UNIVERSITY OF EDINBURGH

FACULTY OF SCIENCE AND ENGINEERING

COMPUTER SCIENCE FOUR

COMPUTATIONAL COMPLEXITY

Thursday 18 April 2002

14:00 to 15:30

Examiners:

Prof. K. Turner (External)

Prof. C. Stirling (Chairman)

INSTRUCTIONS TO CANDIDATES

Answer any TWO questions.

All questions carry equal weight.

1. (a) Explain what it means for a space bound $S(n)$ to be *space constructible.* [*3 marks*]

(b) Let $S_1(n)$ and $S_2(n)$ be space bounds such that $S_2(n)$ is space constructible, $S_1(n), S_2(n) \geq \log n$, and the ratio $S_2(n)/S_1(n)$ tends to ∞ as n tends to ∞. Prove that there exists a language in $\text{DSPACE}(S_2(n))$ that is not in $\text{DSPACE}(S_1(n))$. Make clear in your proof where each of the various conditions on $S_1(n)$ and $S_2(n)$ is used. [*13 marks*]

(c) Prove the following *strict* inclusions between complexity classes. You may assume that all the functions of n that arise are space constructible. You may use standard results about the relationships between complexity classes — in addition to the one given in part (b) — provided they are clearly stated.

i. $\text{DSPACE}(n) \subset \text{DSPACE}(n \log n)$,

ii. $\text{NTIME}(n^2) \subset \text{DSPACE}(n^2 \log n)$, and

iii. $\text{PSPACE} \subset \text{DSPACE}(2^n)$. [*9 marks*]

2. (a) Describe the *Boolean circuit* model of computation. [*4 marks*]

   (b) "The Turing machine is a uniform model of computation, while the Boolean circuit model is non-uniform." Explain this statement, and describe how uniformity may be imposed on the Boolean circuit model. [*5 marks*]

   (c) Recall that NC^k is the class of languages accepted by uniform circuits of size $O(n^c)$ and depth $O((\log n)^k)$, for some constant c. Suppose that AC^k is defined in a similar way to NC^k, but relative to a circuit model in which $\wedge$- and $\vee$- gates have unbounded fan-in. (The $\neg$-gate has just one input, as before.) Prove that

$$\mathrm{NC}^k \subseteq \mathrm{AC}^k \subseteq \mathrm{NC}^{k+1}, \quad \text{for } k = 1, 2, \ldots.$$

[*8 marks*]

   (d) Recall that $\textsc{Con}$ is the language of triples (G, s, t), where G is a directed graph, s and t are distinguished vertices of G, and there is a path from s to t in G. Demonstrate that $\textsc{Con} \in \mathrm{AC}^1$. [*8 marks*]

3. This question concerns the following two optimisation problems:

- MAX=3SAT: Given a CNF Boolean formula φ with *exactly* three literals per clause, what is the maximum number of clauses in φ that can be simultaneously satisfied?
- MINVC: Given a graph G, what is the size of a smallest vertex cover in G? (A *vertex cover* is a set of vertices U such that every edge in G has at least one endpoint in U. In other words the vertices in U "cover" all the edges in G.)

(a) Let α be a number between 0 and 1. What does it mean for an approximation algorithm for MAX=3SAT to achieve performance ratio α? [*4 marks*]

(b) Present a simple randomised approximation algorithm for MAX=3SAT, and show that it achieves (expected) performance ratio $\frac{7}{8}$. [*5 marks*]

(c) What performance ratio would your algorithm from (b) achieve if run on a CNF formula φ with a mixture of two- and three-literal clauses? [*4 marks*]

(d) Suppose

$$\varphi = (\ell_{11} \vee \ell_{12} \vee \ell_{13}) \wedge (\ell_{21} \vee \ell_{22} \vee \ell_{23}) \wedge \ldots \wedge (\ell_{r1} \vee \ell_{r2} \vee \ell_{r3})$$

is an instance of MAX=3SAT. Let $G = G(\varphi)$ be the instance of MINVC defined as follows. The vertex set of G is

$$V(G) = \big\{\ell_{ij} : 1 \leq i \leq r \text{ and } 1 \leq j \leq 3\big\},$$

and the edge set is

$$\begin{aligned} E(G) = \big\{&\{\ell_{i1}, \ell_{i2}\}, \{\ell_{i2}, \ell_{i3}\}, \{\ell_{i3}, \ell_{i1}\} : 1 \leq i \leq r\big\} \\ &\cup \big\{\{\ell_{ij}, \ell_{i'j'}\} : i \neq i' \text{ and } \ell_{ij} \Leftrightarrow \neg\ell_{i'j'}\big\}. \end{aligned}$$

Informally, vertices in G correspond to (instances of) literals in φ; clauses in φ give rise to triangles in G; and vertices in different triangles are connected if they correspond to a variable and its negation.

Demonstrate that φ has an assignment satisfying k clauses iff $G(\varphi)$ has a vertex cover of size $3r - k$. [*12 marks*]

[**Hint:** Try omitting from U some vertices that correspond to true literals.]

[**Remark:** The reduction in part (d) allows one to translate non-approximability results for MAX=3SAT across to MINVC, though doing this is *not* part of the question.]