## Fixpoints and Games

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## Inductive definitions

Suppose $\Phi$ is an operator taking *sets* $W \subseteq \omega$ to sets $\Phi(W)$. If $\Phi$ is monotone, then

$$\varnothing \subseteq \Phi(\varnothing) \subseteq \Phi(\Phi(\varnothing)) \subseteq \dots \textit{ transfinitely}$$

The limit is the set *inductively defined by* $\Phi$.
Put $\Phi^\zeta = \Phi(\bigcup_{\xi<\zeta} \Phi^\xi)$; then $\Phi^\infty = \bigcup_\zeta \Phi^\zeta$ is the limit.

If $\Phi$ is definable as $W \mapsto \{\, w \mid \phi(W, w) \,\}$ for a $\Gamma$ formula $\phi(W, w)$ with a free set variable $W$, then write $\phi^\infty(w)$ for $w \in \Phi^\infty$.
If $\phi$ also has free variable $x$, and $w_0 \in \omega$, can define

$$R(x) \Leftrightarrow \phi^\infty(x, w_0)$$

Such an $R$ is said to be $\Gamma$*-IND*.
If $\phi$ is *positive*, then $R$ is *pos-*$\Gamma$*-IND*.
Kleene showed that

$$pos\text{-}\Pi^0_1\text{-}IND = \Pi^1_1$$

## The fixpoint hierarchy

Take *first-order* arithmetic and add set variables $X$, membership $\in$, and an operator to form inductive definitions:

$$\mu(w, W).\phi(w, W)$$

denotes the set inductively defined by $\phi$ (must be positive).
Define $\Sigma^\mu_0 = \Sigma^0_1$; $\Pi^\mu_n = \neg\Sigma^\mu_n$;
and $P(x)$ to be $\Sigma^\mu_{n+1}$ if there is $\Pi^\mu_n$ $Q(w, W, x)$ such that $P(x) \Leftrightarrow \tau \in \mu(w, W).Q(w, W, x)$.
Note that $\Sigma^\mu_1 = pos\text{-}\Pi^0_1\text{-}IND = \Pi^1_1$.
The fixpoint hierarchy is strict by 'the usual argument'.

## ... came from Computer Science

In modal logic, don't need individual variables: $[]\Phi$ means $s \in []\Phi$ where $s$ is the 'current state'.
So the fixpoint extension of modal logic looks like $\mu Z.\Phi(Z)$, with $Z$ a variable over sets of states.
The fixpoint hierarchy in modal fixpoint logic (alias modal mu-calculus) $\mu Z_1.\nu Z_2.\mu Z_3 \dots$ is important for several reasons – and was early understood to be intimately connected with *Rabin automata* and *parity games*.

## Automata and modal mu-calculus

A *Rabin automaton* is a finite automaton equipped with $m$ pairs $(R_i, G_i)$ of subsets of states.
An infinite run is *accepted* if there some $i$ such that $R_i$ is seen finitely often and $G_i$ is seen infinitely often: or, so to say,

$$\bigvee_{1\le i\le m} (\neg\infty R_i \wedge \infty G_i)$$

Rabin automata correspond to certain fixpoint languages; $m$ corresponds to alternation depth.
In an *alternating* Rabin automaton, we play a Gale–Stewart style game on the automaton, with the above winning condition.

They are equivalent to *alternating parity automata*. Here we have sets $X_1, X_2, X_3, X_4, \dots, X_{2m}$; the winning condition is: the highest $X_i$ seen infinitely often must be even.
These are equivalent to modal mu-calculus: $2m$ corresponds to fixpoint alternation.
Now, the parity condition says:

$$\infty X_{2m} \vee (\neg\infty X_{2m-1} \wedge (\infty X_{2m-2} \vee \dots \wedge (\infty X_2 \vee \neg\infty X_1)\dots))$$

Parity and Rabin conditions are boolean combinations of $\infty$ and $\neg\infty$.

## The game quantifier; determinacy

Let $P(\alpha, x)$ define a family of games; define

$$\Game\alpha.P(\alpha, x) \Leftrightarrow x \in \{\, x \mid \text{Eloise wins } P(\alpha, x) \,\}$$

(so loosely $\Game = \exists\forall\exists\forall\ldots$)
If $P$ is $\Gamma$ then $\Game\alpha.P$ is $\Game\Gamma$.
Martin's theorem says $Det(\boldsymbol{\Delta}^1_1)$: if $P$ is $\boldsymbol{\Delta}^1_1$ then the game $P$ is determined.
If $\Gamma$ is a known class, what do we know about $\Game\Gamma$?
In general: if $Det(\boldsymbol{\Gamma})$ (and . . . ) then $\neg\Game\Gamma = \Game\neg\Gamma$.
For analytical $\Gamma$ we have $\Game\Pi^1_n = \Sigma^1_{n+1}$; and assuming $Det(\boldsymbol{\Sigma}^1_n)$ also $\Game\Sigma^1_n = \Pi^1_{n+1}$.

