Fixpoint alternation and the Wadge hierarchy

Julian Bradfield

Laboratory for Foundations of Computer Science, University of Edinburgh

Jacques Duparc

Ecole des HEC, Université de Lausanne

Sandra Quickert

Laboratory for Foundations of Computer Science, University of Edinburgh

Abstract: In [2] Bradfield found a link between finite differences formed by Σ_2^0 sets and the mu-arithmetic introduced by Lubarski [10]. We extend this approach into the transfinite: in allowing countable disjunctions we show that this kind of extended mu-calculus matches neatly to the transfinite difference hierarchy of Σ_2^0 sets. The difference hierarchy is intimately related to parity games. When passing to infinitely many priorities, it might not longer be true that there is a positional winning strategy. However, if such games are derived from the difference hierarchy, this property still holds true. In the second part, we use the more refined Wadge hierarchy to understand further the links established in the first part, by connecting game-theoretic operations to operations on Wadge degrees.

1 Introduction

Modal mu-calculus, the logic obtained by adding least and greatest fixpoint operators to modal logic, has long been of great practical and theoretical interest in systems verification. The problem of understanding alternating least and greatest fixpoints gave rise to a powerful and elegant theory relating them to alternating parity automata and to parity games, developed by many people including particularly Emerson, Lei, Jutla and Streett. Meanwhile, mu-arithmetic, the logic obtained by adding fixpoints to first-order arithmetic, made a brief appearance in the early 90s when Lubarsky studied its ordinal-defining capabilities – curiously, the logic had not previously been studied *per se* even by logicians. Then Bradfield used mu-arithmetic as a meta-language for modal mu-calculus, in which to prove a theorem on alternating fixpoints. Subsequently, Bradfield looked further into the analogies between mu-arithmetic and modal mu-calculus, and showed

Email addresses: `jcb@inf.ed.ac.uk` (Julian Bradfield), `jacques.duparc@unil.ch` (Jacques Duparc), `squicke1@inf.ed.ac.uk` (Sandra Quickert).

a natural equation between arithmetic fixpoints and the finite difference hierarchy over Σ^0_2, corresponding to the equation between modal fixpoints and parity games. Once in the world of arithmetic, it becomes natural to think about transfinite hierarchies. In this paper, we study the transfinite extension of the connection between mu-arithmetic and the effective difference hierarchy, and connect it to the Wadge hierarchy. We remark here that already in [1] Barua had studied a relationship between automata and difference hierarchies, showing that ω-regular languages over a finite alphabet are contained in the (classical) difference hierarchy, and moreover their rank therein is computable.

PART I

In this part, we introduce the transfinite mu-calculus and its model-checking, relate it to the difference hierarchy, and establish the existence of positional winning strategies for the model-checking game.

2 The Transfinite Mu-Calculus

2.1 Syntax and Semantics of the Transfinite Mu-Calculus

The logic we are considering is an extension of the usual mu-arithmetic, as introduced by Lubarski [10]. First, let us establish basic notation and conventions. ω is the set of non-negative integers; variables $i, j, \ldots, n$ range over ω. The set of finite sequences of integers is denoted ω^*; finite sequences are identified with integers via standard codings; the length of a sequence s is denoted $\mathrm{lh}(s)$. The set of infinite sequences of integers is ${}^\omega\omega$. For $\alpha \in {}^\omega\omega$, $\alpha(i)$ is the i'th element of α, and $\alpha({<}i)$ is the finite sequence $\langle \alpha(0), \ldots, \alpha(i-1) \rangle$. Concatenation of finite and infinite sequences is written with concatenation of symbols or with $\frown$, and extended to sets pointwise. The usual Kleene lightface hierarchy is defined on ω, ${}^\omega\omega$ and their products: $\Sigma^0_1 = \Sigma^1_0$ is the semi-recursive sets, $\Sigma^0_{n+1} = \exists x \in \omega . \Pi^0_n$, $\Pi^i_n = \neg\Sigma^i_n$ and $\Sigma^1_{n+1} = \exists \alpha \in {}^\omega\omega . \Pi^1_n$. The corresponding boldface hierarchy is similar, but starts with $\boldsymbol{\Sigma}^0_1 = \boldsymbol{\Sigma}^1_0$ being the open sets.

Mu-arithmetic has as basic symbols the following: function symbols f, g, h; predicate symbols P, Q, R; first-order variables x, y, z; set variables X, Y, Z; and the symbols $\vee, \wedge, \exists, \forall, \mu, \nu, \neg, \in$. The language has expressions of three kinds, individual terms, set terms, and formulae. The individual terms comprise the usual terms of first-order logic. The set terms comprise set variables and expressions $\mu(x, X).\, \phi$ and $\nu(x, X).\, \phi$, where X occurs positively in ϕ. Here μ binds both an individual variable and a set variable; henceforth we shall often write just $\mu X.\, \phi$, and assume that the individual variable is the lower-case of the set variable. We also use $\mu\!\!\!\nu$ to mean 'μ or ν as appropriate'. The formulae are built by the usual first-order construction, together with the rule that if τ is an individual term and Ξ is a set term, then $\tau \in \Xi$ is a formula.

The semantics of the first-order connectives is as usual; $\tau \in \Xi$ is interpreted naturally; and the set term $\mu X.\, \phi(x, X)$ is interpreted as the least fixpoint of the functional $X \mapsto \{\, m \in \omega \mid \phi(m, X) \,\}$ (where $X \subseteq \omega$).

To produce a transfinite extension, we add the following symbols and formulae. If we have countably many recursively given Φ_i, $i \in \omega$, whose free set variables are contained in the same finite set of set variables, then we allow infinite countable disjunction $\bigvee_{i<\omega} \Phi_i$

and conjunction $\bigwedge_{i<\omega} \Phi_i$. The restriction on free variables means that we can transform any formula to a closed formula by adding finitely many fixpoint operators. The semantics is obvious.

Any formula in the mu-calculus can be rewritten in a prenex normal form:

$$\tau_n \in \mu X_n.\tau_{n-1} \in \nu X_{n-1}.\tau_{n-2} \in \mu X_{n-2} \ldots \tau_1 \in \mu\!\!\!/\nu X_1.\Phi$$

For the transfinite mu-arithmetic we need an extension of this formulation.

Definition 1 By induction on the construction of the formula we say that a formula in the transfinite mu-calculus is written in extended prenex normal form

▷ if it is a formula in the finite mu-calculus and written in prenex normal form, or

▷ if the formula is an infinite disjunction or conjunction of extended prenex normal form formulae, or

▷ if it is some $\mu\!\!\!/\nu X.\Phi$ where Φ is in extended prenex normal form. ◁

Given formulae Φ_i for $i < \omega$ in the mu-arithmetic, we observe that the formula $\bigvee_{i<\omega} \Phi_i$ can be written in extended prenex normal form, simply by writing each Φ_i in prenex normal form. Given an arbitrary formula of the extended arithmetic mu-calculus, an easy proof on induction by the formula's construction shows that it can be written in extended prenex normal form. Furthermore, we can unfold its complexity and represent it by a wellfounded tree on ω^*.

2.2 A Hierarchy of the Transfinite Mu-Calculus

The fixpoint alternation hierarchy of mu-arithmetic is thus: the first order formulae and all set variables form the class Σ_0^μ which is the same as Π_0^μ. For any natural number n let Σ_{n+1}^μ be generated from $\Sigma_n^\mu \cup \Pi_n^\mu$ by closing it under $\vee, \wedge$ and the operation $\mu X.\Phi$ for $\Phi \in \Sigma_{n+1}^\mu$. Π_{n+1}^μ contains all negations of formulae and set terms in Σ_{n+1}^μ. In order to extend the hierarchy we need to describe the limit step. We allow recursively countable disjunctions and conjunctions, but we want to stay in the lightface hierarchy. Therefore we extend the hierarchy to ω_1^{ck}, the first non-recursive ordinal. Let λ be a recursive limit ordinal. In Σ_λ^μ we collect all formulae of earlier stages and close under $\bigvee_{i<\omega}, \vee$ and $\wedge$. Observe that a formula in Σ_λ^μ is equivalent to a formula $\bigvee_{i<\omega} \Phi_i$ where each $\Phi_i \in \Sigma_{\alpha_i}^\mu$ with $\alpha_i < \lambda$. Finally, we let $\Pi_\lambda^\mu = \neg\Sigma_\lambda^\mu$. The transfinite successor stages are built in the same way as the finite successor stages.

Later, this hierarchy will be linked to the effective version of the Hausdorff–Kuratowski difference hierarchy of Σ_2^0-sets: a set is in Σ_α^∂ iff it is of the form

$$\bigcup_{\xi \in \mathrm{Opp}(\alpha)} A_\xi \setminus \bigcup\nolimits_{\zeta<\xi} A_\zeta$$

where $(A_\xi)_{\xi<\alpha}$ is an effective enumeration of a $\subseteq$-increasing sequence of Σ_2^0-sets, $\alpha < \omega_1^{\mathrm{ck}}$, and $\mathrm{Opp}(\alpha)$ is the set of ordinals $< \alpha$ and of opposite parity to α, where the parity of a limit ordinal is even.

