each $n \in \mathbb{N}$, and as before we put $P_\omega = \bigcup_n P_n$, $d_\omega = \bigcup_n d_n$ and define (P,d) as the completion of (P_ω, d_ω). Moreover, a straightforward adaptation of the techniques of [1] yields the following theorem, where $\cong$ denotes isometry.

Theorem 5.1 $P \cong \{p_0\} \cup \mathcal{P}_{fr}\left((\mathcal{A} \times \mathbb{R}^{>0}) \times P\right)$.

6 Denotational Semantics for PEPA

We have thus obtained P as a solution of a domain equation, and can now give denotational semantics for PEPA. We begin by defining semantic operators on P.

Definition 6.1 *The* degree *of a component $p \in P$ is defined inductively by putting $deg(p_0) = 0$, $deg(p) = n$ if $p \in P_n \setminus P_{n-1}$ for some $n \geq 1$, and $deg(p) = \infty$ otherwise. We then say a component p is* finite *if $deg(p) = n$ for some $n \in \mathbb{N}$ and* infinite *otherwise.*

Thus, each $p \in P$ is either finite, in which case $p = p_0$ or it is a finite set satisfying the reactiveness condition and whose elements are of degree at most $deg(p)$, or it is infinite, in which case $p = \lim_{n\to\infty} p_n$, $\langle p_n \rangle_n$ Cauchy, with each p_n of degree n.

We now define the operators $\oplus$, $/$ and $\bowtie_L$ on P to model choice, hiding and cooperation; this is done inductively on the degree of elements of P. Before we can do this, we need to define the *apparent rate of action type* α for all elements of P_ω which will be needed to model cooperation.

Definition 6.2 *For any $\alpha \in \mathcal{A}$ and $p \in P_\omega$ we define* the apparent rate of action type α in P, *denoted $r_\alpha(p)$, as follows: if $p = p_0$ then put $r_\alpha(p) = 0$ for all $\alpha \in \mathcal{A}$; otherwise, $p = \bigcup_{i=1}^{k}\{(\alpha_i, r_i)p_i\}$ for some $k \in \mathbb{N}$, $(\alpha_i, r_i) \in \mathcal{A} \times \mathbb{R}^{>0}$ and $p_i \in P_\omega$ for all $1 \leq i \leq k$, and in this case we put $r_\alpha(p) = \sum_{\alpha_i = \alpha} r_i$.*

The definition of the semantic operator of choice now follows.

Definition 6.3 *Let $p \in P$, $X, Y \in \mathcal{P}_{fr}((\mathcal{A} \times \mathbb{R}^{>0}) \times P_n)$, $\langle p_n \rangle_n$ and $\langle q_n \rangle_n$ Cauchy sequences of finite components and $L \subseteq \mathcal{A}$. Define the choice operator $\oplus$ on P by induction on degree as follows. For the base case put $p \oplus p_0 = p_0 \oplus p = p$. For the induction step, let $X, Y \in P_{n+1}$, i.e. by construction:*

$$X = \bigcup_{i=1}^{k}\{(\alpha_i, r_i)X_i\} \text{ and } Y = \bigcup_{j=1}^{m}\{(\tilde{\alpha}_j, \tilde{r}_j)Y_i\}$$

for some (α_i, r_i),$(\tilde{\alpha}_j, \tilde{r}_j) \in \mathcal{A} \times \mathbb{R}^{>0}$ and $X_i, Y_j \in P_n$ for all $1 \leq i \leq k$ and $1 \leq j \leq m$, and define:

$$\begin{aligned} X \oplus Y \;=\; & \{(\alpha_i, r_i)X_i \mid 1 \leq i \leq k \text{ where } \alpha_i \neq \tilde{\alpha}_j \text{ or } X_i \neq Y_j \text{ for all } 1 \leq j \leq m\} \\ & \cup\{(\tilde{\alpha}_j, \tilde{r}_j)Y_i \mid 1 \leq j \leq m \text{ where } \tilde{\alpha}_j \neq \alpha_i \text{ or } Y_j \neq X_i \text{ for all } 1 \leq i \leq k\} \\ & \cup\{(\alpha_i, r_i + \tilde{r}_j)X_i \mid 1 \leq i \leq k, 1 \leq j \leq m \text{ where } \alpha_i = \alpha_j \text{ and } X_i = Y_j\}. \end{aligned}$$

Finally, define $(\lim_{i\to\infty} p_i) \oplus (\lim_{j\to\infty} q_j) = \lim_{k\to\infty}(p_k \oplus q_k)$.

