Division of Informatics, University of Edinburgh

Laboratory for Foundations of Computer Science

Transforming PEPA Models to Obtain Product Form Bounds

by

Joanna Tomasik-Krawczyk, Jane Hillston

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Keywords :

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Joanna Tomasik Jane Hillston

February 29, 2000

Abstract

This report presents a detailed study of some examples expressed in the PEPA formalism [2]. It has previously been shown that some sub-classes of PEPA models have a *product form* solution, making them amenable to efficient solution. The models we consider here do not have product form solution because the necessary structural conditions are not satisfied. We demonstrate transformations of these models, based on modifications of the PEPA expressions representing them, which result in new models which are product form. Furthermore we investigate the extent to which performance measures derived from the modified models can be regarded as approximations of the measures pertaining to the original model. We show that if a modeller is interested in one particular performance index, he or she may construct two modified models with product form solutions whose values of this measure are lower and upper bounds of the measure of the initial model.

1 Introduction

One of the reasons that Markovian models of modern computers and communication networks are huge is because they have to include the many complex inter-dependencies which occur between the components of real systems. Exact solution of these models is often not possible because generation of the Markovian states, together with the transition rates between them, and solving the associated Markovian equation system, require enormous time/space resources which are not available. Many different approaches to cope with these problems have been studied. Compositional approaches to construction of a global Markov chain (for example, PEPA) ease the problem of its generation and further expansion. However the real benefits of compositional structure are to be gained when the structure within the model can be used as a basis for compositional solution. In this case the underlying Markov chain need not be solved as a single system of equations but can be decomposed into submodels to be solved separately. When the submodels behave as if they are statistically independent and the decomposed solution yields the same steady state probability distribution as the monolithic solution, the model is said to have *product form* solution.

The intractability of large Markov models has lead to considerable research effort over the last twenty five years into techniques to simplify or aggregate the Markov chain. The objective of simplification procedures is to transform the given Markov chain into another that is, in some sense, easier to solve. This is often taken to mean that the new Markov chain has (considerably) fewer states than the original, but may also mean that the new chain lies within a class of models for which efficient solution techniques are known to exist. Ideally the performance measures obtained from the simplified model will be *exact*, meaning that they are identical to what would have been calculated from the original chain. More usually, some approximation is introduced. For example, when a modeller precisely defines which performance measures are of interest it is sometimes possible to simplify the original chain to obtain another which is easier to solve but in which the chosen performed measures are only bounded. This is the approach which we pursue in this report. We present examples of bounding performance measures of one Markov chain by other chains which have product form solution. Moreover the transformations we consider are performed at the PEPA component level.

In Section 2 we explain the class of Markov chains with product form solution which we are interested in and we describe the formal tools which can be used to detect this feature. In Section 3 we present a simple Taxi System and a model of it, expressed in PEPA, which has a product form solution. We also review the definitions and theorems which allow us to recognise PEPA models with product form solution. In the following section we consider variations of the taxi system, the models for which do not have product form solution. We show upper and lower bounds of some performance measures found after transformations of the initial models at the PEPA component level. The obtained approximate results are compared with their exact numerical values. In the last section we present some conclusions and outline further work.

2 Markov chains with product form solution

Let us consider a Markov chain $\{X_t\}_{t \in T}$ with continuous time, $T \subseteq \mathbb{R}$, and with discrete state space, $X_t \in \prod_{k=1}^{K} S^{(k)}, |S^{(k)}| \leq \mathbb{N}$, $K \in \mathbb{N}$. An example of such a two-dimensional chain, containing 6 states, is shown in Figure 1. To find the steady state probability vector $[\pi_0, \pi_1, \pi_2, \pi_3, \pi_4, \pi_5]$ for this chain we have to solve a system of linear equations derived from flux balance equations written for

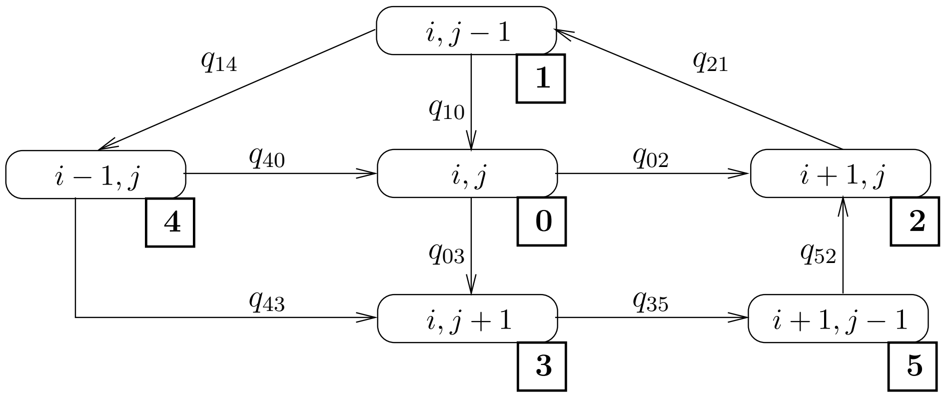

Figure 1: A Markov chain without product form solution.

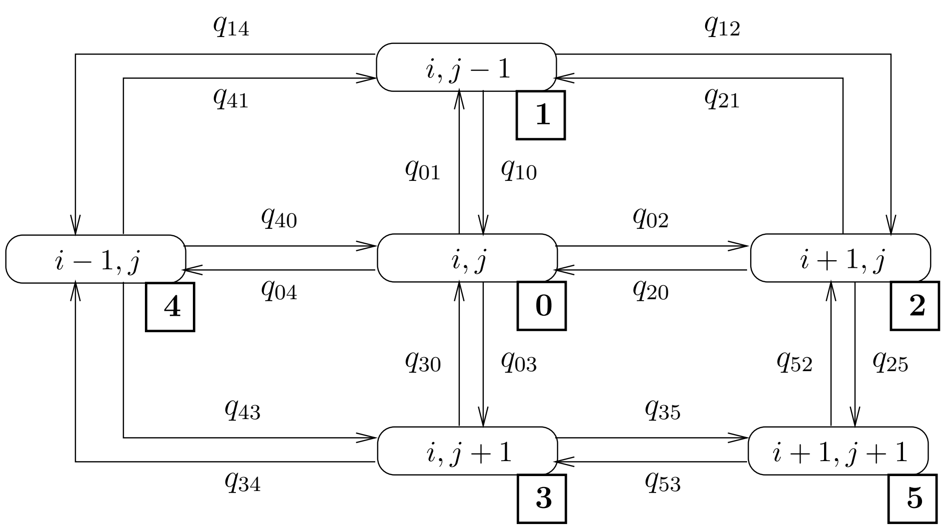

Figure 2: A Markov chain with product form solution.

every state. For example, for the chain shown in Figure 1:

$$\begin{aligned} \pi_0(q_{02}+q_{03}) &= \pi_1 q_{10} + \pi_4 q_{40}, & \pi_1(q_{10}+q_{14}) &= \pi_2 q_{21}, \\ \pi_2 q_{21} &= \pi_0 q_{02} + \pi_5 q_{52}, & \pi_3 q_{35} &= \pi_0 q_{03} + \pi_4 q_{43}, \\ \pi_4(q_{40}+q_{43}) &= \pi_1 q_{14}, & \pi_5 q_{52} &= \pi_3 q_{35}. \end{aligned}$$

These equations, accompanied by the normalisation equation $\sum_{i=0}^{5} \pi_i = 1$, are sufficient to find the π_i.

Another two-dimensional Markov chain is presented in Figure 2. It also consists of 6 states but its flux balance equations, formulated for each state, are much easier to solve than those for the previous chain:

$$\begin{array}{rclrclrcl} \pi'_0 q_{01} &=& \pi'_1 q_{10}, & \pi'_0 q_{02} &=& \pi'_2 q_{20}, & \pi'_0 q_{03} &=& \pi'_3 q_{30}, \\ \pi'_0 q_{04} &=& \pi'_4 q_{40}, & \pi'_2 q_{25} &=& \pi'_5 q_{52} \Rightarrow & \pi'_0 q_{02} q_{25} &=& \pi'_5 q_{52} q_{20}. \end{array}$$

These equations, accompanied by the normalisation equation $\sum_{i=0}^{5} \pi_i = 1$, allow us to calculate π_i. We may notice that pairs of states in the chain in Figure 2 are mutually connected, i.e. if there exists a transition from X_m to X_n, a transition from X_n to X_m must be possible as well. The probability $\pi'_5 = \frac{q_{02}}{q_{20}} \frac{q_{25}}{q_{52}} \pi'_0$ may be derived directly from the graph in Figure 2 tracing the path from state 0 to state 5 passing through state 2 and multiplying corresponding ratios of outgoing and incoming transition rates. However, we can also go $0 \to 3 \to 5$ and then $\pi'_5 = \frac{q_{03}}{q_{30}} \frac{q_{35}}{q_{53}} \pi'_0$, so $q_{02} q_{25} q_{53} q_{30} = q_{03} q_{35} q_{52} q_{20}$, which means that for the cycle $(0, 2, 5, 3, 0)$ the Kolmogorov criterion is fulfilled. Looking at all the possible cycles of the chain we conclude that it is a reversible Markov chain because the Kolmogorov criterion is satisfied for all of them.