3 Model-Checking for the Transfinite Mu-Calculus

The aim of this section is to introduce a model checking game in which all runs are finite, even for formulae of transfinite complexity. Let us assume that the formula to check has set variables' indices smaller than some recursive ordinal α.

We associate to every set variable X_ξ for $\xi < \alpha$ a clock A_ξ. Verifier will be in charge of setting the clocks A_ξ with odd index, while Refuter will be in charge of the clocks A_ξ with even index. Limit ordinals are considered to be even. Moreover, we follow the convention that any variable under the scope of a minimal fixpoint operator is of odd parity, any variable under the scope of a maximal fixpoint operator is of even parity. The rules of the model checking game with clocks G_{clock} between the players Verifier and Refuter are as follows:

At the very start, Verifier has the role of $\exists$-player and Refuter the one of $\forall$-player. Verifier chooses for each odd clock A_ξ some value α_ξ, and Refuter chooses for each even clock A_ξ some value α_ξ. Then the usual model checking game starts. Each time some set variable X_ξ is seen, the clock A_ξ must be set by Verifier to a smaller value if ξ is odd, or else it is Refuter's task to lower the value of A_ξ. At the same time, the clocks A_ζ with $\zeta < \xi$ can be reset to any value $< \omega_1$ by the respective players in charge of them. More precisely, given a mu-arithmetic formula Φ and an assignment s to free individual variables of Φ, the game $G_{clock}(\Phi, s)$ goes as follows:

- ▷ first move: Verifier plays $(\alpha_\xi \mid \xi < \gamma \text{ and } \xi \text{ odd})$ with $\alpha_\xi < \omega_1$.
- ▷ second move: Refuter plays $(\alpha_\xi \mid \xi < \gamma \text{ and } \xi \text{ even})$ with $\alpha_\xi < \omega_1$.
- ▷ third move: Verifier plays $(\Phi, s, (\alpha_\xi \mid \xi < \gamma))$.
- ▷ nth move: Assume the play is at the position $(\varphi, s, (\beta_\xi \mid \xi < \gamma))$ where φ is a subformula of Φ. We need to consider several cases:
  - ◇ $\varphi = \varphi_0 \wedge/\vee \varphi_1$: $\forall/\exists$-player chooses either $(\varphi_0, s, (\beta_\xi \mid \xi < \gamma))$ or $(\varphi_1, s, (\beta_\xi \mid \xi < \gamma))$
  - ◇ $\varphi = \bigwedge / \bigvee_{\beta<\gamma} \varphi_\beta$: $\forall/\exists$-player chooses exactly one $\beta_0 < \gamma$ and plays $(\varphi_{\beta_0}, s, (\alpha_\xi \mid \xi < \gamma))$
  - ◇ $\varphi = \neg\varphi_0$: $\exists$-player chooses $(\varphi_0, s, (\beta_\xi \mid \xi < \gamma))$, Verifier and Refuter change their roles as $\exists$-player and $\forall$-player
  - ◇ $\varphi = \exists/\forall x \varphi_0$: $\exists/\forall$-player chooses some a for x and plays $(\varphi_0, s[a/x], (\beta_\xi \mid \xi < \gamma))$
  - ◇ $\varphi = \nu/\mu X \varphi_0$: $\forall/\exists$-player plays $(\varphi_0, s, (\beta_\xi \mid \xi < \gamma))$
  - ◇ $\varphi = x \in P$: the game stops. Verifier wins iff $s \in P$ in case she is the current $\exists$-player, and $s \notin P$ in case she is the current $\forall$-player
  - ◇ $\varphi = x_\xi \in X_\xi$ with odd/even ξ: if $\alpha_\xi = 0$, the game stops and Verifier/Refuter loses. Otherwise, Verifier/Refuter needs to choose some $\beta'_\xi < \beta_\xi$ and for each odd/even $\zeta < \xi$ some $\beta'_\zeta < \omega_1$. Refuter/Verifier chooses for each even/odd $\zeta < \xi$ some $\beta'_\zeta < \omega_1$. Verifier/Refuter plays $(\Phi, s, (\beta'_\zeta \mid \zeta \leq \xi)^\frown(\beta_\zeta \mid \xi < \zeta < \gamma))$

Observe that by the requirement that each X_ξ be in the scope of an even number of negations, Verifier is always in the role of the $\exists$-player when encountering set variables.

Clearly, the game $G_{clock}(K, \Phi)$ is just like the usual model checking game $G(K, \Phi)$ with the additional requirement that there are clocks which must be reset. This has an important impact on the winning conditions: while it is possible to have infinite runs in $G(K, \Phi)$, we will easily see that each run of $G_{clock}(K\Phi)$ is finite.

Claim 2 The payoff set for each player in $G_{clock}(K, \Phi)$ is clopen in the usual topology on trees. In other words, the winner of each run of the game is decided after finitely many moves.

Proof. If there was an infinite play, then we claim that there is a maximal ordinal ξ such that X_ξ is seen infinitely often, but only finitely many times propositional variables with higher indices are seen. This claim is easily proved by induction on the formula's construction: assume that $\Phi = \bigvee_{\xi<\zeta} \Phi_\xi$ where ζ is a countable limit ordinal and the claim holds for any Φ_ξ. At the very first step of the model checking game Verifier needs to choose one of the formulae Φ_ξ, and the model checking game after the choice will only use subformulae from the chosen formula. Thus, the claim holds for Φ. All other induction steps are similarly simple to check.

Let us fix such an X_ξ provided by the claim. Along the play there is a point where some propositional variable with higher index than ξ is seen for the last time. This is the last time the clock A_ξ can be reset. By well-foundedness of the ordinals, the clock must reach 0 at some point, meaning that the play cannot be infinite. □

Claim 3 For every Kripke structure K we have $K \models \Phi$ iff Verifier has a winning strategy in $G_{clock}(K, \Phi)$.

Proof. It suffices to show that Verifier has a winning strategy in the usual model checking game $G(K, \Phi)$ iff she has a winning strategy in $G_{clock}(K, \Phi)$.

Assume $(M, s) \models \Phi$. Then Verifier has a winning strategy in the usual model checking game $G(K, \Phi)$, i.e. she can guarantee that either the maximal priority seen infinitely often is an even one in case of an infinite run, or that the run reaches a subformula involving only propositional variables at some point $s \in K$ where the subformula holds true. Let f be her winning strategy. We will transform it to a strategy f_c for the game G_{clock}.

For each odd ξ and each position p in the model checking game consistent with f let T^p_ξ be the set of all initial plays starting at p and being consistent with f such that X_ξ is not dominated. In the usual way, by exploiting the fact that X_ξ is only seen finitely often on any maximal branch in T^p_ξ, we can attach a rank to the tree T^p_ξ: all leaves and all nodes where X_ξ is seen for the last time (i.e. the cone of T^p_ξ above the node does not contain X_ξ any more) get rank 1. Recursively each node whose cone of T^p_ξ after removing the cones of ranked nodes does not have any leaves gets rank of the supremum of all nodes in its cone. Verifier sets the clock A_ξ to the rank of the root of T^p_ξ for each odd ξ at the beginning of the game. The strategy f_c is thus: Verifier simply follows strategy f, and in addition each time some odd X_ξ is encountered, she computes the trees T^p_ξ from her actual position p and sets her clocks equal to the rank of the roots of these trees. Moreover, each time some X_ξ is encountered at some position p independently of the parity, for each odd $\zeta < \xi$ Verifier computes the rank of the tree T^p_ζ and sets A_ζ to this value. In doing so, she will always produce legal moves: if she encounters some even X_ξ at some position p, the rank of the root of T^p_ξ is strictly less than the actual value of the clock A_ξ, because the latter was the rank of a tree containing T^p_ξ.

Now it is easy to conclude that f_c is a winning strategy for Verifier: we know already that each play is finite. If the game ends when examining some propositional variable, then Verifier wins since f_c was derived from a winning strategy f for the usual model checking game by just attaching a strategy for dealing with clocks. If the game ends because some clock is set to 0, then it must be Refuter's loss, because f_c guarantees that the minimal value played by Verifier is at least 1.

With the same arguments Similarly, we can easily show that Refuter has a winning strategy in $G(K, \Phi)$ iff he has a winning strategy in $G_{clock}(K, \Phi)$. By determinacy, this means that if Verifier has a winning strategy in $G_{clock}(K, \Phi)$, then she has a winning strategy in $G(K, \Phi)$. $\Box$

The ordinal clocks here are similar in spirit to the progress measures introduced by Klarlund [8] for Rabin automata-theoretic properties.