Intuitively, the three clauses in the definition of $X \oplus Y$ above are needed to ensure that reactiveness is preserved. The first two clauses select all the elements of X and Y such that their union will satisfy the reactiveness condition, whereas the third clause combines the remaining elements of X and Y into a set satisfying the reactiveness condition by summing up the rates of activities which have the same action type and successor component. Recall that this is needed for full abstraction. For example, if $X = \{(\alpha, r_1)p\} \cup \{(\beta, r_2)q\}$ and $Y = \{(\alpha, r_3)p\}$ then $X \cup Y$ and $X \oplus Y$ are identified by the strong equivalence. We do not allow $X \cup Y$ as a valid denotation, but instead we take:

$$X \oplus Y = \{(\alpha, r_1 + r_3)p\} \cup \{(\beta, r_2)q\}$$

and it is easy to see that $X \oplus Y$ satisfies reactiveness condition.

We now define the operators $/$ (hiding) and $\bowtie_L$ (cooperation). To ensure their well-definedness on P, i.e. that the reactiveness condition is preserved, we shall make use of the $\oplus$ operator. For convenience, for any $\{x_i \,|\, 1 \leq i \leq k\}$ we introduce the following notation:

$$\oplus\{x_i \,|\, 1 \leq i \leq k\} \stackrel{def}{=} x_1 \oplus x_2 \ldots \oplus x_n.$$

We now proceed with the remaining operators. There is a commonality between the definitions, in that they are roughly in three parts: we first show how to combine elements of P, then lift the operators to sets, and then finally we cater for the infinite processes by combining limits of two Cauchy sequences.

Definition 6.4 *Let $p \in P$, $X, Y \in \mathcal{P}_{fr}\big((\mathcal{A} \times \mathbb{R}^{>0}) \times P_n\big)$, $\langle p_n \rangle_n$ and $\langle q_n \rangle_n$ Cauchy sequences of finite components and $L \subseteq \mathcal{A}$.*

(Hiding) *For the base case put $p_0/L = p_0$, and for induction put*

$$X/L = \oplus\{x/L \,|\, x \in X\},$$

where each $x \in X$ is by definition of the form $x = (\alpha, r)\tilde{X}$ for some $(\alpha, r) \in \mathcal{A} \times \mathbb{R}^{>0}$ and $\tilde{X} \in P_n$, and we can define:

$$((\alpha, r)\tilde{X})/L = \begin{cases} (\tau, r)(\tilde{X}/L) & \text{if } \alpha \in L \\ (\alpha, r)(\tilde{X}/L) & \text{if } \alpha \notin L \end{cases}$$

Finally, put $(\lim_{i\to\infty} p_i)/L = \lim_{i\to\infty}(p_i/L)$.

(Cooperation) *We first define $\underset{L}{\bowtie}$ on singleton elements of P_{n+1}, that is, elements of the form $(\alpha, r)\tilde{X}$ where $(\alpha, r) \in \mathcal{A} \times \mathbb{R}^{>0}$ and $\tilde{X} \in P_n$ by putting:*

$$((\alpha, r)\tilde{X}) \underset{L}{\bowtie} p_0 = \begin{cases} (\alpha, r)(\tilde{X} \underset{L}{\bowtie} p_0) & \text{if } \alpha \notin L \\ p_0 & \text{if } \alpha \in L, \end{cases}$$

and

$$p_0 \underset{L}{\bowtie} ((\alpha, r)\tilde{X}) = \begin{cases} (\alpha, r)(p_0 \underset{L}{\bowtie} \tilde{X}) & \text{if } \alpha \notin L \\ p_0 & \text{if } \alpha \in L. \end{cases}$$