Fulfilling the Kolmogorov criterion for every cycle $(i_0, i_1, \ldots, i_n, i_0)$ and the complementary one $(i_0, i_n, \ldots, i_1, i_0)$ in a Markov chain graph is a necessary and sufficient condition for obtaining the balance equations in their detailed form, i.e. $\pi_m q_{mk} = \pi_k q_{km}$ for all k, m. The above conclusion is based upon the theorem [1, 5] as follows:

Theorem 2.1 *Let $\{X_t\}_{t \in T}$ be an irreducible N-dimensional Markov chain with infinitesimal generator matrix $\boldsymbol{Q}$ and with steady state solution $\boldsymbol{\pi} = [\pi_0, \pi_1, \ldots, \pi_{N-1}]$. The following statements are equivalent:*

1. *$\{X_t\}_{t \in T}$ is reversible,*
2. *the linear equation system $\boldsymbol{\pi Q}$ is a system of detailed balance equations,*
3. *for any pair of cycles $C : (i_0, i_1, \ldots, i_n, i_0)$ and $\underline{C} : (i_0, i_n, \ldots, i_1, i_0)$ we have $q_{0,1} q_{1,2} \cdots, q_{n-1,n} q_{n,0} = q_{0,n} q_{n,n-1} \cdots, q_{2,1} q_{1,0}$ (the Kolmogorov criterion).*

The consequence is that the chain depicted in Figure 2 has a product form solution, i.e. $\pi'_{(m,n)} = C \pi'_{(m)} \pi'_{(n)}$, where $\pi'_{(m)}$, $\pi'_{(n)}$ are the probabilities of the states of the unimodal chains making up the multidimensional chain and C is a normalisation constant which makes all the probabilities $\pi'_{(m,n)}$ sum to 1.0.

In [1] van Dijk presented a method of transforming one Markov chain into another one in which the detailed balance equations are satisfied, and is hence simpler to solve. Results obtained from the simpler model may serve as bounds of some interesting performance measures of the initial model. His idea is that if there is an outgoing transition from a state i but there is no incoming transition to this state either we add an extra incoming transition to the state i or we remove the already existing transition. The procedure is analogous if there is an incoming transition to the state i but there is no outgoing transition from this state—we either remove the existing transition or add an outgoing one. We can consider the system presented in Figure 2 as a reversible version of that presented in Figure 1. Adding/removing "unpaired" arcs lead to different models whose performance measures approximate the ones of the initial system.

3 The initial Taxi System with a product form solution

The initial Taxi Rank System (Figure 3) whose model has a product form solution comes originally from [6]. The component C_1 captures the behaviour of the customer who can go to a market either on foot or hiring a taxi, while the component T_1 represents the behaviour of the taxi. A model of the system in PEPA is stated as $TS \stackrel{\text{def}}{=} (C_1||C_1) \underset{L}{\bowtie} T_1$, where the cooperation set $L = \{$*to_market*, *from_market*$\}$. It depicts a system with two customers and only one taxi available.

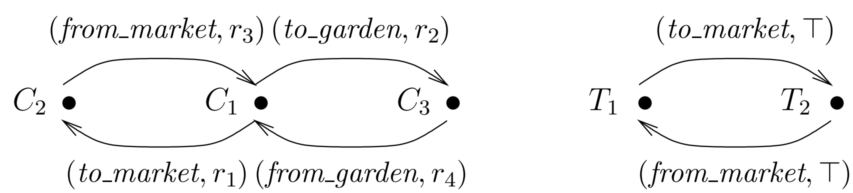

Figure 3: The initial Taxi System, $TS \stackrel{\text{def}}{=} (C_1||C_1) \underset{L}{\bowtie} T_1$, $L = \{$*to_market*, *from_market*$\}$.

We remark that this model is made up of three components which are input-output linear components. The formal definition of an input-output component included below is a slight modification of the one stated in [3]. This definition is preceded by another formulating the idea of a reverse pair and followed by another considering only birth-death components (both of these also appear in [3]). We denote the countable set of all possible action types by $\mathcal{A}$ and the set of activities by $\mathcal{A}ct$, $\mathcal{A}ct \subseteq \mathcal{A}ct \times \mathbb{R}^+$, where $\mathbb{R}^+$ is the set of positive real numbers together with the symbol $\top$ indicating unspecified transition rate. $\mathcal{A}ct(P)$ denotes the set of activities enabled in the component (syntactic term) P. We also use the term *sequential component* for a component S such that $S ::= (\alpha, r).S \mid S + S \mid X$, where X denotes a constant which is a sequential component.

Definition 3.1 *A PEPA component P is said to enable* a reverse pair $(\alpha, -\alpha)$, *if $(\alpha, r) \in \mathcal{A}ct(P)$ and for every (α, r)-derivative P' ($P \xrightarrow{(\alpha,r)} P'$), there exists $(-\alpha, s) \in \mathcal{A}ct(P')$ such that P is an $(-\alpha, s)$-derivative of P', where $r, s \in \mathbb{R}^+$.*

Definition 3.2 *A sequential PEPA component P, with initial component P_0, is* an input-output component *if*

1. *for all $\alpha \in \mathcal{A}(P_0)$, such that $P_0 \xrightarrow{(\alpha,r)} P_0'$ for some r, α forms part of the reverse pair $(\alpha, -\alpha)$,*

2. *for all $P_i \in \mathrm{ds}(P_0)$, for all $\alpha_i \in \mathcal{A}(P_i)$, such that $P_i \xrightarrow{(\alpha_i,r)} P_i'$ for some r, α_i forms part of the reverse pair $(\alpha_i, -\alpha_i)$.*

Definition 3.3 *A PEPA component P_0 is* a linear input-output component *if for every $P_i \in \mathrm{ds}(P_0)$ and for all Q such that $P_i \xrightarrow{a} Q$, P_i and Q communicate exclusively via the actions a and $-a$ such that $P_i \xrightarrow{a} Q$ and $Q \xrightarrow{-a} P_i$.*

A particular type of input-output component is a birth-death component P_0 for which every component P_i, $P_i \in \mathrm{ds}(P_0)$ has at most two one-step derivatives other than itself. The components of the TS model are birth-death components.

The main result of [3] is that a model formed as the cooperation of linear input-output components, cooperating on elements of reverse pairs, is also reversible. Thus it has a product form solution.

4 Taxi Systems without product form solutions

In this section we present taxi systems derived from the initial one in which the reversibility of the corresponding Markov chain is destroyed. Each system is modelled in PEPA but these models are solved directly. Instead the desired results are bounded by values obtained from calculations for other PEPA models whose underlying Markov chains do have product form solutions. These bounding PEPA models are constructed according to the rules given in [1] and outlined in the Section 2.

4.1 The Taxi System with Breakdowns

The initial system, *TS*, has been changed by adding extra states (T_3, T_4) in the Taxi component. The new system is called *TSBD* (Taxi System with Breakdowns) and is presented in Figure 4. The additional states indicate a failure of the car (T_3 — the breakdown occurred at the rank, T_4 — it occurred at the market) and they are connected to already existing states by new transitions labelled with: (fail, s_1), (repair, s_2), and (tow, w). A customer who has hired the taxi stays inside it when it is broken down and towed away from the market to the taxi rank where it will undergo repair. This fact is described by the transition $(\mathit{tow}, \top)$ in the C_1 component. Notice that if this transition was not introduced the corresponding global Markov chain of the *TSBD* system would not be irreducible. The global Markov chain does not have a product form solution because the Kolmogorov condition is not satisfied for the T_1 component. It is composed of 16 states listed and numbered as presented in Table 1 using state vector notation [4].