4 Parity Games

The use of parity games in model checking has been described by many authors. A very detailed survey is given by Niwiński [13]. Let us mention that we follow the convention that if the maximal priority seen infinitely often is odd, then player I wins. When looking at a formula in the transfinite mu-calculus, we need to play a parity game with infinitely many priorities: for each set variable we need a distinct priority. If we take the binary tree and attach to each node a priority in an arbitrary fashion, then, when playing a parity game on this tree, we might end up having a "wild" payoff set for player I, and we might also lose the nice property of having a memoryless winning strategy [6]. Furthermore, it might be that there is no maximum among the priorities seen infinitely often, and infinite runs might even meet each priority only finitely many times. However, as we will see, a labelling derived from a model checking game of a transfinite mu-calculus formula avoids all these undesired effects. Moreover, such a labelling describes some set of the transfinite difference hierarchy $\bigcup_{\alpha<\omega_1^{\text{ck}}} \Sigma_\alpha^\partial$ and vice versa.

5 Connecting the Transfinite Difference Hierarchy and the Transfinite Mu-Calculus

Our aim is to extend the following theorem of Bradfield [2]:

Theorem 4 *For every natural number n the equality $\Game\Sigma_n^\partial = \Sigma_{n+1}^\mu$ holds true.*

Here $\Game$ is the game quantifier, defined so that $\Game\alpha.\phi(x, \alpha)$ is the set of x such that player I has a winning strategy for the game with payoff set $\{\alpha\colon \phi(x, \alpha)\}$. Informally, $\Game\alpha.\phi(x, \alpha) = \exists a_0.\forall a_1.\exists a_2.\forall a_3. \ldots \phi(x, a_0 a_1 a_2 a_3 \ldots)$.

The extension into the transfinite is our main result.

Theorem 5 *For every recursive ordinal α the equality $\Game\Sigma_\alpha^\partial = \Sigma_{\alpha+1}^\mu$ holds true.*

Proof. Let α be a recursive limit ordinal, and let $\mu X_{\alpha+1}. \bigvee_{i<\omega} \Phi_i \in \Sigma_{\alpha+1}^\mu$, so in particular each Φ_i is in some Σ_β^μ for some $\beta < \alpha$. We need to find a game with payoff set in Σ_α^∂ whose winning positions for player I are calculated by this formula.

Assume that the formula $\mu X_{\alpha+1}. \bigvee_{i<\omega} \Phi_i$ describes a nonempty subset of ω, and choose some witness n for this nonemptyness. Now consider the game tree which results from the parity game played as a model checking game. We might think of it as a subtree in ω^*, each node labelled with the position in the model checking game. In extending the tree in an appropriate way we may assume that it does not contain finite maximal branches, and in further simplifying the tree we may assume that each node marks a loop-back, i.e. we see some X_β at each node. This can be done because any infinite branch must hit such nodes infinitely many times. In omitting which element $n' \in \omega$ is inspected in the model checking game, and in only keeping track on the index of the inspected subformula Φ_i and the index of the set variable X_β at a loopback we get a tree T which is simply labelled by pairs (i, β) with $i < \omega$ and a successor ordinal $\beta \in \alpha \cup \{\alpha + 1\}$. Thus, a node $s \in T$ is of the form $s = s(0)s(1)s(2)\ldots s(n)$ with $s(k) = (i, \beta)$. Observe that the labelling has the structure of a set in $\Sigma^\partial_{\alpha+1}$. Let us describe the payoff set for player I.

For $i < \omega$ and a successor ordinal $\beta \in \alpha$ we define

$$A_{i,\beta} = \{x \in [T] \mid \exists n \forall m > n\ x(m)_1 = i \wedge \exists n \forall m > n\ x(m)_2 \leq \beta\}$$

Clearly, each $A_{i,\beta} \in \Sigma^0_2$. Let

$$C_\beta = \bigcup_{i<\omega} A_{i,\beta}$$

and

$$C = \bigcup_{\beta<\alpha \text{ odd}} C_\beta \setminus \textstyle\bigcup_{\zeta<\beta} C_\zeta.$$

Thus, $C \in \Sigma^\partial_\alpha$. We have to show that C is the payoff set for player I.

Let $x \in C$. Fix some $\beta < \alpha$ such that $x \in C_\beta \setminus \bigcup_{\zeta<\beta} C_\zeta$. Fix some i such that $x \in A_{i,\beta} \setminus \bigcup_{\zeta<\beta} C_\zeta$. Since for $j \neq k$ the sets $A_{j,\xi}$, $A_{k,\zeta}$ are disjoint, we obtain $x \in A_{i,\beta} \setminus \bigcup_{\zeta<\beta} A_{i,\zeta}$. Thus, the path $x \in [T]$ belongs to a run where the model checking game eventually remains inside the subformula Φ_i and where β is the maximal ordinal seen infinitely many often. Therefore, x belongs to the payoff set for player I.

To show the other inclusion, assume that x is a winning run for player I. Thus, $\alpha + 1$ is only seen finitely many often as second component. Therefore, there is some Φ_i which will eventually only be inspected by the run. Being a winning run, $x \in A_{i,\beta} \setminus \bigcup_{\zeta<\beta} A_{i,\zeta}$. Again, by disjointness we obtain $x \in C$. □

In classical descriptive set theory, the Hausdorff–Kuratowski theorem states that for any n (or indeed any $\alpha < \omega_1$), the difference hierarchy over $\mathbf{\Sigma}^0_n$ up to level ω_1 exhausts the class $\mathbf{\Delta}^0_{n+1}$. There is a natural effectivization of this result: the difference hierarchy over Σ^0_n up to level ω_1^{ck} exhausts Δ^0_{n+1}. Although natural, and 'obviously right', it is only very recently that a proof has been published, by MedSalem and Tanaka [12]. Level 2 of this theorem, together with the result above, immediately give us the very attractive result:

Corollary 6 $\Sigma^\mu_{\omega_1^{\mathrm{ck}}} = \partial\Delta^0_3$

6 Nicely Behaving Labellings

When extending the mu-arithmetic into the transfinite we need to check whether we keep key properties, namely the existence of positional winning strategies. This leads to

Definition 7 Let P be a parity game with priorities in some $\alpha < \omega_1$. P is called max-closed iff for every infinite run the set of all labels seen infinitely often is non-empty and contains a maximum. ◁

Clearly, the rules of the model checking game ensure that the parity game derived from a model checking game of a transfinite mu-calculus formula is max-closed.

Theorem 8 *Each max-closed parity game admits a positional winning strategy for one of the players.*

Proof. We proceed by induction on the set of labels α. Of course, any set of countably many labels can be relabelled by natural numbers, but max-closedness is not preserved in general. In the sequel l will always denote the labelling function, $l : \alpha \to V$ where V is the set of vertices in the considered game graph.

Let us first consider the easier case, i.e. α is a limit ordinal. Assume player I has a winning strategy f; we need to find a positional winning strategy.

Let T be the tree of all possible plays. We define

$$A_0 = \{ s \in T \mid \exists \beta < \alpha . \forall t \in T[s] . \, l(t) \leq \beta \}$$

i.e. the cone of T above s is labelled with values up to β. Observe that by max-closedness, A_0 is dense in T. Otherwise, we could select a cone $T[t]$ having an empty intersection with A_0, meaning that every subcone of this cone is labelled with values cofinal in α. Since α is countable, there is a sequence $(\alpha_i)_{i<\omega}$ with each $\alpha_i < \alpha$ and $\bigcup_{i<\omega} \alpha_i = \alpha$. In the cone $T[t]$ it is easy to construct an infinite path x s.t. for each i there is some n_i with $l(x(n_i)) > \alpha_i$, contradicting the max-closedness.

Although A_0 is dense, it might be that the complement still contains infinite paths. Thus, we define by recursion:

$$A_\beta = \{ s \in (T \setminus \bigcup_{\zeta<\beta} A_\zeta) \mid \exists \xi < \alpha . \forall t \in (T[s] \setminus \bigcup_{\zeta<\beta} A_\zeta) . \, l(t) \leq \xi \}$$

The process stops at some countable γ, and then $\bigcup_{\beta<\gamma} A_\beta = [T]$ (since otherwise we can construct a path with labels cofinal in α, contradicting maxclosedness). From these sets we can easily determine the set of winning positions for player I. We let H_0 be the set of all elements in A_0 such that player I can win the game starting at that position. Since the labels in the cone of the game tree are bounded by some $\beta < \alpha$, by induction hypothesis player I has a positional winning strategy within H_0. In particular, the game stays within H_0. In general we let H_β be the subset of A_β such that player I has a winning strategy as long as the game stays within A_β, and as soon as the game leaves A_β, some $H_{\beta'}$ is entered with $\beta' < \beta$. Again, within H_β player I has a positional winning strategy. Analogously to Section 5 the process stabilizes at some countable γ, and $H_\gamma = \bigcup_{\beta<\gamma} H_\beta$ is the set of all winning positions of player I, and it can be described by a formula of the transfinite mu-arithmetic provided the set of labels does not exceed ω_1^{ck}. It is fairly easy to describe a positional winning strategy for player I: as long as the game takes place in some H_β, she follows the positional winning strategy within H_β. It might be that player I cannot force the game to stay inside H_β, but if this set is left, then some $H_{\beta'}$ is entered with $\beta' < \beta$, and from that moment on player I follows the positional winning strategy for $H_{\beta'}$. Since all the H_β are pairwise disjoint, the concatenation of all positional winning strategies for the H_β gives a positional winning strategy for all her winning positions.