Now lift the above to sets, i.e. for any $Y \in P_{n+1}$ put:

$$((\alpha, r)\tilde{X}) \underset{L}{\bowtie} Y = (\alpha, r)(\tilde{X} \underset{L}{\bowtie} Y) \ \text{ and } \ Y \underset{L}{\bowtie} ((\alpha, r)\tilde{X}) = (\alpha, r)(Y \underset{L}{\bowtie} \tilde{X}).$$

Next if $\{(\alpha, r)\tilde{X}\} \in P_{n+1}$ and $\{(\alpha, \hat{r})\tilde{Y}\} \in P_{n+1}$, define an auxiliary operator $\overset{(XY)}{\underset{L}{\bowtie}}$:

$$((\alpha, r)\tilde{X}) \overset{(XY)}{\underset{L}{\bowtie}} ((\alpha, \hat{r})\tilde{Y}) = (\alpha, R)(\tilde{X} \underset{L}{\bowtie} \tilde{Y}) \ \text{ where } R = \frac{r}{r_\alpha(X)} \frac{\hat{r}}{r_\alpha(Y)} \min\{r_\alpha(X), r_\alpha(Y)\}.$$

We are now in a position to define $\underset{L}{\bowtie}$ on all elements of P as follows:

$$p_0 \underset{L}{\bowtie} p_0 = p_0, \ \ X \underset{L}{\bowtie} p_0 = \oplus\{x \underset{L}{\bowtie} p_0 \,|\, x \in X\} \ \ p_0 \underset{L}{\bowtie} X = \oplus\{p_0 \underset{L}{\bowtie} x \,|\, x \in X\}$$

and

$$\begin{aligned} X \underset{L}{\bowtie} Y \ = \ \oplus\{ & \{x \underset{L}{\bowtie} Y \,|\, x \in X,\, x = (\alpha, r)\tilde{X} \text{ and } \alpha \notin L\} \\ & \cup \{X \underset{L}{\bowtie} y \,|\, y \in Y,\, y = (\alpha, r)\tilde{Y} \text{ and } \alpha \notin L\} \\ & \cup \{x \overset{(XY)}{\underset{L}{\bowtie}} y \,|\, x \in X,\, y \in Y,\, x = (\alpha, r_1)\tilde{X},\, y = (\alpha, r_2)\tilde{Y} \text{ and } \alpha \in L\}\}. \end{aligned}$$

Finally, define $(\lim_{i\to\infty} p_i) \bowtie_L (\lim_{j\to\infty} q_j) = \lim_{k\to\infty}(p_k \bowtie_L q_k)$.

We now investigate the continuity properties of the semantic operators defined above. Continuity (at least on guarded terms i.e. terms in which every constant is prefixed) is needed to ensure the existence of limits which model recursive components, and one would expect continuity to hold for PEPA operators. For example, if $\langle p_i \rangle_i$ and $\langle q_j \rangle_j$ are Cauchy sequences in P, then $\oplus$ was defined by $(\lim_{i\to\infty} p_i) \oplus (\lim_{j\to\infty} q_j) = \lim_{k\to\infty}(p_k \oplus q_k)$. This is well defined only if the limit of $\langle p_k \oplus q_k \rangle_k$ exists. If $\oplus$ is continuous, then for all $n, m \in \mathbb{N}$:

$$d(p_n \oplus q_n, p_m \oplus q_m) \leq d(p_n, p_m) + d(q_n, q_m)$$

and since $\langle p_i \rangle_i$ and $\langle q_j \rangle_j$ are Cauchy, we have $\langle p_k \oplus q_k \rangle_k$ is Cauchy and thus it can be shown that the limit of $\langle p_k \oplus q_k \rangle_k$ exists. We begin by considering $\oplus$; unfortunately, in general $\oplus$ is not continuous, as can be seen from the counter-example in Figure 3.

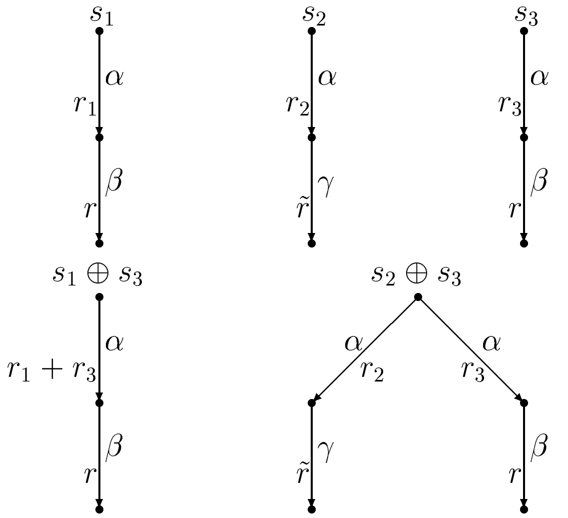

Figure 3 Counter-example to continuity of $\oplus$.