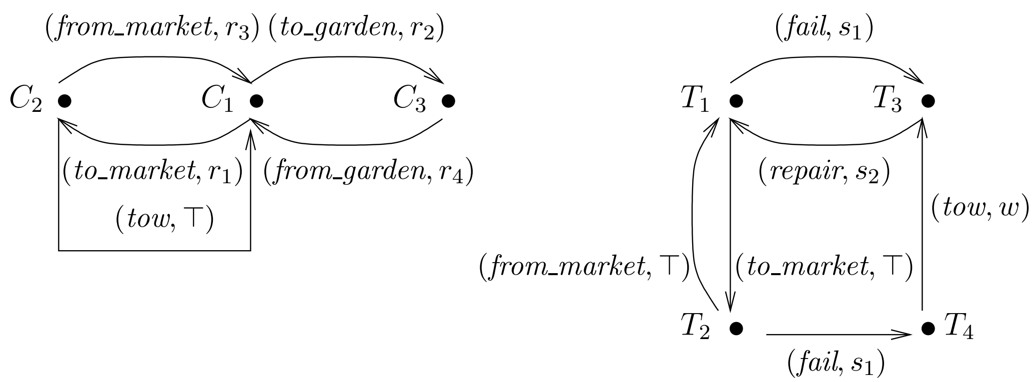

Figure 4: The Taxi System with Breakdowns, $TSBD \stackrel{\text{def}}{=} (C_1||C_1) \underset{L}{\bowtie} T_1$, $L = \{$*to_market, from_market, tow*$\}$.

The reverse pairs in the *TSBD* system are: (*to_market, from_market*) in both the components C_1, T_1, (*to_garden, from_garden*), (*to_market, tow*) in the C_1 component only, and (*fail, repair*) in the T_1 component only.

No	State	No	State	No	State	No	State
0	(C_1, C_1, T_1)	4	(C_1, C_3, T_1)	8	(C_2, C_3, T_2)	12	(C_3, C_2, T_2)
1	(C_1, C_1, T_3)	5	(C_1, C_3, T_3)	9	(C_2, C_3, T_4)	13	(C_3, C_2, T_4)
2	(C_1, C_2, T_2)	6	(C_2, C_1, T_2)	10	(C_3, C_1, T_1)	14	(C_3, C_3, T_1)
3	(C_1, C_2, T_4)	7	(C_2, C_1, T_4)	11	(C_3, C_1, T_3)	15	(C_3, C_3, T_4)

Table 1: The states of the Markov chain of the *TSBD* model

Let us suppose that we want to bound the probability that the taxi is not broken down, i.e. either a customer is using it or a customer can hire it:

$$\pi_{\text{USE}} = \sum_{(i,j),(k,l)} \left(\pi(C_i, C_j, T_1) + \pi(C_k, C_l, T_2)\right), \tag{1}$$

where $(i, j) \in \Big\{(1, 1), (1, 3), (3, 1), (3, 3)\Big\}$, $(k, l) \in \Big\{(1, 2), (2, 1), (2, 3), (3, 2)\Big\}$.

4.1.1 An upper bound of π_{USE} for the *TSBD* system

To find an upper bound of this probability we have to increase probabilities of staying either in (C_i, C_j, T_1) or (C_k, C_l, T_2) states (see Formula 1). This goal may be reached by removing all those taxi breakdown states which cannot be reached/left by an action with a reverse pair element. In the system *TSBD* we remove the T_4 state together with $T_2 \xrightarrow{(fail,s_1)} T_4$ because a direct transition back, $T_4 \rightarrow T_2$, does not exist. The T_4 state becomes a state without any incoming transition, so it should be removed together with the (tow, w) action which is its outgoing transition. Because $tow \in L$ and the corresponding transitions may take place in the C_1 component, the $(tow, \top)$ action must be removed from the C_1 component and *tow* should be excluded from the set L. The new system, $\mathit{TSBD}_{\mathrm{Up}}$, considers taxi failures (Figure 5) but they are less frequent than for the system *TSBD*. Its Markov chain consists of 12 states whose ordering numbers make up a subset of those for *TSBD* listed in Table 1, $\{0, 1, 2, 4, 5, 6, 8, 10, 11, 12, 14, 15\}$.

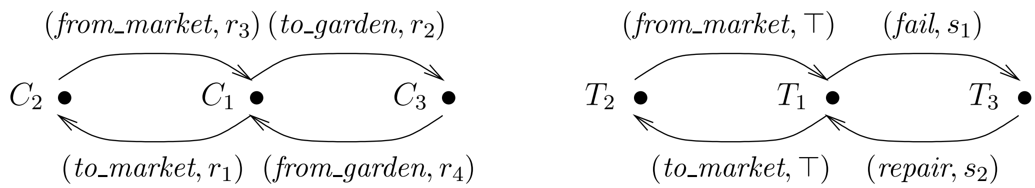

Figure 5: The Taxi System bounding the greatest value of the probability π_{USE} of the Taxi System with Breakdowns, $\mathit{TSBD}_{\mathrm{Up}} \stackrel{\mathrm{def}}{=} (C_1||C_1) \underset{L}{\bowtie} T_1$, $L = \{\mathit{to\_market}, \mathit{from\_market}\}$.

After setting the local balance equations for the $\mathit{TSBD}_{\mathrm{Up}}$ system we may express the state probabilities as functions of the probability of one freely chosen and one fixed state (π_0 in this case):

$$\begin{array}{llllllll}
\pi_1 = S_{12}\pi_0, & \pi_2 = R_{13}\pi_0, & \pi_4 = R_{24}\pi_0, & \pi_5 = S_{12}R_{24}\pi_0, \\
\pi_6 = R_{13}\pi_0, & \pi_8 = R_{13}R_{24}\pi_0, & \pi_{10} = R_{24}\pi_0, & \pi_{11} = S_{12}R_{24}\pi_0, \\
\pi_{12} = R_{24}^2\pi_0, & \pi_{14} = R_{24}^2\pi_0, & \pi_{15} = S_{12}R_{24}^2\pi_0. &
\end{array}$$

where: $R_{13} = r_1/r_3$, $R_{24} = r_2/r_4$, $S_{12} = s_1/s_2$. After solving the normalisation equation, one gets $\pi_0 = 1/d_{\mathrm{Up}}$ where

$$d_{\mathrm{Up}} = 1 + S_{12} + 2R_{13} + 2R_{24} + 2S_{12}R_{24} + R_{13}R_{24} + 2R_{24}^2 + S_{12}R_{24}^2,$$

and, eventually

$$\begin{aligned}
\pi_{\mathrm{USE_{Up}}} &= \pi_0 + \pi_2 + \pi_4 + \pi_6 + \pi_8 + \pi_{10} + \pi_{12} = \\
&= \frac{1 + 2R_{13} + 2R_{24} + 2R_{24}^2 + R_{13}R_{24}}{d_{\mathrm{Up}}}. \qquad (2)
\end{aligned}$$

4.1.2 A lower bound of π_{USE} for the *TSBD* system

To find a lower bound for the probability π_{USE} we are interested in increasing the probabilities of the global states (C_i, C_j, T_3) and (C_k, C_l, T_4). The outgoing transitions from the failure states T_3, T_4 are: $T_3 \xrightarrow{(repair, s_2)} T_1$ and $T_4 \xrightarrow{(tow, w)} T_3 \xrightarrow{(repair, s_2)} T_1$. We may go back $T_1 \xrightarrow{(fail, s_1)} T_3$ and (*fail, repair*) is a reverse pair. However, an analogous return path cannot be set for T_4 because we observe that the path $T_2 \xrightarrow{(fail, s_1)} T_4 \xrightarrow{(tow, w)} T_3$ in the component T_1 of the *TSBD* system does not have a complementary one (one cannot go back: $T_3 \to T_4 \to T_2$). The transition $T_4 \xrightarrow{(tow, w)} T_3$ is labelled with activity type *tow* which forms a reverse pair (*to_market, tow*) in the C_1 component but which appears individually in the T_1 component.

In order to transform the T_1 component into a linear input-output component with relatively longer sojourn times in the T_3 and T_4 states we remove the transition $T_4 \xrightarrow{(tow, w)} T_3$ and add the transition $T_4 \xrightarrow{(repair, z)} T_2$. The introduced transition is labelled with activity type *repair* which forms a reverse pair with *fail*. Thus we get: $T_2 \xrightarrow{(fail, s_1)} T_4 \xrightarrow{(repair, z)} T_2$. The transition rate z is chosen to estimate the transition $T_4 \xrightarrow{(tow, w)} T_3 \xrightarrow{(repair, s_2)} T_1$ as $z = \frac{ws_2}{w+s_2}$. Because $tow \in L$ and it shows up only in the C_1 component now, it should be excluded from the set L and the transition $C_2 \xrightarrow{(tow, \top)} C_1$ in the C_1 component must be cut (Figure 6).