Analogously, if player *II* has a winning strategy, then he has a positional winning strategy as well.

Now let us consider the successor case. Assume $\alpha = \beta + 1$ is odd, thus player *I* needs to make sure that β is seen only finitely many times. Assume player *I* has a winning strategy, we need to find a positional winning strategy.

Let H_0 consist of those vertices s such that, starting from s, player *I* has a winning strategy which never leads to any vertex labelled with β. Such vertices must exist, otherwise player *II* has a winning strategy. In particular, being at such a node player *I* can force the game to stay in H_0. We claim that within H_0 player *I* has a positional winning strategy. Consider the game tree starting from $s \in H_0$ and remove all nodes outside H_0 together with the cones above those nodes. The remaining tree is labelled with values smaller than β, and by induction hypothesis on this subtree (and the corresponding subgraph) player *I* has a positional winning strategy. This positional winning strategy is clearly a winning strategy for the whole game. Constructing H_γ analogously to the limit case yields a positional winning strategy for the whole game.

Now assume player *II* has a winning strategy. This means that either he can manage to see β infinitely often, or, if player *I* keeps the occurrence of β finite, he wins the induced subgame. A positional strategy is described as follows: if a vertex belongs to player *II*'s winning region, and if he has a winning strategy which guarantees him to reach some vertex labelled with β, then he plays in a way that he never leaves his winning region and after finitely many steps he will reach β. Clearly, to reach some node within the winning region labelled with β is an open condition, thus there exists a positional strategy for achieving that goal. If, after reaching β, player *II* can still reach another β within his winning region, he goes for it. At some point it might be that he still has a winning strategy, but he cannot make sure that β is seen again. At this stage consider the subgraph S which consists of all nodes in player *II*'s winning region with labels smaller than β – the edge relation restricted to S stays the same. Observe that by being a winning region player *I* can only leave the subgraph S in moving to a vertex which is still in player *II*'s winning region, but from where player *II* can reach β memoryless again while remaining in his winning region. As long as the run stays in S, by induction hypothesis player *II* has a positional winning strategy. Thus, in concatenating the positional winning strategies for the different regions we obtain a positional winning strategy for the whole game.

The remaining case, $\alpha = \beta+1$ even, is handled similarly. We can construct a positional winning strategy for player *I* as in the odd case for player *II* and vice versa. $\square$

Corollary 9 For any formula in the transfinite mu-arithmetic, model checking with parity games admits positional winning strategies.

PART II

In this part we will examine the result from a set theoretic angle. In examining Wadge degrees we will gain a better understanding of the link between the game operator $\Game$ and the difference hierarchy. The first two instances of the result are $\Sigma^\mu_1 = \Game\Sigma^0_1$, $\Sigma^\mu_2 = \Game\Sigma^0_2$, but then, $\Sigma^\mu_n = \Game\Sigma^0_n$ fails for $n > 2$, and must be replaced with $\Sigma^\mu_n = \Game\Sigma^\partial_{n-1}$. At first glance it seems there is no logic behind this. However, the effective Wadge hierarchy, a refinement of the effective difference hierarchy, gives the solution.

The reader may find most of the basic material in [7]. The effective Wadge hierarchy is studied in [9]. For the reader unfamiliar with the Wadge hierarchy we will recall basic definitions and facts.

7 The Wadge Ordering

A natural improvement of the Hausdorff–Kuratowski hierarchy was induced by Wadge's [15] work based on a reduction relation defined in terms of continuous functions. That is, a natural way to compare the topological complexity of sets A and B was to say $A \leq_W B$ – intuitively meaning A is topologically less complicated than B – if the problem of knowing whether x belongs to A *reduces* to knowing whether $f(x)$ belongs to B for some *simple* function, where simple meant continuous. The effective version deals with recursive functions instead:

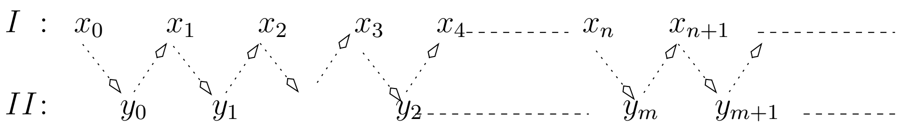
$$A \leq_W B \quad \text{iff} \quad \exists \text{ recursive } f : \Sigma_A^\omega \to \Sigma_B^\omega .\, f^{-1}B = A$$

The Wadge ordering ($\leq_W$) induces the strict ordering ($<_W$) and the Wadge equivalence ($\equiv_W$):

$$A <_W B \quad \text{iff} \quad A \leq_W B \wedge B \not\leq_W A$$
$$A \equiv_W B \quad \text{iff} \quad A \leq_W B \leq_W A$$

When restricted to Borel sets, or in the effective case, Kleene pointclasses, this ordering becomes a quasi-well-ordering, i.e. it is well-founded, and has antichains of length at most two. Moreover, if A and B are incomparable, then $A \equiv_W B^{\complement}$. The reason for this is that all these properties derive from Borel Determinacy [11]. Indeed, Wadge defined the relation $A \leq_W B$ in terms of the existence of a winning strategy in a suitable game: the Wadge game.

Definition 10 (The Wadge game) Let $A \subseteq \Sigma_A^\omega$, $B \subseteq \Sigma_B^\omega$, $\mathbf{W}(A, B)$ is an infinite two-player game where players (I and $I\!I$) take turn playing letters in Σ_A for I, and in Σ_B for $I\!I$. Unlike I, player $I\!I$ is allowed to skip any finite number of consecutive moves (that is, she plays no letter), though she must eventually move. Thus a run of the game has the form $(\Sigma_A \Sigma_A^* \Sigma_B)^\omega$, and its projections to Σ_A and Σ_B are ω-words.



At the end of a run (in ω moves), I has produced an ω-word $x \in \Sigma_A^\omega$ and $I\!I$ has produced $y \in \Sigma_B^\omega$. The winning conditions are:

$$I\!I \text{ wins } \mathbf{W}(A, B) \quad \text{iff} \quad (x \in A \Leftrightarrow y \in B)$$

Notation: In general, we consider sets A, B over different alphabets Σ_A, Σ_B; however, in order to reduce notational clutter, we will wherever possible assume a fixed alphabet and suppress the subscript on Σ. $\triangleleft$

Now a winning strategy for $I\!I$ is nothing more than a function which given input $x \in A$ (resp. $A^{\complement}$) produces a $y \in B$ (resp. $B^{\complement}$), using only finitely many letters of x to determine each successive letter of y, i.e. a continuous function f such that $A = f^{-1}B$. Hence $I\!I$ wins $\mathbf{W}(A, B)$ iff $A \leq_W B$. Conversely, a continuous reduction yields a winning strategy.

Let us define the equivalence relation $\sim$ by

$$A \sim B \quad \text{iff} \quad A \equiv_W B \;\text{ or }\; A \equiv_W B^{\complement} \;\text{ or }\; A \equiv_W 0B \cup 1B^{\complement}$$

Quotiented by $\sim$, and using determinacy, the Wadge ordering $\leq_W$ turns into a well-ordering (denoted by $\leq_{/\sim}$) whose minimal elements are all clopen sets. This induces the notion of the Wadge degree defined inductively:

$$d^{\circ}A = 0 \quad \text{iff} \quad A \text{ is clopen}$$

$$d^{\circ}A = \sup\{d^{\circ}B + 1 : \; B <_{/\sim} A\}$$

where $<_{/\sim}$ stands for the strict Wadge ordering $<_W$ quotiented by $\sim$.

8 Multiplication by ω_1^{ck} or by ω_1

Now, given a topological class, that is a class closed under pre-image by recursive functions (such as Σ^0_1, Σ^{∂}_n), a set A is complete for the class if it reduces all sets in it. As usual, a complete set is a set of maximal complexity, therefore of maximal Wadge degree. In other words, the Wadge degree of a complete set of a given class is a measure of the topological complexity of this class.

If we look at the sequence of Wadge degrees of complete sets for respectively Σ^0_1, $\Sigma^0_2 = \Sigma^{\partial}_1$, Σ^{∂}_2, $\Sigma^{\partial}_3, \ldots$, we find $1, \omega_1^{\text{ck}}, (\omega_1^{\text{ck}})^2, (\omega_1^{\text{ck}})^3, \ldots$. Surprisingly, the progression is precisely multiplication by ω_1^{ck}.

In the boldface hierarchy Duparc [4], extending Wadge's [15] definition on self-dual sets, defined an operation which increases the Wadge rank of a given set by the factor ω_1. Namely,

$$A \bullet \omega_1 = (\Sigma \cup \{a_+, a_-\})^* a_+ A \;\cup\; (\Sigma \cup \{a_+, a_-\})^* a_- A^{\complement}$$

for a_+, a_- two different letters not in Σ. Intuitively, a player (either *I* or *II*) in charge of $A \bullet \omega_1$ in a Wadge game is exactly like the same player being in charge of A with the extra possiblity to erase all his moves and decide to start all over again being in charge of $A^{\complement}$ instead of A, and erase everything again and switch from A to $A^{\complement}$, and so on. Playing a_+ or a_- takes care of both the initialization of the play, and the choice between A and $A^{\complement}$. A word containing infinitely many a_+ or a_- is not in $A \bullet \omega_1^{\text{ck}}$, so eventually the player must settle down to a genuine Wadge game.