Observe that by definition of the metric d: $d(s_1, s_2) = \frac{1}{2}(\frac{1}{\mathbf{R}}|r_1 - r_2| + 1)$ while

$$\begin{aligned} & d(s_1 \oplus s_3, s_2 \oplus s_3) \\ &= \max\left\{ \frac{1}{2}\left(\frac{1}{\mathbf{R}}|(r_1 + r_3) - r_2| + 1\right), \frac{1}{2}\left(\frac{1}{\mathbf{R}}|(r_1 + r_3) - r_3| + \frac{1}{2}\frac{1}{\mathbf{R}}|r - r|\right) \right\} \\ &= \frac{1}{2}\max\left\{ \frac{1}{\mathbf{R}}|(r_1 + r_3) - r_2| + 1, \frac{1}{\mathbf{R}}|r_1| \right\} \\ &= \frac{1}{2}\left(\frac{1}{\mathbf{R}}|(r_1 + r_3) - r_2| + 1\right) \quad \text{since by definition } \mathbf{R} > r_1 \text{ and thus } 1 > \frac{1}{\mathbf{R}}|r_1|. \end{aligned}$$

Therefore, if $r_1 > r_2$ we have $|(r_1 + r_3) - r_2| > |r_1 - r_2|$, and thus

$$d(s_1 \oplus s_3, s_2 \oplus s_3) > d(s_1, s_2)$$

from which it follows that $\oplus$ is not continuous. It should be noted that the above situation does not arise under guarded recursion. For reasons similar to the above it follows that hiding is not continuous either.

Continuity of $\oplus$ and $/$ can be guaranteed in the restricted setting of those elements of P for which union and hiding preserve the reactiveness condition, i.e. $X \cup Z$ and $Y \cup Z$, $\{x/L \mid x \in X\}$ and $\{y/L \mid y \in Y\}$ must satisfy the reactiveness condition. By the definition of the Hausdorff distance it can be shown that

$$d(X \oplus Z, Y \oplus Z) \leq d(X,Y) \text{ and } d(X/L, Y/L) \leq d(X,Y)$$

and hence $\oplus$ and $/$ are continuous. It should be noted that this restriction corresponds to imposing deterministic choice at the level of syntax, i.e. components of the form $(\alpha, r_1).\, E + (\alpha, r_2).\, F$ are disallowed.

The difficulty with continuity of cooperation is of a different nature. In general, $\underset{L}{\bowtie}$ is not continuous due to the fact that relative probabilities are incorporated into the operational semantics. More precisely, this happens when the activity rate is of the form:

$$R = \frac{r_1}{r_\alpha(E)} \frac{r_2}{r_\alpha(F)} \min(r_\alpha(E), r_\alpha(F)).$$

However, if X,Y and $Z \in P_\omega$ are such that Z is (unweighted) *passive* with respect to the components X and Y, that is, $r_\alpha(Z) \geq r_\alpha(X), r_\alpha(Y)$ for all actions α, then it can be shown that $\underset{L}{\bowtie}$ is continuous in this restricted setting.

Lemma 6.1 *For any X, Y and $Z \in P_\omega$ such that $X \cup Z$ and $Y \cup Z$ satisfy the reactiveness condition and $r_\alpha(Z) \geq r_\alpha(X), r_\alpha(Y)$ (and the same similarly holds for all the subcomponents of X, Y and Z) we have:*

$$d(X \underset{L}{\bowtie} Z, Y \underset{L}{\bowtie} Z) \leq d(X,Y).$$

Proof. The proof is by induction on $n = \max\{deg(X), deg(Y)\} + deg(Z)$. If $n = 0$, or one or more of X, Y and Z equals p_0, the result is trivial. Now suppose the lemma holds for $n = k$, and consider $n = k + 1$.