## Games and induction

Kechris & Moschovakis showed that

$$\Game\Sigma^0_1 = \Pi^1_1$$

Solovay showed

$$\Game\Sigma^0_2 = pos\text{-}\Sigma^1_1\text{-}IND$$

Now $\Sigma^1_1 = \neg\Pi^1_1 = \neg\Sigma^\mu_1$. So putting it another way:

$$\begin{aligned} \Game\Sigma^0_1 &= \Sigma^\mu_1 \\ \Game\Sigma^0_2 &= \Sigma^\mu_2 \end{aligned}$$

The computer science version suggests how to continue . . .

## Playing games with arithmetic fixpoints

Take an arithmetic fixpoint formula of the form

$$\mu X_{2m-1}.\nu X_{2m-2}.\ldots\nu X_2.\mu X_1.\phi$$

We can define a game $P$ on $\omega$ such that Eloise wins iff the formula is true.
How? It is exactly an 'interpreter': build interpreter machine, with states coded as integers. For correctness, need exactly that the plays satisfy a parity condition: highest $X_i$ seen infinitely often is even. (Why? By transferring via modal mu-calculus; or directly.)
Therefore *any* fixpoint property $Q$ is $\Game\alpha.P$ for some parity condition $P$.
What is a parity condition? '$\infty X_i$ in $\alpha$' says that $\forall j.\exists k > j.$'$X_i$ seen at $\alpha(k)$'.
So it is $\Pi^0_2$; so parity is $\nabla^0_2$ (boolean closure of $\Sigma^0_2$).

## From fixpoints via parity to difference hierarchies

So we have $\Sigma^\mu_n \subseteq \Game\nabla^0_2$ for all $n$.
Can we refine this?
The *difference hierarchy over* $\Sigma^0_2$ is defined by

$$\Sigma^\partial_1 = \Sigma^0_2 \qquad \Sigma^\partial_{n+1} = \Sigma^0_2 \wedge \Pi^\partial_n$$

We may as well define $\Sigma^\partial_0 = \Sigma^0_1$. Then Kechris–Moschovakis and Solovay give us

$$\Sigma^\mu_1 = \Game\Sigma^\partial_0 \qquad \Sigma^\mu_2 = \Game\Sigma^\partial_1$$

Is this the right formulation for the generalization?
By inspection the parity condition of rank $m$ is $\Sigma^\partial_m$.
By more careful inspection it is actually $\Sigma^\partial_{m-1}$.
(Why? Because rank 1 is 'finitely often $X_1$', which can only be true if play terminates: $\exists j.$'play stops at $\alpha(j)$' which is only $\Sigma^0_1$.)

## From differences to fixpoints

So we now have

$$\Sigma^\mu_n \subseteq \Game\Sigma^\partial_{n-1}$$

The converse is harder. The idea is to extend Solovay, which analysed Wolfe's proof of $Det(\boldsymbol{\Sigma}^0_2)$.
Suppose $P(a) \Leftrightarrow (\exists i.Q(i,\alpha)) \wedge R(\alpha)$ is $\Sigma^\partial_{n+1}$,
so $Q$ is $\Pi^0_1$ and $R$ is $\Pi^\partial_n$.
We define inductively 'easy winning positions' $u = a_0 \ldots a_k$ for Eloise by
$W^\zeta = \{\, u \mid \exists i.$'Eloise wins $H^\zeta_i$ from $u$' $\}$,
where $H^\zeta_i$ is defined in terms of $R$, $W^{<\zeta}$ and $Q$.
It can be shown that Eloise wins from $u$ iff $u \in W^\infty$. Then $W$ is an inductive definition with a (by induction) $\Pi^\mu_{n+1}$ body, so is $\Sigma^\mu_{n+2}$, Q.E.D.
So we have the theorem

$$\Sigma^\mu_{n+1} = \Game\Sigma^\partial_n.$$

## Onwards and Upwards . . .

It is 'well-known' that $\Delta^0_3 = \bigcup_{\zeta<\omega_1^{CK}} \Sigma^\partial_\zeta$ (and similarly for higher $\Delta^0_n$), for the transfinite difference hierarchy.
So is it the case that

$$\Sigma^\mu_{\zeta+1} = \Game\Sigma^\partial_\zeta?$$

(For $\zeta < \omega_1^{CK}$ and appropriate transfinite extension of fixpoint hierarchy.)
Yes . . . though now fixpoint satisfaction no longer corresponds to general transfinite parity games, but rather to certain well-behaved transfinite parity games.

(Joint work with Jacques Duparc and Sandra Quickert)

## . . . with WWW

Another way of understanding what happens as we climb the difference hierarchy.

The effective Wadge degrees of $\Sigma_n^{\partial}$ are $(\omega_1^{\mathrm{CK}})^n$ (and we hope this continues transfinitely).

There is an operation on Wadge games which has the effect of increasing the degree by a multiplicative factor of $\omega_1^{\mathrm{CK}}$.

This operation is roughly: allow one player to cancel the game and start again, perhaps switching to the complement of 'his' domain. He's allowed to do this a finite number of times.

In a loose sense, this corresponds at the strategy computation level to wrapping an inductive definition around your existing strategy.

Question: what does the game quantifier do to Wadge degrees?

(Joint work with Jacques Duparc and Sandra Quickert)