This operation preserves the Wadge ordering

$$A \leq_W B \Rightarrow A \bullet \omega_1 \leq_W B \bullet \omega_1$$

and satisfies:

$$d^{\circ}(A \bullet \omega_1) = d^{\circ}(A) \;\cdot \omega_1.$$

Thus, this operation is the set theoretic counterpart of multiplication by ω_1. Now, turning back to the lightface hierarchy where the mu-calculus lives, can we find a similar operation transforming a set A with $d^{\circ}A = \alpha < {\omega_1^{\text{ck}}}^{\omega}$ into a set B whose Wadge rank relatively to the lightface hierarchy is $d^{\circ}B = \alpha \cdot \omega_1^{\text{ck}}$?

The answer is indeed yes, and luckily, it is the same operation. Let us denote the operation $\bullet\,\omega_1$ with $\bullet\,\omega_1^{\text{ck}}$ if restricted to the lightface hierarchy.

Lemma 11 Let A be a set of the lightface hierarchy with Wadge degree $d^\circ A = \alpha < \omega_1^{\mathrm{ck}^\omega}$ where the Wadge degree is computed relatively to the lightface hierarchy. Then $A \bullet \omega_1^{\mathrm{ck}}$ is in the lightface hierarchy, too, and $d^\circ A = \alpha \cdot \omega_1^{\mathrm{ck}}$.

To see that the same operation works for the lightface hierarchy, we need a result of Selivanov's [14]. A slight rephrasing of his result is:

Lemma 12 Let A be a set of the lightface hierarchy with Wadge degree $d^\circ A = \alpha < (\omega_1^{\mathrm{ck}})^\omega$, and let $\alpha = (\omega_1^{\mathrm{ck}})^{m_0}\alpha_0 + \ldots + (\omega_1^{\mathrm{ck}})^{m_k}\alpha_k$ be the unique canonical representation of α with base ω_1^{ck}. Then in the boldface hierarchy $d^\circ A = \omega_1^{m_0}\alpha_0 + \ldots + \omega_1^{m_k}\alpha_k$. □

Thus, the initial segment of the Wadge hierarchy of the lightface hierarchy up to $(\omega_1^{\mathrm{ck}})^\omega$ is embeddable in the initial segment of the boldface hierarchy up to $(\omega_1)^\omega$ in a very intuitive way.

It is an easy exercise to show that if A is hyperarithmetical with $d^\circ A = \alpha < (\omega_1^{\mathrm{ck}})^\omega$ in Σ^ω, then $A \bullet \omega_1^{\mathrm{ck}}$ is hyperarithmetical in $\{\Sigma \cup \{a_+, a_-\}\}^\omega$. Let $\alpha = (\omega_1^{\mathrm{ck}})^{m_0}\alpha_0 + \ldots + (\omega_1^{\mathrm{ck}})^{m_k}\alpha_k$. By Lemma 12, in the boldface hierarchy $d^\circ A = \omega_1^{m_0}\alpha_0 + \ldots + \omega_1^{m_k}\alpha_k$. By [4], $d^\circ A \bullet \omega_1 = d^\circ A \bullet \omega_1^{\mathrm{ck}} = (\omega_1^{m_0}\alpha_0 + \ldots + \omega_1^{m_k+1}\alpha_k) \cdot \omega_1$ where the rank is computed in the boldface hierarchy. Again by Lemma 12, $d^\circ A \bullet \omega_1^{\mathrm{ck}} = \alpha \cdot \omega_1^{\mathrm{ck}}$, establishing Lemma 11.

For the full transfinite extension of the mu-calculus we need the extension of Selivanov's result [14] for higher ranks to prove that the operation $\bullet\omega_1^{\mathrm{ck}}$ still works. However, this has not (yet) been shown.

9 Division by ω_1^{ck} or by ω_1

For inductive proofs on the degree of sets, an inverse operation, division of the Wadge degrees by ω_1^{ck} or ω_1 respectively, is needed. It must be a set theoretical counterpart of division by ω_1^{ck} or ω_1 respectively: given any set A the set A/ω_1 must satisfy:
(1) $A \leq_W B \Rightarrow A/\omega_1 \leq_W A/\omega_1$
(2) $d^\circ A = \alpha \cdot \omega_1^{\mathrm{ck}} \ \Rightarrow\ d^\circ A/\omega_1 = \alpha$

We will only deal with the boldface class and introduce an operation how to decrease the Wadge degree by ω_1. The reason for this is the following: Our construction uses in a crucial way a structural theorem of [4]. The effective version is not proved. Again, however, for degrees less than $(\omega_1^{\mathrm{ck}})^\omega$ it follows from Selivanov's result [14] that the effective version holds true. We are confident that the result can be extended for the whole effective hierarchy. However, this requires a nontrivial and probably somewhat cumbersome proof. The aim of this section is to show the main technique of how to decrease degrees.

In fact, there is a whole theory to develop how to decrease the Wadge degree by division. We will concentrate on what we need, the case of division by ω_1 for the boldface hierarchy, but we will indicate how variants of our technique may yield division by other ordinals.

We will obtain our result in two steps: First, we will define an operation turning a set A into $A/_{\mathrm{w}}\omega_1$ such that

$$d^\circ A = \alpha \cdot \omega_1 \ \Rightarrow\ d^\circ A/_{\mathrm{w}}\omega_1 = \alpha + 1$$

Then we set an easy operation on non self dual sets $A \longmapsto (A - \mathbf{1})$ that satisfies:

$$d^\circ A = \alpha + 1 \ \Rightarrow\ d^\circ (A - \mathbf{1}) = \alpha$$

Finally, combining both operations gives the result.

10 Question Trees

We will start with the general framework needed for both operations. We shall define an extended form of game, in which a player may ask the opponent whether they will play within a particular specified tree. The opponent must either agree to play within the tree, or declare that they will play outside it, specifying the route out.

Let us fix a finite alphabet Σ with $\{0, 1\} \in \Sigma$. We now define three constructions on the sets on which Wadge games are played:

Definition 13 Let $A, B \subseteq \Sigma$.

We denote $[A \vee B]$ for any set equivalent to $0A \cup 1B$. Furthermore, we denote $\varnothing \to A$ for any set equivalent to 0^*1A, and $\varnothing^{\complement} \to A$ for any set equivalent to $0^\omega \cup 0^*1A$.

Playing on the set $[A \vee B]$ corresponds to a game in which the player starts by choosing whether to play the main game in A or B. It is clearly Wadge equivalent to the stronger of A and B, if A and B are Wadge comparable. If A and B are not comparable, $B \equiv_W A^{\complement}$ holds, therefore $[A \vee B] \equiv_W \left[A \vee A^{\complement}\right]$ which is (up to Wadge equivalence) the least set above A and B. Moreover, $\left[A \vee A^{\complement}\right]$ is self-dual. (A set A is *self-dual* with respect to Wadge equivalence iff $A \leq_W A^{\complement}$ iff $A^{\complement} \leq_W A$, and so A is non-self-dual iff $A \not\leq_W A^{\complement}$.)

Playing on $\varnothing \to A$ corresponds to being allowed to defer (for a finite time) actually entering A, while the opponent plays normally; playing on $\varnothing^{\complement} \to A$ adds the possibility to defer for ever.

These two constructions give us the following useful property:

Lemma 14 A set A is non-self-dual iff $(A \equiv_W \varnothing^{\complement} \to A \ \vee\ A \equiv_W \varnothing \to A)$.

Proof. It follows from the definitions that A is non-self-dual iff player I has a winning strategy in the games $\mathbf{W}(A, A^{\complement})$ and $\mathbf{W}(A^{\complement}, A)$. Consider the game $\mathbf{W}(A^{\complement}, A)$. In the run of this game in which II keeps on skipping (which violates the rules, and so guarantees she loses), I's winning strategy defines an infinite word x. Since the strategy is winning, at any point in the play, I can still win even if II starts playing real moves instead of skipping.

Firstly, suppose that $x \in A^{\complement}$, and consider the game $\mathbf{W}(A^{\complement}, \varnothing \to A)$. I can win this game, by playing the following strategy: as long as II plays 0 or 1, or skips, play the next letter of x. As soon as II starts playing from Σ, play according to the winning strategy for $\mathbf{W}(A^{\complement}, A)$ from the current position in that game, and win. So I wins $\mathbf{W}(A^{\complement}, \varnothing \to A)$, meaning that $A^{\complement} \not\leq_W \varnothing \to A$, and so $A \equiv_W \varnothing \to A$.