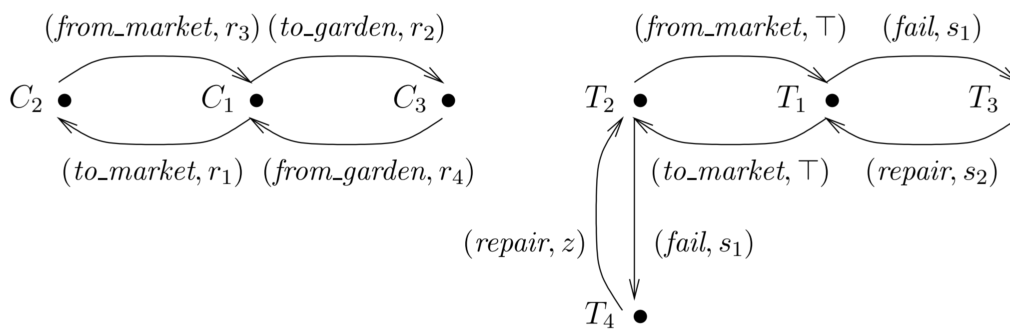

Figure 6: The Taxi System bounding the smallest value of the probability π_{USE} of the Taxi System with Breakdowns, $TSBD_{\mathrm{Low}} \stackrel{\mathrm{def}}{=} (C_1 || C_1) \underset{L}{\bowtie} T_1$, $L = \{to\_market,\ from\_market\}$.

Notice that there is the possibility of repetitions of the cycle $T_2 \xrightarrow{(fail, s_1)} T_4 \xrightarrow{(repair, z)} T_2$ which increases the time spent in the breakdown state T_4. The probability of choosing T_4 when being at T_2 is equal to $p = \frac{s_1}{s_1+r_3}$ and of choosing T_1 is

$q = \frac{r_3}{s_1+r_3}$. The probability that the cycle is not repeated at all is equal to q; that it is repeated only once is equal to pq; that it is repeated twice is equal to p^2q; and so on. The mean number of consecutive cycle repetitions is equal to

$$\begin{aligned}
\sum_{i=1}^{\infty} ip^iq &= (pq + p^2q + p^3q + \cdots + p^iq + \cdots) + \\
&\quad +(p^2q + p^3q + \cdots + p^iq + \cdots) + (p^3q + \cdots + p^iq + \cdots) + \cdots \\
&= pq(1 + p + \cdots + p^i + \cdots) + p^2q(1 + p + \cdots + p^i + \cdots) + \\
&\quad +p^3q(1 + p + \cdots + p^i + \cdots) + \cdots \\
&= (pq + p^2q + \cdots + p^iq + \cdots)\frac{1}{1-p} = \frac{pq}{(1-p)^2}, \quad |p| < 1.
\end{aligned}$$

The corresponding Markov chain consists of 16 states as listed in Table 1 for the *TSBD* model. The local balance equations are as follows (using the same shorthands as before and, additionally, $Q = s_1/z$):

$$\begin{aligned}
\pi_1 &= S_{12}\pi_0, & \pi_2 &= R_{13}\pi_0, & \pi_3 &= QR_{13}\pi_0, & \pi_4 &= R_{24}\pi_0, \\
\pi_5 &= S_{12}R_{24}\pi_0, & \pi_6 &= R_{13}\pi_0, & \pi_7 &= QR_{13}\pi_0, & \pi_8 &= R_{13}R_{24}\pi_0, \\
\pi_9 &= QR_{13}R_{24}\pi_0, & \pi_{10} &= R_{24}\pi_0, & \pi_{11} &= S_{12}R_{24}\pi_0, & \pi_{12} &= R_{13}R_{24}\pi_0, \\
\pi_{13} &= QR_{13}R_{24}\pi_0, & \pi_{14} &= R_{24}^2\pi_0, & \pi_{15} &= S_{12}R_{24}^2\pi_0.
\end{aligned}$$

Solving the normalisation equation gives one $\pi_0 = 1/d_{\text{Low}}$, where

$$d_{\text{Low}} = 1+S_{12}+2R_{13}+2R_{24}+2S_{12}R_{24}+2R_{13}R_{24}+2QR_{13}+R_{24}^2+2QR_{13}R_{24}+S_{12}R_{24}^2.$$

The obtained lower bound for π_{USE} is equal to

$$\begin{aligned}
\pi_{\text{USE}_{\text{Low}}} &= \pi_0 + \pi_2 + \pi_4 + \pi_6 + \pi_8 + \pi_{10} + \pi_{12} + \pi_{14} \\
&= \frac{1 + 2R_{13} + 2R_{24} + 2R_{13}R_{24} + R_{24}^2}{d_{\text{Low}}}.
\end{aligned} \tag{3}$$

4.1.3 Some numerical results

The bounds calculations were performed under the assumption that $r_1 = r_3$, $r_2 = r_4$, which is equivalent to the equation $R_{13} = R_{24} = 1$. Seven data sets were taken into consideration:

1. assuming that $s_1 = s_2$ ($S_{12} = 1$), the values of s_1, s_2, w are $s_1 = s_2 = w = 1.0$, i.e. these values are of the same order,

2. assuming that $s_1 = s_2$ ($S_{12} = 1$), the values of s_1, s_2, w are $s_1 = s_2 = 1.0$ and $w = 1000.0$, i.e. $w \gg s_1, s_2$,

3. assuming that $s_1 = s_2$ ($S_{12} = 1$), the values of s_1, s_2, w are $s_1 = s_2 = 1.0$ and $w = 0.001$, i.e. $w \ll s_1, s_2$,

Input Values	Lower Bound	Exact Value	Upper Bound
Point 1	0.400	0.430783	0.6667
Point 2	0.500	0.500000	0.6667
Point 3	0.002	0.003093	0.6667
Point 4	0.250	0.287501	0.5000
Point 5	0.500	0.548329	0.8000
Point 6	0.200	0.252743	0.5000
Point 7	0.400	0.465661	0.8000

Table 2: The probability π_{USE} for the *TSBD* system

4. the values of s_1, s_2, w are $s_1 = 2.0$, $s_2 = 1.0$ ($S_{12} = 2.0$), $w = 1.0$, i.e. the taxi is more prone to breakdowns,
5. the values of s_1, s_2, w are $s_1 = 1.0$, $s_2 = 2.0$ ($S_{12} = 0.5$), $w = 1.0$, i.e. the mean repair time is shorter,
6. the values of s_1, s_2, w are $s_1 = 2.0$, $s_2 = 1.0$ ($S_{12} = 2.0$), $w = 0.5$, i.e. as in Point 4 but towing is slower,
7. the values of s_1, s_2, w are $s_1 = 1.0$, $s_2 = 2.0$ ($S_{12} = 0.5$), $w = 0.5$, i.e. as in Point 5 but towing is slower.

For each case we computed the bounds and the exact probability value obtained by solving the Markov chain of the initial *TSBD* model. The results are presented in Table 2. Notice that the upper bounds are the same in the cases 1–3 because the formula (2) is not a function of w.

4.1.4 A cyclic component in a system to bound the initial one

The lower bound of the probability π_{USE} for the *TSBD* model was obtained by transforming the initial model into one which was a linear input-output component with all the states unchanged. We might be tempted to decrease the number of states in each component C_1, leading to a smaller global Markov chain.

Let the taxi failure be indicated by the state T_3/C_4 only. We now have to distinguish in which taxi state this failure has happened. This is done by introducing two activities: *fail_at_rank*, *fail_at_market* instead of the more general activity *fail*. The repaired taxi is to return either to the rank or to the market. A return to the rank is made by the activity previously used for this purpose, *repair*, and (*fail_at_rank*, *repair*) is a reverse pair. Another activity reflecting repair and towing, under the name of *maintenance* is added in order to find a complementary element of reverse pair with *fail_at_market*. The action of activity

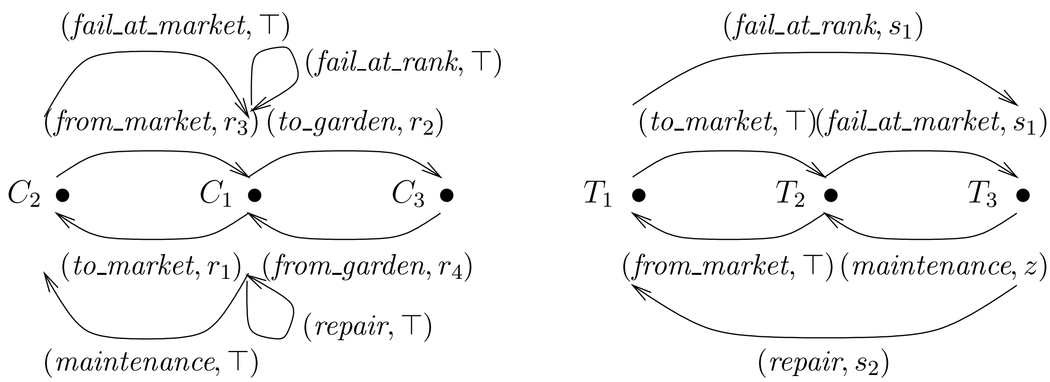

Figure 7: The Taxi System bounding the smallest value of the probability π_{USE} of the Taxi System with Breakdowns, with the cyclic component C_1, $TSBD_{\mathrm{LowCyclic}} \stackrel{\mathrm{def}}{=} (C_1||C_1) \underset{L}{\bowtie} T_1$, $L = \{$*to_market*, *from_market*, *fail_at_market*, *fail_at_rank*, *repair*, *maintenance*$\}$.