From Lemma 4.1,

$$d(X \underset{L}{\bowtie} Z, Y \underset{L}{\bowtie} Z) \leq 1$$

and thus if $d(X,Y) = 1$ the result follows. It therefore remains to consider the case of $d(X,Y) < 1$. From the definition of d and the Hausdorff distance, in this case X and Y must be of the form:

$$X = \bigcup_{i=1}^{k} \{(\alpha_i, r_i) X_i\} \text{ and } Y = \bigcup_{j=1}^{\tilde{k}} \{(\alpha_j, \tilde{r}_j) Y_j\} \tag{1}$$

for some $k, \tilde{k} \in \mathbb{N}$, where $(\alpha_i, r_i), (\alpha_j, \tilde{r}_j) \in \mathcal{A} \times \mathbb{R}^{>0}$ and $X_i, Y_j \in P_\omega$ for all $1 \leq i \leq k$ and $1 \leq j \leq \tilde{k}$. Moreover using the Hausdorff distance we have the following property:

$$\forall\, 1 \leq i \leq k, \exists 1 \leq j \leq \tilde{k} \text{ such that } d(X,Y) \geq d((\alpha_i, r_i)X_i, (\alpha_j, \tilde{r}_j)Y_j) \text{ and } \alpha_i = \alpha_j. \tag{2}$$

On the other hand, if we consider any $z \in X \underset{L}{\bowtie} Z$, by definition of $\underset{L}{\bowtie}$ there are three cases to consider, and using (1) and the fact $X \cup Z$ is reactive, the cases are of the following form:

(i) $z = (\alpha_i, r_i)(X_i \underset{L}{\bowtie} Z)$, then from (1), (2) and the fact $Y \cup Z$ is reactive,

$$\tilde{z} = (\alpha_j, \tilde{r}_j)(Y_j \underset{L}{\bowtie} Z) \in Y \underset{L}{\bowtie} Z,$$

where $d(X,Y) \geq d((\alpha_i,r_i)X_i,(\alpha_j,\tilde{r}_j)Y_j)$ and $\alpha_i = \alpha_j$. By the definition of $\bowtie_L$ and d:

$$\begin{aligned} d(z,\tilde{z}) &= d((\alpha_i,r_i)(X_i \bowtie_L Z),(\alpha_j,\tilde{r}_j)(Y_j \bowtie_L Z)) \\ &= \frac{1}{2}\left(\frac{1}{\mathbf{R}}|r_i - \tilde{r}_j| + d(X_i \bowtie_L Z, Y_j \bowtie_L Z)\right) \\ &\leq \frac{1}{2}\left(\frac{1}{\mathbf{R}}|r_i - \tilde{r}_j| + d(X_i, Y_j)\right) \quad \text{by induction} \\ &= d((\alpha_i,r_i)X_i,(\alpha_j,\tilde{r}_j)Y_j) \leq d(X,Y) \ \text{ by (2).} \end{aligned}$$

(ii) $z = X \bowtie_L ((\alpha,r)\tilde{Z})$, then, $\tilde{z} = Y \bowtie_L ((\alpha,r)\tilde{Z}) \in Y \bowtie_L Z$, and we have:

$$\begin{aligned} d(z,\tilde{z}) &= d((\alpha,r)(X \bowtie_L \tilde{Z}),(\alpha,r)(Y \bowtie_L \tilde{Z})) \\ &= \frac{1}{2}\left(\frac{1}{\mathbf{R}}|r - r| + d(X \bowtie_L \tilde{Z}, Y \bowtie_L \tilde{Z})\right) \\ &= \frac{1}{2}\left(\frac{1}{2} d(X \bowtie_L \tilde{Z}, Y \bowtie_L \tilde{Z})\right) \\ &\leq \frac{1}{2} d(X,Y) \ \text{ by induction} \\ &\leq d(X,Y). \end{aligned}$$