On the other hand, if $x \notin A^{\complement}$, consider instead the game $\mathbf{W}(A^{\complement}, \varnothing^{\complement} \to A)$. I can win *this* game, by playing the same strategy as before. If II ever does switch into playing on Σ, then we win by the same argument as before; and if II plays 0 for ever, then we have played x ($\notin A^{\complement}$) against 0^ω ($\in \varnothing^{\complement} \to A$) and so won the play. Therefore we have $A \equiv_W \varnothing^{\complement} \to A$

The converse arguments also go through. □

Definition 15 We define the degree d° of certain sets by: $d^\circ\varnothing = 1$;
if A is a **non self-dual** set, then $d^\circ A = \sup\{d^\circ B :\ B <_W A \text{ and } B \text{ non self-dual}\}$;
if A is a **non-self-dual** set, then $d^\circ \left[A \vee A^{\complement}\right] = d^\circ A = d^\circ A^{\complement}$. ◁

Note that in contrast to the official definition of a Wadge rank we define the rank only on non self-dual sets and on self-dual sets of the form $\left[A \vee A^{\complement}\right]$. Note further that since we work with a finite alphabet, the space is compact, and any infinite limit of non-selfdual sets is non-selfdual. Thus, in this case the rank is defined for all Borel sets.

The next step is to define subtraction and division operators in terms of the games and question trees. However, for technical reasons, we shall first have to define some *weak* versions of the operations.

The idea behind the definitions of both $(A - \mathbf{1})$ and $A/_{\mathrm{w}}\omega_1$ is the same: a player in a Wadge game in charge of $A/_{\mathrm{w}}\omega_1$ or $(A - \mathbf{1})$ is like a player being in charge of $A \subseteq \Sigma^\omega$ but having his opponent asking him questions whether or not the infinite word x he is actually constructing step by step, will remain in a certain tree $T_i \subseteq \Sigma^*$. The player must either say he will not stay in T_i, and specify how he will move out of it; or say he will stay in T_i. The opponent is allowed to ask questions about as many trees as he wants as long as the player answers no. Once the player answers yes and agrees to stay in the tree, no more questioning is allowed.

Different conditions posed on these trees will give different strengths for reducing the Wadge degree. We will present two variants, one for the operation $(A - \mathbf{1})$, and one for the operation $A/_{\mathrm{w}}\omega_1$. Clearly, this is the right place to play with other variants to obtain other kinds of reducing the rank.

The precise definition is as follows. A *pruned* tree is one with no terminal nodes, and $[T] \subseteq \Sigma^\omega$ denotes the set of infinite branches of T. Given $x \in \Sigma^\omega$, we write x_{even} for the word $x(0)x(2)x(4)\dots$.

Definition 16 A *qtree* $\mathscr{T}$ on Σ is a non-empty pruned tree that satisfies the following properties for any $u \in \mathscr{T}$ and any integer $n < \mathrm{lh}(u)$:

if n is even: then $u(n) \in \Sigma$ (these are the nodes that correspond to the main run), and

if n is odd: then $u(n)$ is an auxiliary move with three different options. To understand the meaning of these options, let $u' = u(0)\dots u(n-1)$. In the tree $\mathscr{T}$, the position u' has various immediate successors, of which $u(n)$ is one. If no question is asked after u', then $u(n)$ is the only successor of u', and has the form $\langle - \rangle$, meaning 'no question'. Otherwise, the question is put: 'will you stay in a certain tree in the main run?', and an answer is given. The answer is either $\langle \mathtt{no}, v\rangle$, meaning 'no, and v is the way I will proceed outside the tree', or $\langle \mathtt{yes}\rangle$, meaning 'yes, I will stay in this tree (my *commitment tree*)'. In the case of a 'no' answer, an explicit witness v is included in the node $u(n)$; in the case of a 'yes' answer, the commitment tree is *implicitly* given as the complement of all the witnesses in the 'no' siblings of $u(n)$.

Option $\langle - \rangle$: this is the 'no question' case. Formalizing the description above, we require $\mathscr{T}$ to satisfy:

$$\text{if } (u \restriction n)\,\langle - \rangle \in \mathscr{T} \text{ then } (u \restriction n)\,\langle \mathtt{yes}\rangle \notin \mathscr{T} \text{ and for any } v \in \Sigma^* \;\; (u \restriction n)\,\langle \mathtt{no}, v\rangle \notin \mathscr{T}.$$

Option $\langle \mathtt{no} \rangle$: in this, the 'no' case, $u(n) = \langle \mathtt{no}, v\rangle$ for some $v \in \Sigma^*$ and $u_{\text{even}} \subseteq v$. Then v is the witness to the promise to stay out, and formalizing the description above, we require that any position w in $\mathscr{T}$ that extends u must satisfy $w_{\text{even}} \subseteq v$ or $v \subseteq w_{\text{even}}$. In addition, we require that all the 'no' siblings of $u(n)$ are independent of it, i.e. that $\mathscr{T}$ satisfies the following condition: if $(u \restriction n)\,\langle \mathtt{no}, v'\rangle$ belongs to $\mathscr{T}$ with $v' \neq v$, then both $v' \subseteq v$ and $v \subseteq v'$ must fail. Or, to say it differently, T must satisfy the condition:

$$\Big(u \in \mathscr{T} \ \wedge\ u\,\langle \mathtt{no}, v\rangle \in \mathscr{T} \wedge\ u\,\langle \mathtt{no}, v'\rangle \in \mathscr{T}\Big) \Rightarrow v = v' \ \vee\ v \perp v'.$$

Option $\langle \mathtt{yes}\rangle$: in this case $u(n) = \langle \mathtt{yes}\rangle$. This is a commitment to stay in the tree defined by the complement of all the 'no' siblings. Formally, this means that any w in $\mathscr{T}$ that extends $(u \restriction n)\,\langle \mathtt{yes}\rangle$ must satisfy

$$v \subseteq w_{\text{even}} \textbf{ fails for any } \boldsymbol{v} \text{ such that } (u \restriction n)\,\langle \mathtt{no}, v\rangle \in \mathscr{T}.$$

◁

The previous definition is set up in a way that we get

Lemma 17 If for two branches $x, y \in [\mathscr{T}]$ we have $x_{\text{even}} = y_{\text{even}}$, then $x = y$.

Proof. This follows immediately from the additional conditions posed on options $\langle \mathtt{no}\rangle$ and $\langle -\rangle$. □

If at position $u \in \mathscr{T}$ the player chooses $u\,\langle \mathtt{yes}\rangle$, he commits himself to stay in the tree defined by *avoiding* all the 'no'-witnesses he could have played at that point. Each 'no'-witness v defines an open (and indeed clopen) subtree of $\mathscr{T}$; hence avoiding all of them means staying in a closed subtree. This commitment tree $C_{u\langle \mathtt{yes}\rangle}$ is alternatively describable as

$$C_{u\langle \mathtt{yes}\rangle} = [T_{u\langle \mathtt{yes}\rangle}]$$

where

$$T_{u\langle \mathtt{yes}\rangle} = \{w \in \Sigma^* :\ v \subseteq w \textbf{ fails for any } \boldsymbol{v} \text{ such that } u\,\langle \mathtt{no}, v\rangle \in \mathscr{T}\}.$$

Equally, when a player gives a no answer, he *avoids* the closed tree $C_{u\langle \mathtt{yes}\rangle}$. In what follows, *avoid* refers to avoiding commitment, i.e. only taking no and $-$ choices.

Definition 18 We call $u \in \mathscr{T}$ an **avoiding** branch if for all n with $2n+1 \leq \text{lh}(u)$ we have $u(2n+1) \neq \langle \mathtt{yes}\rangle$.

For such an avoiding branch, we denote by $\text{Av}\,u$ the union of all closed sets avoided:

$$\text{Av}\,u = \bigcup_{2n+1 \leq \text{lh}(u)} C_{(u \restriction 2n)\langle \mathtt{yes}\rangle}$$

where by convention, $C_{(u\restriction 2n)\langle \mathtt{yes}\rangle} = \varnothing$ in case $(u \restriction 2n)\,\langle \mathtt{yes}\rangle \notin \mathscr{T}$.

The same definition applies to an infinite branch $y \in [\mathscr{T}]$ (with $\text{lh}(u) = \omega$). ◁

Clearly, $\text{Av}\,u$ is closed, being a finite union of closed sets, while $\text{Av}\,y$ is a Σ^0_2-set.

We will use this framework to define both $(A{-}\mathbf{1})$ and $A/_{\text{w}}\omega_1$, by imposing appropriate additional conditions on the tree $\mathscr{T}$.