type *fail_at_market* may be fired when a customer is in the C_2 state. It brings him back to the rank and the action of activity type *maintenance* (i.e. the second element of the reverse pair) transports him to the market where a possible failure has happened. The components cooperate over the introduced activities, so these activities have to be elements of the cooperation set L (Figure 7) and the corresponding Markov chain consists of 12 states. The C_1 component has become a cyclic one. In order to fulfil the Kolmogorov condition for this component, the equality $r_1 s_1 s_2 = s_1 r_3 z$ must hold, so $z = \frac{r_1 s_2}{r_3}$.

The local balance equations for the $TSBD_{\mathrm{LowCyclic}}$ model with the states numbered as in Table 1 and with the shorthands used in the previous subsections are:

$$\begin{array}{llllllllllll}
\pi_1 &=& S_{12}\pi_0, & \pi_2 &=& R_{13}\pi_0, & \pi_4 &=& R_{24}\pi_0, & \pi_5 &=& S_{12}R_{24}\pi_0,\\
\pi_6 &=& R_{13}\pi_0, & \pi_8 &=& R_{13}R_{24}\pi_0, & \pi_{10} &=& R_{24}\pi_0, & \pi_{11} &=& S_{12}R_{24}\pi_0,\\
\pi_{12} &=& R_{13}R_{24}\pi_0, & \pi_{14} &=& R_{24}^2\pi_0, & \pi_{15} &=& S_{12}R_{24}^2\pi_0. & & &
\end{array}$$

Solving the normalisation equation gives one $\pi_0 = 1/d_{\mathrm{LowCyclic}}$, where

$$d_{\mathrm{LowCyclic}} = 1 + S_{12} + 2R_{13} + 2R_{24} + 2S_{12}R_{24} + 2R_{13}R_{24} + R_{24}^2 + S_{12}R_{24}^2.$$

The obtained lower bound for π_{USE} is equal to

$$\begin{aligned}
\pi_{\mathrm{USE}_{\mathrm{LowCyclic}}} &= \pi_0 + \pi_2 + \pi_4 + \pi_6 + \pi_8 + \pi_{10} + \pi_{12} + \pi_{14} = \\
&= \frac{1 + 2R_{13} + 2R_{24} + 2R_{13}R_{24} + R_{24}^2}{d_{\mathrm{LowCyclic}}}. \qquad (4)
\end{aligned}$$

The probability $\pi_{\mathrm{USE}_{\mathrm{LowCyclic}}}$ (Equation 4) is a function of S_{12}, R_{13}, and R_{24} only. Assuming that $s_1 = s_2$, $r_1 = r_3$, $r_2 = r_4$: $\pi_{\mathrm{USE}_{\mathrm{LowCyclic}}} = 0.75$ which does not properly bound the probability π_{USE} (see Table 2).

4.2 The Taxi System with Mandatory/Optional Stop

In this section we consider a version of the system in which the initial system has been changed by adding an extra state on one of the edges (C_4/T_3). It indicates a stop at a service station when going from the rank to the market. The taxi either must stop (Figure 8) or it may stop (Figure 9).

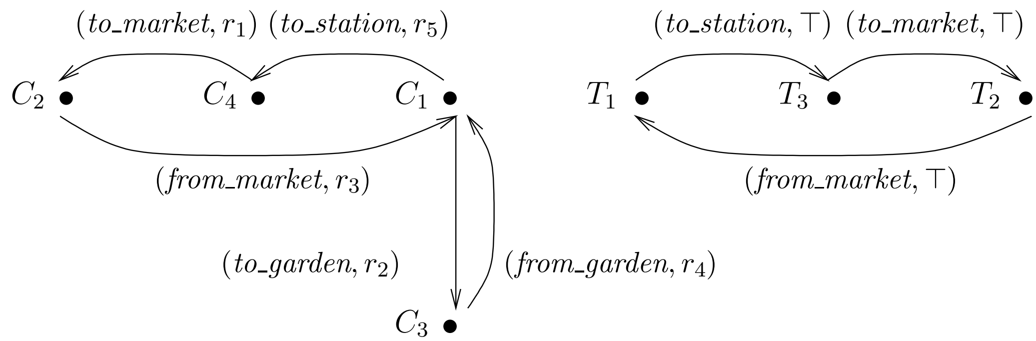

Figure 8: The Taxi System with Mandatory Stop, $TSS \stackrel{\text{def}}{=} (C_1||C_1) \underset{L}{\bowtie} T_1$, $L = \{to\_market,\ from\_market,\ to\_station\}$.

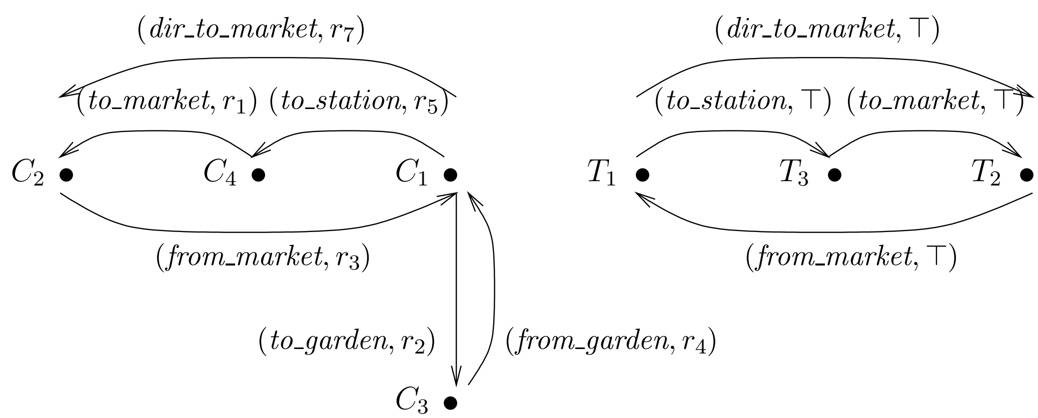

Figure 9: The Taxi System with Optional Stop, $TSSO \stackrel{\text{def}}{=} (C_1||C_1) \underset{L}{\bowtie} T_1$, $L = \{to\_market,\ from\_market,\ to\_station,\ dir\_to\_market\}$.

The modeller's goal in both cases is to bound the probability that the taxi is hired by a customer (i.e. the T_2 or T_3 state in the T_1 component). The corresponding Markov chains for both the cases consist of 12 states numbered as listed in Table 3.

No	State	No	State	No	State	No	State
0	(C_1, C_1, T_1)	3	(C_1, C_4, T_3)	6	(C_3, C_1, T_1)	9	(C_3, C_4, T_3)
1	(C_1, C_2, T_2)	4	(C_2, C_1, T_2)	7	(C_3, C_2, T_2)	10	(C_4, C_1, T_3)
2	(C_1, C_3, T_1)	5	(C_2, C_3, T_2)	8	(C_3, C_3, T_1)	11	(C_4, C_3, T_3)

Table 3: The states of the Markov chain of the *TSS*/*TSSO* systems

4.2.1 An upper bound of $\pi_{\text{IN-USE}}$ for the *TSS* system

We are interested in increasing the time spent in the states T_2 and T_3 of the component T_1. We want to create an additional reverse pair whose elements belong to the cooperation set L. We reach this goal by replacing the transition $C_2 \xrightarrow{(from\_market, r_3)} C_1$ by two transitions: $C_2 \xrightarrow{(from\_market, R_3)} C_4$ and $C_4 \xrightarrow{(from\_station, R_6)} C_1$. Because the former transition $C_2 \xrightarrow{(from\_market, r_3)} C_1$ was executed together with the transition $T_2 \xrightarrow{(from\_market, \top)} T_1$ and $from\_market \in L$, the analogous split has to be made for the T_1 component: $T_2 \xrightarrow{(from\_market, \top)} T_3$ and $T_3 \xrightarrow{(from\_station, \top)} T_1$. To preserve the simultaneous transitions $C_2 \to C_4 \to C_1$ and $T_2 \to T_3 \to T_1$ respectively, the added activity type, i.e. *from_station*, has to be an element of the cooperation set L (Figure 10). By doing this we make longer the possible sojourn times in the states T_2 and T_3 (C_2 and C_4 respectively).