(iii) $z = ((\alpha_i,r_i)X_i) \overset{(XZ)}{\underset{L}{\bowtie}} ((\alpha_i,r)\tilde{Z})$, then similarly to case *(i)*:

$$\tilde{z} = ((\alpha_j,\tilde{r}_j)Y_j) \overset{(XZ)}{\underset{L}{\bowtie}} ((\alpha_i,r)\tilde{Z}) \in Y \bowtie_L Z,$$

and moreover, by definition of $\bowtie_L$

$$z = (\alpha_i,R)(X_i \bowtie_L \tilde{Z}) \text{ where } R = \frac{r_i}{r_{\alpha_i}(X)} \frac{r}{r_{\alpha_i}(Z)} min\{r_{\alpha_i}(X), r_{\alpha_i}(Z)\} \tag{3}$$

$$\tilde{z} = (\alpha_i,\tilde{R})(Y_j \bowtie_L \tilde{Z}) \text{ where } \tilde{R} = \frac{\tilde{r}_j}{r_{\alpha_j}(Y)} \frac{r}{r_{\alpha_i}(Z)} min\{r_{\alpha_j}(Y), r_{\alpha_i}(Z)\}. \tag{4}$$

By hypothesis, $r_\alpha(Z) \geq r_\alpha(X), r_\alpha(Y)$, and thus substituting this into (3) and (4) we reach:

$$\begin{aligned} d(z,\tilde{z}) &= d((\alpha_i,R)(X_i \bowtie_L \tilde{Z}),(\alpha_i,\tilde{R})(Y_j \bowtie_L \tilde{Z})) \\ &= \frac{1}{2}\left(\frac{1}{\mathbf{R}}\left|r_i \frac{r}{r_{\alpha_i}(Z)} - \tilde{r}_j \frac{r}{r_{\alpha_i}(Z)}\right| + \frac{1}{2} d(X_i \bowtie_L \tilde{Z}, Y_j \bowtie_L \tilde{Z})\right) \\ &= \frac{1}{2}\left(\frac{1}{\mathbf{R}} \frac{r}{r_{\alpha_i}(Z)} |r_i - \tilde{r}_j| + d(X_i \bowtie_L \tilde{Z}, Y_j \bowtie_L \tilde{Z})\right) \\ &\leq \frac{1}{2}\left(\frac{1}{\mathbf{R}}|r_i - \tilde{r}_i| + d(X_i \bowtie_L \tilde{Z}, Y_j \bowtie_L \tilde{Z})\right) \ \text{ since } \frac{r}{r_{\alpha_i}(Z)} \leq 1 \\ &\leq \frac{1}{2}\left(\frac{1}{\mathbf{R}}|r_i - \tilde{r}_i| + d(X_i, Y_j)\right) \ \text{ by induction} \\ &= d((\alpha_i,r_i)X_i,(\alpha_j,\tilde{r}_j)Y_j) \leq d(X,Y) \ \text{ by (2).} \end{aligned}$$

Therefore, for all $z \in X \bowtie_L Z$ there exists $\tilde{z} \in Y \bowtie_L Z$ such that $d(z,\tilde{z}) \leq d(X,Y)$, and by definition of the Hausdorff distance (see Definition 5.1):

$$d(z, Y \bowtie_L Z) \leq d(X,Y) \ \text{ for all } z \in X \bowtie_L Z.$$

Moreover, by symmetry

$$d(\tilde{z}, X \bowtie_L Z) \leq d(X,Y) \ \text{ for all } \tilde{z} \in Y \bowtie_L Z.$$

Therefore, again by definition of the Hausdorff distance, $d(X \bowtie_L Z, Y \bowtie_L Z) \leq d(X,Y)$ as required. □

We now define denotational metric semantics for the set of components $\mathcal{C}$ of PEPA. As usual, we introduce the semantic map $\mathcal{M} : \mathcal{C} \rightarrow (\mathcal{E} \rightarrow P)$ parameterised by environments $\mathcal{E}$. Environments, ranged over by ρ, are simply assignments of variables $\mathcal{X}$ to components and are defined by $\mathcal{E} = \mathcal{X} \rightarrow P$. They are needed to handle free variables. The meaning $\mathcal{M}(E)$ of a component E is a function from environments to components.