Definition 19 Given a set A, we call any set C A-**trivial** if it satisfies $C \cap A = \varnothing \vee C \subseteq A$). If A is clear from context, we will simply say that C is *trivial.* ◁

Definition 20 Given a set $A \subseteq \Sigma^\omega$, and a qtree $\mathscr{T}$ on Σ, the pair $\langle A, \mathscr{T}\rangle$ is called -1-weak if for **all** branches $u \in \mathscr{T}$ the two following conditions hold:

(1) Along the way exactly one question is asked, i.e.

$$(\exists n \in \mathbb{N}\ (u(2n+1) = \langle \mathtt{yes} \rangle\ \vee\ u(2n+1) = \langle \mathtt{no}, v \rangle) \Longrightarrow\ \forall k \neq n\ u(2k+1) = \langle - \rangle)$$

with $v \in \Sigma^*$ and

$$\exists n \in \mathbb{N}\ (y(2n+1) = \langle \mathtt{yes} \rangle\ \vee\ y(2n+1) = \langle \mathtt{no}, v \rangle)$$

(2) On each infinite avoiding branch y, the set $\mathrm{Av}\, y$ of queried commitments (which is closed, because there is at most one queried commitment) is A-trivial. That means that as soon as we make a commitment, we are forced either to end up in A or to end up in $A^{\complement}$. ◁

Definition 21 Given a set $A \subseteq \Sigma^\omega$ and a qtree $\mathscr{T}$ on Σ, the pair $\langle A, \mathscr{T} \rangle$ is called ω_1-weak if it satisfies both:

(1) $\forall k\ \forall n > k\ \ x(2k+1) = \langle \mathtt{yes} \rangle\ \Rightarrow\ x(2n+1) = \langle - \rangle$.
That is, if the player takes option $\langle \mathtt{yes} \rangle$, thereby agreeing to remain in the given closed set until the end of the game, then no further questions are allowed. Hence, until the end of the game, only the main run counts, together with the fact that it *must* remain in the closed set $C_{(x \upharpoonright 2k+1)}$.

(2) **for all** infinite **avoiding** branches $y \in [\mathscr{T}]$, the intersection of the complement of the union of all the closed sets avoided with A is clopen, i.e. $A \cap (\mathrm{Av}\, y)^{\complement} \in \Delta^0_1$.
In the special case of $A \subseteq \mathrm{Av}\, y$ we say the branch y is $\emptyset\omega_1$-weak, and in the special case of $A^{\complement} \subseteq \mathrm{Av}\, y$ we say the branch y is $\emptyset^{\complement}\omega_1$-weak. If in a tree every avoiding branch is $\emptyset\omega_1$-weak, we call the tree $\emptyset\omega_1$-weak. If every avoiding branch is $\emptyset^{\complement}\omega_1$-weak, then we call the tree $\emptyset^{\complement}\omega_1$-weak. ◁

In both cases we restrict the possibilities of when questions are allowed to be asked, and further we demand a simple relationship between A and $\mathrm{Av}\, y$. The reason will become clear in what follows: for every A there is a canonical set B with $d^\circ A = d^\circ B$, and for every such B there is a natural question tree $\mathscr{T}_B$ such that $B^{\mathscr{T}_B}$ has its rank decreased in the intended way. The restrictions we put on the question trees make sure that the canonical tree $\mathscr{T}_B$ is 'legal' and has maximal impact on the reduction of the rank among all 'legal' trees.

Finally, we are able to define the operations we are after.

Definition 22 Let $A \subseteq \Sigma^\omega$, and $\mathscr{T}$ be a qtree on Σ,

(1) $A^{\mathscr{T}} = \{x \in [\mathscr{T}] :\ x_{\text{even}} \in A\}$
(2) $(A{-}\mathbf{1}) =$ a $\leq_W$-minimal element in $\{A^{\mathscr{T}} : \mathscr{T}$ a -1-weak qtree on $\Sigma\}$
(3) $A/_{\mathrm{w}}\omega_1 =$ a $\leq_W$-minimal element in $\{A^{\mathscr{T}} : \mathscr{T}$ a ω_1-weak qtree on $\Sigma\}$
(4) $A/\omega_1 = (A/_{\mathrm{w}}\omega_1{-}\mathbf{1})$ ◁

We need to show that these operations decrease the Wadge rank in the intended way.

11 Subtraction by 1

We will start with the less complex operation -1-weak.

The main idea is as follows: Given a non self-dual set A with $d^\circ A = \alpha + 1$, there is a non self-dual set B with either $A \equiv_W \varnothing \to \left[B \vee B^{\complement}\right]$ or $A \equiv_W \varnothing^{\complement} \to \left[B \vee B^{\complement}\right]$. Let us assume $A \equiv_W \varnothing \to \left[B \vee B^{\complement}\right]$. The question tree on $\varnothing \to \left[B \vee B^{\complement}\right]$ simply asks whether the player will always play 0's. The delay of the answer is exactly what gives us the increase in rank if we compare B with $\varnothing \to \left[B \vee B^{\complement}\right]$. Thus, the question tree takes away this extra power, and after finitely many steps the player finds himself being in charge of either B or $B^{\complement}$ or has committed himself to playing only 0's. The rank of the payoff set is α, and this is what we wanted. We will use continuous mappings to argue that -1-weak question trees do the same job for A.

Again, for notational convenience we fix an alphabet Σ. Let $A \subseteq \Sigma^\omega$, $B \subseteq \Sigma^\omega$, and $A \leq_W B$. We need to show the following propositions:

Proposition 23 Given $\mathscr{T}_B$ any -1-weak qtree on Σ, there exists a -1-weak qtree $\mathscr{T}_A$ on Σ such that $A^{\mathscr{T}_A} \leq_W B^{\mathscr{T}_B}$. Moreover, if for all branches $y \in \mathscr{T}_B$ we have $\operatorname{Av} y \cap B = \varnothing$, then for all branches $x \in \mathscr{T}_A$ we have $\operatorname{Av} x \cap A = \varnothing$. Equally, if for all branches $y \in \mathscr{T}_B$ we have $\operatorname{Av} y \subseteq B$, then for all branches $x \in \mathscr{T}_A$ we have $\operatorname{Av} x \subseteq A$.

Proposition 24 Up to Wadge equivalence, there exists only one $\leq_W$-minimal element in $\{A^{\mathscr{T}} : \mathscr{T}$ a -1-weak qtree on $\Sigma\}$.

Proposition 25 The operation $X \longmapsto (X{-}\mathbf{1})$ preserves the Wadge ordering:

$$A \leq_W B \Rightarrow (A{-}\mathbf{1}) \leq_W (B{-}\mathbf{1})$$

Proposition 26 $d^\circ A = \alpha + 1 \ \Rightarrow \ d^\circ\,(A{-}\mathbf{1}) = \alpha$

Proof of Proposition 23. Let σ be a winning strategy for *II* in the game $\mathbf{W}(A, B)$, and σ^* be the continuous function induced by σ. This function $\sigma^*\colon \Sigma^\omega \longmapsto \Sigma^\omega$ satisfies ${\sigma^*}^{-1} B = A$.

We will use σ^* to derive from $\mathscr{T}_B$ a suitable $\mathscr{T}_A$, a -1-weak qtree on Σ. The preimages under σ^* of closed sets questioned in $\mathscr{T}_B$ will become the closed sets questioned in $\mathscr{T}_A$.

To be more precise, let us define $\mathscr{T}_A$ recursively by:

▷ $\mathscr{T}_A \restriction 1 = \Sigma$.

▷ Assume $\mathscr{T}_A \restriction 2n + 1$ is defined. Let $u \in \mathscr{T}_A$ be of length $2n + 1$. Lemma 17 provides us with a unique initial path s in $\mathscr{T}_B$ with $s_{\text{even}} = \sigma(u_{\text{even}})$. If $s\,\langle - \rangle \in \mathscr{T}_B$, then set $\mathscr{T}_A[u] = \{u\,\langle - \rangle\}$. If there is some $k < 2n + 1$ with $\sigma(u \restriction k) = s$, then set as well $\mathscr{T}_A[u] = \{u\,\langle - \rangle\}$. Otherwise, set $\mathscr{T}_A[u] = \{u\,\langle \mathtt{yes} \rangle\} \cup \{u\,\langle \mathtt{no}, v\rangle \mid \sigma(v)\Sigma_A^* \cap \operatorname{Av} s = \varnothing$ and v minimal$\}$. Finally, let us define $\mathscr{T}_A \restriction 2n + 2 = \bigcup\{\mathscr{T}_A[u] \mid u \in \mathscr{T}_A \restriction 2n + 1\}$.

▷ Assume $\mathscr{T}_A \restriction 2n + 2$ is defined. If there is some odd $k < 2n + 2$ with $u(k) = \langle \mathtt{no}, v\rangle$ and $\operatorname{lh}(u) < \operatorname{lh}(v)$, then let $\mathscr{T}_A[u] = \{u^\frown v(2n + 2)\}$. Otherwise, let $\mathscr{T}_A[u] = u\Sigma$. Finally, set $\mathscr{T}_A \restriction 2n + 3 = \bigcup\{\mathscr{T}_A[u] \mid u \in \mathscr{T}_A \restriction 2n + 2\}$.

Observe that since σ^* is continuous, we know that the closed set of $\operatorname{Av} u = {\sigma^*}^{-1}(\operatorname{Av} s)$ is A-trivial: If $\operatorname{Av} s \subseteq B$, then $\operatorname{Av} u = {\sigma^*}^{-1}(\operatorname{Av} s) \subseteq {\sigma^*}^{-1}(B) = A$. Moreover, if $\operatorname{Av} s \cap B = \varnothing$, then $\operatorname{Av} u \cap A = {\sigma^*}^{-1}(\operatorname{Av} s \cap B) = \varnothing$. Thus, $\mathscr{T}_A$ is a -1-weak qtree on Σ.