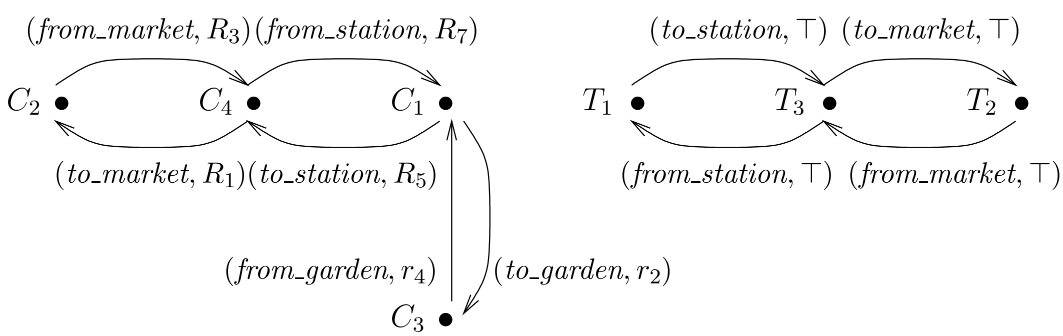

Figure 10: The Taxi System bounding the greatest value of the probability $\pi_{\text{IN-USE}}$ of the Taxi System with Mandatory Stop, $TSS_{\text{Up}} \stackrel{\text{def}}{=} (C_1||C_1) \bowtie_L T_1$, $L = \{to\_market, from\_market, to\_station, from\_station\}$.

To find which values of R_3 and R_6 should be chosen, we analyse two simple Markov chains depicted in Figure 11. The probabilities for the first chains are:

$$[\pi_0, \pi_1, \pi_2] = [\frac{r_1 r_3}{d}, \frac{r_3 r_5}{d}, \frac{r_1 r_5}{d}], \quad \text{where} \quad d = r_1 r_3 + r_3 r_5 + r_1 r_5.$$

Putting $R_{51} = r_5/r_1$ and $R_{53} = r_5/r_3$ we may rewrite π_0 as $\pi_0 = 1/(1+R_{51}+R_{53})$. On the other hand, the probabilities for the second chain are:

$$[\pi_0^*, \pi_1^*, \pi_2^*] = [\frac{1}{d^*}, \frac{Q_{56}}{d^*}, \frac{Q_{13}Q_{56}}{d^*}], \qquad \text{where} \qquad d^* = 1 + Q_{56} + Q_{13}Q_{56}$$

and $Q_{56} = r_5/R_6$, $Q_{13} = r_1/R_3$. We want to maximize the sum $\pi_1^*+\pi_2^*$, so instead

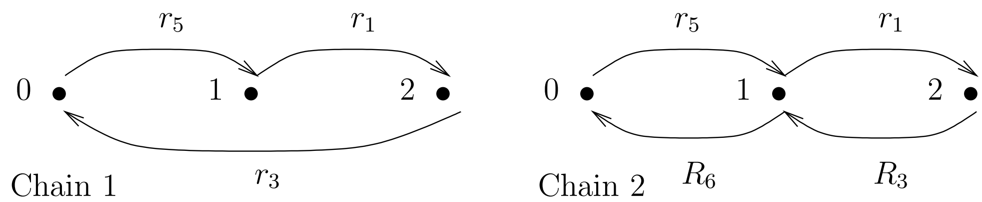

Figure 11: Approximation of transition rates for upper bounds of $\pi_{\text{IN-USE}}$ of the *TSS* and *TSSO* systems.

the minimisation of π_0^* is performed:

$$1 + Q_{56} + Q_{13}Q_{56} > 1 + R_{51} + R_{53} \tag{5}$$

Let $Q_{56} > R_{51}$, so under this assumption

$$R_6 < r_1. \tag{6}$$

To fulfil the inequality (5) with the condition (6) satisfied, it is sufficient that $Q_{13}Q_{56} = R_{53}$, from which we deduce

$$R_3 = \frac{r_1 r_3}{R_6}. \tag{7}$$

The local balance equations for TSS_{Up}, with the states numbered as in Table 3, are:

$$\begin{array}{rclrclrcl}
\pi_1 &=& Q_{13}Q_{56}\pi_0, & \pi_2 &=& R_{24}\pi_0, & \pi_3 &=& Q_{56}\pi_0, \\
\pi_4 &=& Q_{13}Q_{56}\pi_0, & \pi_5 &=& R_{24}Q_{13}Q_{56}\pi_0, & \pi_6 &=& R_{24}\pi_0, \\
\pi_7 &=& R_{24}Q_{13}Q_{56}\pi_0, & \pi_8 &=& R_{24}^2\pi_0, & \pi_9 &=& R_{24}Q_{56}\pi_0, \\
\pi_{10} &=& Q_{56}\pi_0, & \pi_{11} &=& R_{24}Q_{56}\pi_0. & & &
\end{array}$$

where $R_{24} = r_2/r_4$, $Q_{13} = r_1/R_3$, $Q_{56} = r_5/R_6$. After the normalisation procedure: $\pi_0 = 1/d_{\text{Up}}$, where

$$d_{\text{Up}} = 1 + 2R_{24} + 2Q_{56} + 2R_{24}Q_{56} + 2Q_{13}Q_{56} + 2R_{24}Q_{13}Q_{56} + R_{24}^2 \tag{8}$$

and the bound for the probability is equal to:

$$\pi_{\text{IN-USE}_{\text{Up}}} = 1 - (\pi_0 + \pi_2 + \pi_6 + \pi_8) = 1 - \frac{(1 + R_{24})^2}{d_{\text{Up}}}. \tag{9}$$

4.2.2 An upper bound of $\pi_{\text{IN-USE}}$ for the *TSSO* system

Transformation of the *TSSO* model is made in two steps. First, we split the transition $C_1 \xrightarrow{(dir\_to\_market, r_7)} C_2$ ($T_1 \rightarrow T_2$, respectively) into two, passing by the state C_4 (T_3). This is done by incorporating the rate of the transition with activity type *dir_to_market* into two existing ones $C_1 \xrightarrow{(to\_station, R_5)} C_4$ and $C_4 \xrightarrow{(to\_market, R_1)} C_2$ ($T_1 \rightarrow T_3$ and $T_3 \rightarrow T_2$, respectively) as depicted in Figure 12. We are interested in increasing the probability of staying in either of the states C_4 (T_3) or C_2 (T_2), therefore the rate of leaving the state C_1 (T_1) should not be decreased. We achieve this by setting

$$R_5 = r_5 + r_7 \qquad \text{and} \qquad R_1 = r_1$$

because the total rate of leaving the state C_1 (T_1) is preserved. The activity type *dir_to_market* no longer exists in any component, so it should be removed from the cooperation set L.

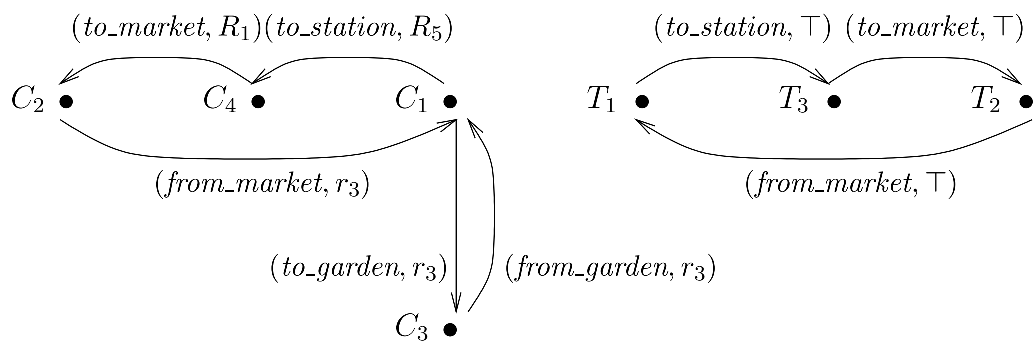

Figure 12: The intermediate Taxi System for bounding the greatest value of the probability $\pi_{\text{IN-USE}}$ of the Taxi System with Optional Stop, $TSSO^*_{\text{Up}} \stackrel{\text{def}}{=} (C_1||C_1) \bowtie_L T_1$, $L = \{to\_market,\ from\_market,\ to\_station\}$.