Definition 6.5 *Define $\mathcal{M} : \mathcal{C} \rightarrow (\mathcal{E} \rightarrow P)$ inductively on the structure of the components of PEPA as follows:*

$$\begin{aligned}
\mathcal{M}((\alpha,r).\,E)(\rho) &= \{(\alpha,r)\mathcal{M}(E)(\rho)\} \\
\mathcal{M}(E+F)(\rho) &= \mathcal{M}(E)(\rho) \oplus \mathcal{M}(F)(\rho) \\
\mathcal{M}(E \bowtie_L F)(\rho) &= \mathcal{M}(E)(\rho) \bowtie_L \mathcal{M}(F)(\rho) \\
\mathcal{M}(E/L)(\rho) &= \mathcal{M}(E)(\rho)/L \\
\mathcal{M}(X)(\rho) &= \rho(X) \\
\mathcal{M}(A)(\rho) &= \lim_{k \rightarrow \infty} \mathcal{M}^k(A)(\rho)
\end{aligned}$$

where if $A \stackrel{\text{def}}{=} E$, then $\mathcal{M}^0(A)(\rho) = p_0$ and

$$\mathcal{M}^{k+1}(A)(\rho) = \mathcal{M}(E)(\rho\{\mathcal{M}^k(A)(\rho)/A\}).$$

In addition, under the assumption of guardedness of the terms it can be shown that:

$$\mathcal{M}(\mathrm{A})(\rho) = \lambda X.\, \mathcal{M}(E)(\rho\{X/A\}),$$

i.e. recursion is interpreted as the usual (unique) fixed point. It should be pointed out that guardedness is sufficient to rule out the failure of continuity.

Finally, we state the full abstraction theorem for PEPA. Its import is that PEPA components are equivalent with respect to strong equivalence precisely if their denotations coincide. The proof is a direct result of the lemma below.

Lemma 6.2 *For all $E \in \mathcal{C}$, $S \in \mathcal{C}/\cong$ and $\alpha \in \mathcal{A}$, $q[E,S,\alpha] = r > 0$ if and only if $(\alpha,r)p \in \mathcal{M}(E)(\rho)$ where $p = \bigcup_{F \in S} \mathcal{M}(F)(\rho) \stackrel{\text{def}}{=} \mathcal{M}(S)(\rho)$.*

Proof. The proof is by induction on the structure of E.

- ▷ $E = (\beta,\tilde{r})F$; then $\mathcal{M}(E)(\rho) = \{(\beta,\tilde{r})\mathcal{M}(F)(\rho)\}$. On the other hand, $q[E,S,\alpha] = r > 0$ if and only $\alpha = \beta$, $r = \tilde{r}$ and $F \in S$. Moreover, by the inductive hypothesis on F and the definition of $\cong$ (see Definition 2.2) it is straightforward to show $\mathcal{M}(S)(\rho) = \mathcal{M}(F)(\rho)$, and thus this case is proved.

▷ $E = E_1 + E_2$; then it is straightforward to show that for any $S \in \mathcal{C}/\cong$ and $\alpha \in \mathcal{A}$ $q[E,S,\alpha] = q[E_1,S,\alpha] + q[E_2,S,\alpha]$. Hence, consider any $S \in \mathcal{C}/\cong$ and $\alpha \in \mathcal{A}$ such that $q[E,S,\alpha] = r > 0$. There are three cases: $q[E_2,S,\alpha] = r_1 > 0$ and $q[E,S,\alpha] = r_2 > 0$ such that $r_1 + r_2 = r$; $q[E_1,S,\alpha] = r$ and $q[E_2,S,\alpha] = 0$; and $q[E_1,S,\alpha] = 0$ and $q[E_2,S,\alpha] = r$. We will just consider the first case. By induction on the structure of E, the lemma holds for E_1 and E_2. Therefore, $q[E_1,S,\alpha] = r_1 > 0$ if and only if $(\alpha, r_1)p_1 \in \mathcal{M}(E_1)(\rho)$ and $p_1 = \mathcal{M}(S)(\rho)$, and $q[E_2,S,\alpha] = r_2 > 0$ if and only if $(\alpha, r)p_2 \in \mathcal{M}(E_2)(\rho)$ and $p_2 = \mathcal{M}(S)(\rho)$. Therefore, $p_1 = p_2$ and by definition of $\oplus$ (see Definition 6.3) and the reactiveness condition, $q[E_1,S,\alpha] = r_1 > 0$ and $q[E_2,S,\alpha] = r_2 > 0$ if and only if $(\alpha, r_1 + r_2)p_1 \in \mathcal{M}(E_1)(\rho) \oplus \mathcal{M}(E_2)(\rho) = \mathcal{M}(E_1 + E_2)(\rho) = \mathcal{M}(E)(\rho)$ and $p_1 = \mathcal{M}(S)(\rho)$ as required.