Performing the same procedure for the intermediate system $TSSO^*$ as was done for the *TSS* system, we split the transition $C_2 \xrightarrow{(from\_market, r_3)} C_1$ ($T_2 \rightarrow T_1$) into two: $C_2 \xrightarrow{(from\_market, R_3)} C_4$ and $C_4 \xrightarrow{(from\_station, R_7)} C_1$ ($T_2 \rightarrow T_3$, $T_3 \rightarrow T_1$, respectively) and add a new activity type *from_station* in order to ensure the existence of a reverse pair with element *to_station*. This new type must be included in the cooperation set L (Figure 13). The transition rates are set as for the *TSS* system (compare with (6), (7)):

$$R_7 < R_1 \qquad \text{and} \qquad R_3 = \frac{r_3 R_1}{R_7}.$$

Calculations for this bound are performed according to formulæ (8), (9) for the *TSS* system, where Q_{56} is replaced by Q_{57}.

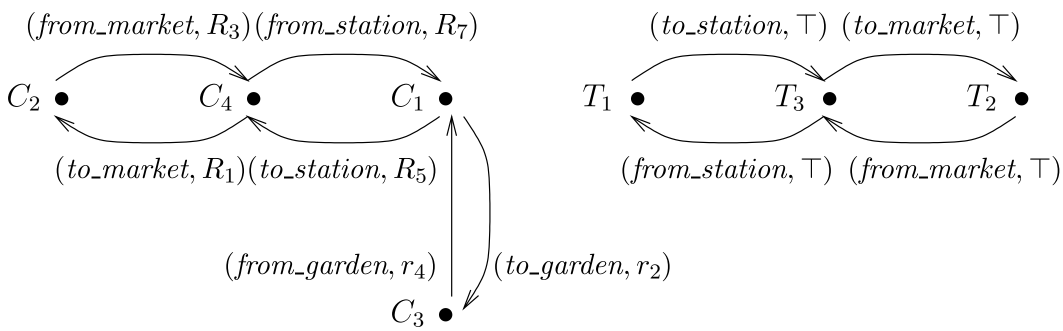

Figure 13: The Taxi System bounding the greatest value of the probability $\pi_{\text{IN-USE}}$ of the Taxi System with Optional Stop, $TSSO_{\text{Up}} \stackrel{\text{def}}{=} (C_1||C_1) \underset{L}{\bowtie} T_1$, $L = \{$*to_market, from_market, to_station, from_station*$\}$.

4.2.3 A lower bound of the probability $\pi_{\text{IN-USE}}$ for the TSS system

In order to find the lower bound of $\pi_{\text{IN-USE}}$ we have to construct a model in which the sojourn time in the T_2 state is relatively shorter than in the original one. This may be done by allowing a direct path from the state C_1 to the state C_2 in the C_1 component. This path corresponds to the complementary path $T_1 \to T_2$ in the T_1 component because both the components cooperate while changing states from $C_1(T_1)$ to $C_2(T_2)$. Two consecutive transitions $C_1 \xrightarrow{(to\_station, r_5)} C_4$ and $C_4 \xrightarrow{(to\_market, r_1)} C_2$ are concatenated and one activity type is assigned to the new transition $C_1 \xrightarrow{(to\_market, R_1)} C_2$ (Figure 14). In this case *to_market* has been chosen but it does not really matter. The important thing is that the new transition $C_1 \xrightarrow{(to\_market, R_1)} C_2$ and the existing one $C_2 \xrightarrow{(from\_market, r_3)} C_1$ form the shortest possible cycle and we get a reverse pair (*to_market, from_market*). The activity types *to_market* and *from_market* belong to the cooperation set L. Because the *to_station* activity type no longer appears in any component, it should be removed from the set. The new set L is $\{$*to_market, from_market*$\}$ and both its elements are also elements of the same reverse pair. The value R_1 is chosen as the rate of two transitions with exponential coefficients r_5, r_1 respectively, executed consecutively, i.e. $R_1 = \frac{r_1 r_5}{r_1 + r_5}$. Notice that this choice preserves the mean only.

The corresponding Markov chain has 8 states and, if they are numbered according to the schema in Table 3, the local balance equations are:

$$\begin{array}{llll} \pi_1 = R_{13}\pi_0, & \pi_2 = R_{24}\pi_0, & \pi_4 = R_{13}\pi_0, & \pi_5 = R_{13}R_{24}\pi_0, \\ \pi_6 = R_{24}\pi_0, & \pi_7 = R_{13}R_{24}\pi_0, & \pi_8 = R_{24}^2\pi_0, & \end{array}$$

where $R_{13} = R_1/r_3$ and $R_{24} = r_2/r_4$. The probability π_0 is equal to $1/d_{\text{Low}}$, where

$$d_{\text{Low}} = 1 + 2R_{13} + 2R_{24} + 2R_{13}R_{24} + R_{24}^2.$$

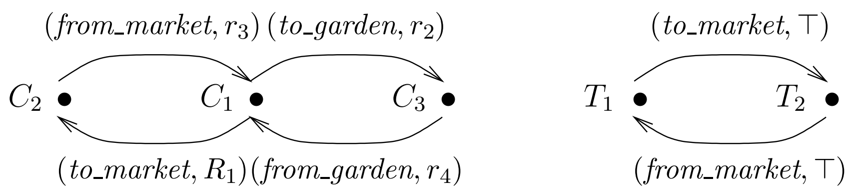

Figure 14: The Taxi System bounding the smallest value of the probability $\pi_{\text{IN-USE}}$ of the Taxi System with Mandatory/Optional Stop, $TSS_{\text{Low}} \stackrel{\text{def}}{=} TSSO_{\text{Low}} \stackrel{\text{def}}{=} (C_1||C_1) \underset{L}{\bowtie} T_1$, $L = \{to\_market,\ from\_market\}$.

The obtained lower bound for $\pi_{\text{IN-USE}}$ is equal to

$$\pi_{\text{IN-USE}_{\text{Low}}} = \pi_1 + \pi_4 + \pi_5 + \pi_7 = \frac{2R_{13}(1 + R_{24})}{d_{\text{Low}}}. \tag{10}$$

4.2.4 A lower bound of $\pi_{\text{IN-USE}}$ for the *TSSO* system

The natural way to get a lower bound of this probability is to remove the additional states $C_4(T_3)$ indicating the taxi stop. As before, concatenation of two transitions between the states $C_1 \xrightarrow{(to\_station, r_5)} C_4$ and $C_4 \xrightarrow{(to\_market, r_1)} C_2$ ($T_1 \rightarrow T_3$, $T_3 \rightarrow T_2$, respectively) requires a choice of one of them (it does not matter which one because both the activity types can be fired in all the components and both these types belong to the set L). The activity type which is dropped has to be excluded from the cooperation set L. By doing this we get two transitions $C_1 \xrightarrow{(to\_market, \frac{r_1 r_5}{r_1 + r_5})} C_2$ and $C_1 \xrightarrow{(dir\_to\_market, r_7)} C_2$ (and two edges $T_1 \rightarrow T_2$ respectively). The doubled edges can be replaced by one edge labelled by either *to_market* or *dir_to_market*. In this case *to_market* is chosen and the complementary action type *dir_to_market* is removed from the set L. The new transition rate R_1 is calculated by adding the rates of two replaced transitions: $R_1 = r_7 + \frac{r_1 r_5}{r_1 + r_5}$ (Figure 14). Notice that TSS_{Low} and $TSSO_{\text{Low}}$ are equal to each other from topological point of view, only the value of R_1 is different.

4.2.5 Some numerical results

1. all transition rates r_i, $i = 1, 2, \ldots 5$ and r_7 are equal to 1.0,

   (a) *TSS*: $R_1 = 1.0$ and to calculate an upper bound we have chosen $R_6 = 0.5 < r_1$, so $R_3 = 2.0$, $Q_{13} = 0.5$, and $Q_{56} = 2.0$,

   (b) *TSSO*: in this case $R_1 = 1.0$, $R_5 = 2.0$, and the chosen value of R_7 for an upper bound is $R_7 = 0.5 < R_1$, so $R_3 = 2$ and $R_{24} = 1.0$, $Q_{13} = R_1/R_3 = 0.5$, $Q_{57} = R_5/R_7 = 4.0$,

The *TSS* system			
Input Values	Lower Bound	Exact Value	Upper Bound
Point 1a	0.3333	0.703518	0.7500
Point 2a	0.4000	0.640370	0.7500
Point 3a	0.5000	0.779582	0.8000
The *TSSO* system			
Input Values	Lower Bound	Exact Value	Upper Bound
Point 1b	0.6000	0.779715	0.8571
Point 2b	0.6250	0.746889	0.8000
Point 3b	0.6667	0.824477	0.8571

Table 4: The probability $\pi_{\text{IN-USE}}$ for the *TSS* and *TSSO* systems

2. all transition rates r_i, $i = 2, 3, 4, 5$ and r_7 are equal to 1.0, $r_1 = 2.0$ i.e. the time spent at the service station is shorter,
   (a) *TSS*: $R_1 = 2/3$, for an upper bound $R_6 = 1.0 < r_1$ $R_3 = 1.0$,
   (b) *TSSO*: for a lower bound $R_1 = 5/3$, $R_{13} = 5/3$, for an upper bound $R_1 = 2.0$, $R_5 = 2.0$, $R_7 = 1.0 < R_1$, $R_3 = 2.0$, $Q_{13} = 1.0$, $Q_{57} = 2.0$,
3. transition rates r_i, $i = 2, 3, 4$ and r_7 are equal to 1.0, $r_1 = r_5 = 2.0$ i.e. a customer is encouraged to take a taxi rather than to walk and the taxi route via a service station is actually shorter,
   (a) *TSS*: for a lower bound $R_1 = 1.0$, $R_{13} = 1.0$, for an upper bound $R_6 = 1.0 < r_1$, $R_3 = 2$, $Q_{13} = 1.0$, $Q_{56} = 2.0$,
   (b) *TSSO*: for a lower bound $R_1 = 2.0$, for an upper bound $R_5 = 3.0$, $R_1 = 2.0$, $R_7 = 1.0 < R_1$, $R_3 = 2.0$, $Q_{13} = 1.0$, $Q_{57} = 3.0$.

Notice that for the same data set the bounds of $\pi_{\text{IN-USE}}$ for the *TSSO* system are shifted upwards compared to the ones for the *TSS* system (Table 4). Clearly this is because in the *TSSO* system the total rate of leaving the "not-in-use" state $C_1(T_1)$ is greater than in the *TSS* system.

4.2.6 A cyclic component in a system to bound the initial *TSSO* system

It will be shown in this subsection that a system containing a cyclic component may have a product form solution which is an upper bound for the *TSSO* system.

The activity type *from_market* appearing in the initial system *TSSO* (Figure 9) will be called *dir_from_market*. The activity types (*dir_to_market*, *dir_from_market*) form a reverse pair. Two other activities have to be added to balance the *to_market* and *to_station* activity types. They will be named *from_market* and *from_station* respectively and they will be included in the cooperation set L because they have to be fired simultaneously in both the components C_1 and T_1 (Figure 15). The problem now is to choose the values of R_3 and R_7. For the Kolmogorov condition to be fulfilled in the cyclic component C_1 we require $R_3 R_7 = \frac{r_1 r_3 r_5}{r_7}$; this gives a wide scope of possibilities. To bound the probability that the taxi is in-use we have to slow down possible transitions towards the state $C_1(T_1)$. This goal may be reached by setting $R_3 \gg R_7$. However, we do not have any formal formula about how large the difference must be. Notice that a great difference between matrix elements may cause problems for obtaining a numerical solution.

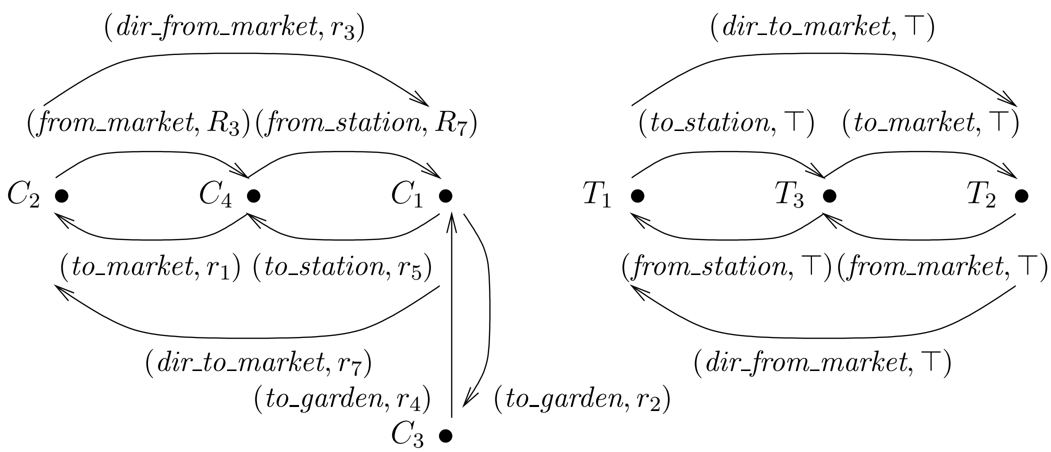

Figure 15: The Taxi System bounding the greatest value of the probability $\pi_{\text{IN-USE}}$ of the Taxi System with Optional Stop, with the cyclic component C_1, $TSSO_{\text{UpCyclic}} \stackrel{\text{def}}{=} (C_1||C_1) \underset{L}{\bowtie} T_1$, $L = \{$*to_market*, *from_market*, *to_market*, *from_station*, *to_station*, *dir_from_market*, *dir_to_market*, $\}$.

The corresponding Markov chain has 12 states and they are ordered as in Table 3. The local balance equations are:

$$\begin{array}{llllllll}
\pi_1 = R_{73}\pi_0, & \pi_2 = R_{24}\pi_0, & \pi_3 = Q_{57}\pi_0, & \pi_4 = R_{73}\pi_0, \\
\pi_5 = R_{24}R_{73}\pi_0, & \pi_6 = R_{24}\pi_0, & \pi_7 = R_{24}R_{73}\pi_0, & \pi_8 = R_{24}^2\pi_0, \\
\pi_9 = R_{24}Q_{57}\pi_0, & \pi_{10} = Q_{57}\pi_0, & \pi_{11} = R_{24}Q_{57}\pi_0. &
\end{array}$$

where: $R_{73} = r_7/r_3$, $R_{24} = r_2/r_4$, $Q_{57} = r_5/R_7$. After solving the normalisation equation one gets $\pi_0 = 1/d_{\text{UpCycl}}$, where

$$d_{\text{UpCycl}} = 1 + 2R_{24} + 2R_{73} + 2Q_{57} + 2R_{24}R_{73} + 2R_{24}Q_{57} + R_{24}^2,$$

and, eventually

$$\pi_{\text{IN-USE}_{\text{UpCycl}}} = 1 - (\pi_0 + \pi_2 + \pi_6 + \pi_8) = \frac{2(R_{73} + Q_{57})(1 + R_{24})}{d_{\text{UpCycl}}}. \tag{11}$$

Let $r_i = 1.0$, $i \in \{1, 2, 3, 4, 5, 7\}$. Firstly, we put $R_3 = R_7 = 1.0$. In this case $\pi_{\text{IN-USE}_{\text{UpCycl}}} = 0.6667$ and this value is too small (compare with 1b in Table 4). Secondly, we put $R_3 = 2.0$, $R_7 = 0.5$, $R_3 > R_7$, so $\pi_{\text{IN-USE}_{\text{UpCycl}}} = 0.7500$ and this value is still too small. Finally, we put $R_3 = 10.0$, $R_7 = 0.1$, $R_3 > R_7$, so $\pi_{\text{IN-USE}_{\text{UpCycl}}} = 0.9167$. In this case the value is large enough: it is even too large according to the bound already calculated. Moreover, the difference of magnitude between R_3 and R_7 is 100 which is not recommended for numerical purposes.

5 Conclusions and further work

We have found that following the scheme of Van Dijk [1] we can construct product form PEPA models whose performance measures are lower/upper bounds for the corresponding measures of the initial model. The bounding models were built at the PEPA component level, taking advantage of the fact described in [3] that a cooperation of reversible linear input-output PEPA components is also reversible, and so has a product form solution. Such an approach allows the modeller to transform the initial system because he or she deals with unimodal Markov processes of reasonable size instead of the entire global Markov chain.

It is important to note that the transformations which are applied are heavily influenced by the performance measure of interest. It is possible that in order to find upper bounds of two different performance measures, two different bounding models would be needed. However, more generally the structure of the bounding models is likely to be the same but the activity rates of additional activities will differ.

The obtained results presented in this paper are encouraging enough to encourage us to continue to develop this method of finding bounds. In our future work we aim to investigate formal transformation techniques for PEPA components in order to be able to find bounding components automatically. We must also establish when replacing one component in a model, by a "bounding component" will result in a "bounding model".

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