The other cases follow similarly, for example the case when $E = E_1 \underset{L}{\bowtie} E_2$, uses induction on E_1 and E_2 together with the transition rules and the definition of the semantic operator $\underset{L}{\bowtie}$. □

Theorem 6.1 *Let $\cong$ be the strong equivalence relation of PEPA, and $\mathcal{M}$ the semantic map defined above. Then for all E,$F \in \mathcal{C}$*

$$E \cong F \text{ if and only if } \mathcal{M}(E)(\rho) = \mathcal{M}(F)(\rho).$$

7 Conclusion

We have succeeded in constructing a fully abstract metric denotational model for the PEPA calculus, thus transferring the compositionality principle to a different level of abstraction. This paper should not be viewed as an end in itself, but rather as the first step towards finding "the right" compositional representation for languages such as PEPA.

We have omitted the discussion of unspecified rates, and instead concentrated on discussing the issues arising from an attempt to model the PEPA calculus with respect to the strong equivalence. Unspecified rates introduce relative probabilities of the transitions, and thus would suffer from problems similar to the difficulty with continutity of cooperation we discussed earlier. As an example, consider the following denotations of components: $\tilde{s}_1 = (\alpha, 1\top)p_0$ and $\tilde{s}_2 = (\alpha, 2\top)p_0$. Then, the probability of both $\tilde{s}_1$ and $\tilde{s}_2$ performing the action α and then terminating is 1. Therefore, since they exhibit exactly the same behaviour, we want the distance between these denotations of components to be zero. However, if we consider the denotations $\tilde{s}_1 \oplus \tilde{s}$ and $\tilde{s}_2 \oplus \tilde{s}$ where $\tilde{s} = (\alpha, \top)(\alpha, \top)p_0$, then the probabilities of $\tilde{s}_1 \oplus \tilde{s}$ and $\tilde{s}_2 \oplus \tilde{s}$ performing the action α and then terminating are

$$\frac{1\top}{r_\alpha(\tilde{s}_1 \oplus \tilde{s})} = \frac{1\top}{1\top + 1\top} = \frac{1}{2} \quad \text{and} \quad \frac{2\top}{r_\alpha(\tilde{s}_2 \oplus \tilde{s})} = \frac{2\top}{1\top + 2\top} = \frac{2}{3}$$

respectively. Thus, since $\tilde{s}_1 \oplus \tilde{s}$ and $\tilde{s}_2 \oplus \tilde{s}$ have different behaviour, the distance between them should be greater than zero. This implies

$$d(\tilde{s}_1 \oplus \tilde{s}, \tilde{s}_2 \oplus \tilde{s}) > 0 = d(\tilde{s}_1, \tilde{s}_2)$$

and thus the operator $\oplus$ is not continuous. A possible solution might be to introduce explicit probabilistic distributions into the domain equation, as e.g. done in [12], but it is not yet clear whether this can be achieved in a compositional fashion. The above mentioned difficulties with the continuity / contractivity deserve further study. They arise due to the nature of the exponential distribution, and in particular properties of composition of thereof, and also from the inclusion of relative probabilities of performing actions when calculating the rates of transitions. In addition, we intend to study the relationship of our model and that generated by the PEPA workbench, as making comparisons with other representations should give a better insight into their respective advantages and disadvantages.

A number of different stochastic process algebras have been proposed recently. This includes: Extended Markovian Process Algebra (EMPA) [3], Markovian Process Algebra (MPA) [2] and Markovian Timed Performance Algebra (MTIPP) [8]. The syntax and operational semantics of these stochasic process algebras is very similar to that of PEPA, the main difference being in the semantics of the parallel or *cooperation* operator. These similarities imply that our metric model for PEPA, with some alteration, could be used to give metric denotational semantics for (non-probabilistic kernels of) the other stochastic process algebras as well.

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