It remains to show that player *II* has a winning strategy in the Wadge game $\mathbf{W}(A^{\mathscr{T}_A}, B^{\mathscr{T}_B})$. However, this is easy: on the even moves which correspond to the main run player *II* simply follows σ. On the odd moves she may be confronted with a question about a closed set. If so, she answers "yes" if player *I* answered the same way, and if player *I* answered with $u\,\langle \mathtt{no}, v\rangle$, player *II*'s next move will be $\sigma(u)\,\langle \mathtt{no}, \sigma(v)\rangle$. □

Proof of Proposition 24. We need to show that up to Wadge equivalence there exists only one $\leq_W$-minimal element in $\{A^{\mathscr{T}} : \mathscr{T} \text{ a } -1\text{-weak qtree on } \Sigma\}$.

We first prove the following claim, which uses Lemma 14's characterization of non-self-dual sets.

Claim 27 Let A be non-self-dual and $\mathscr{T}$ be a -1-weak qtree on Σ. If $A \equiv_W \varnothing^{\complement} \to A$ and for every avoiding branch y we have $\operatorname{Av} y \cap A = \varnothing$, then $A^{\mathscr{T}} \equiv_W A$. Consequently, questions outside A do not let us decrease the rank. If $A \equiv_W \varnothing \to A$ and for every avoiding branch y we have $\operatorname{Av} y \subseteq A$, then $A^{\mathscr{T}} \equiv_W A$. Thus, if $(A-\mathbf{1}) <_W A$, then $(A-\mathbf{1})$ is not created by those trees.

Proof. Let A be non self-dual with $A \equiv_W \varnothing^{\complement} \to A$, and let $\mathscr{T}$ verify $\operatorname{Av} y \cap A = \varnothing$ for every avoiding branch y. Proposition 23 yields some tree $\mathscr{T}'$ with $(\varnothing^{\complement} \to A)^{\mathscr{T}'} \leq_W A^{\mathscr{T}}$ and $\operatorname{Av} x \cap (\varnothing^{\complement} \to A) = \varnothing$ for every avoiding branch $x \in \mathscr{T}'$. In particular, the sequence $0^\omega \notin \operatorname{Av} x$. Since $\operatorname{Av} x$ is closed, there is some k such that $0^k\{0 \cup \Sigma\}^* \cap \operatorname{Av} x = \varnothing$. Now consider the Wadge game $\mathbf{W}(A, (\varnothing^{\complement} \to A)^{\mathscr{T}'})$. In this game the following is a winning strategy for player *II*: start with playing only 0's until a question about some closed set $C_{000\ldots0}$ is asked, answer with $\langle \mathtt{no}, 0^k\rangle$ for an appropriate k, and then just copy player *I*'s moves from the very start. Thus, we get $A \leq_W (\varnothing^{\complement} \to A)^{\mathscr{T}'} \leq_W A^{\mathscr{T}} \leq_W A$. An analogous argument yields the second part of the claim. □

We now prove a second result.

Claim 28 Let A be non self-dual, and both $A^{\mathscr{T}'}$ and $A^{\mathscr{T}''}$ be $\leq_W$-minimal in $\{A^{\mathscr{T}} : \mathscr{T} \text{ a } -1\text{-weak qtree on } \Sigma\}$, then

$$A^{\mathscr{T}'} \equiv_W A^{\mathscr{T}''}$$

Proof. Towards a contradiction, we assume $A^{\mathscr{T}'} \perp A^{\mathscr{T}''}$. As a first case we assume further that $A \equiv_W \varnothing \to A$. Because of incomparability it must be the case that the rank is decreased, i.e. $(A-\mathbf{1}) <_W A$, because trivially $A^{\mathscr{T}} \leq_W A$ is always true for any question tree $\mathscr{T}$; they cannot be incomparable.

By the previous claim, we know that for every avoiding branch y in both $\mathscr{T}'$ and $\mathscr{T}''$ we have $\operatorname{Av} y \subseteq A$. By finiteness of the alphabet Σ both trees $\mathscr{T}'$ and $\mathscr{T}''$ can only ask questions about finitely many closed sets $\operatorname{Av} y$: if one identifies nodes $u\,\langle \mathtt{no}, v\rangle$, $u\,\langle \mathtt{no}, v'\rangle$ inside the question trees, both trees become compact. Since only one question per branch is allowed, there is only room for finitely many questions. Thus, $C = \bigcup\{\operatorname{Av} y \mid y \in [\mathscr{T}']\} \cup \bigcup\{\operatorname{Av} y \mid y \in [\mathscr{T}'']\}$ is closed with $C \subseteq A$. Let $\mathscr{T}$ be the -1-weak qtree on

Σ which asks one question about C on every path at height 1. It is easy to see that $A^{\mathscr{T}} \leq_W A^{\mathscr{T}'}$: player *II* simply copies the moves of player *I* on even moves as long as possible. On odd moves there is nothing to do unless player *II* reaches the only odd position with a question. By that time player *I* must already have answered his question about a closed set $\mathrm{Av}\, y$. If the answer was $\langle \mathtt{no}, v\rangle$, then because of $\bigcup\{\mathrm{Av}\, y \mid y \in [\mathscr{T}']\} \subseteq C$ and the incompatibility requirement on answers how to leave a closed set there is some $v' \sqsubseteq v$ such that *II* can answer with $\langle \mathtt{no}, v'\rangle$. Afterwards she simply continues copying player *I*'s moves. If the answer was $\langle \mathtt{yes}\rangle$, player *II* answers with $\langle \mathtt{yes}\rangle$ as well. *II* may not be able any more to copy player *I*'s moves. However, since $C \subseteq A$, player *I* will produce a sequence in $A^{\mathscr{T}} \cap A$, while player *II* will produce a sequence in $A^{\mathscr{T}'} \cap A$.

Analogously, one proves $A^{\mathscr{T}} \leq_W A^{\mathscr{T}''}$, contradicting the minimality of both $A^{\mathscr{T}'}$ and $A^{\mathscr{T}''}$ which were supposed to be incomparable. By symmetric arguments, in taking the intersection of all involved closed sets, one shows the same for the case $A \equiv_W \emptyset^{\mathsf{C}} \to A$, which gives the complete proof of the claim. □

So far we have proved that any two $\leq_W$-minimal elements in the set

$$\{A^{\mathscr{T}} \mid \mathscr{T} \text{ a } -1\text{-weak qtree on } \Sigma\}$$

are Wadge equivalent in case A is non self-dual. It remains to show that the result holds in case A is self-dual and of the form $\left[B \vee B^{\mathsf{C}}\right]$ for some non self-dual B.

Let A be self-dual, and let B be non self-dual such that $A \equiv_W \left[B \vee B^{\mathsf{C}}\right]$. Let $\mathscr{T}_B$ be $\leq_W$-minimal in $\{B^{\mathscr{T}} : \mathscr{T} \text{ a } -1\text{-weak qtree on } \Sigma\}$, and $\mathscr{T}_{(B^{\mathsf{C}})}$ be $\leq_W$-minimal in $\{(B^{\mathsf{C}})^{\mathscr{T}} \mid \mathscr{T} \text{ a } -1\text{-weak qtree on } \Sigma\}$. Let $\mathscr{T}$ be the unique -1-weak qtree on Σ defined by

$$\left[B^{\mathscr{T}_B} \vee (B^{\mathsf{C}})^{\mathscr{T}_{(B^{\mathsf{C}})}}\right] = \left[B \vee B^{\mathsf{C}}\right]^{\mathscr{T}}$$

Proposition 23 yields some $\mathscr{T}_A$, a -1-weak qtree on Σ, satisfying $A^{\mathscr{T}_A} \leq_W \left[B \vee B^{\mathsf{C}}\right]^{\mathscr{T}}$.

Claim 29 Up to Wadge equivalence $\mathscr{T}_A$ is the unique $\leq_W$-minimal element in the set $\{A^{\mathscr{T}} : \mathscr{T} \text{ a } -1\text{-weak qtree on } \Sigma\}$.

Proof. Let $\mathscr{T}'$ be any -1-weak qtree on Σ. We show that $A^{\mathscr{T}_A} \leq_W A^{\mathscr{T}'}$. Again by Proposition 23, one obtains $\mathscr{T}''$ a -1-weak qtree on Σ that satisfies

$$\left[B \vee B^{\mathsf{C}}\right]^{\mathscr{T}''} \leq_W A^{\mathscr{T}'}.$$

Moreover, this construction yields $\mathscr{T}_0$ and $\mathscr{T}_1$ two -1-weak qtrees on Σ such that

$$\left[B \vee B^{\mathsf{C}}\right]^{\mathscr{T}''} = \left[B^{\mathscr{T}_0} \vee (B^{\mathsf{C}})^{\mathscr{T}_1}\right].$$

Now, by $\leq_W$-minimality,

$$B^{\mathscr{T}_B} \leq_W B^{\mathscr{T}_0} \text{ and } (B^{\mathsf{C}})^{\mathscr{T}_{(B^{\mathsf{C}})}} \leq_W (B^{\mathsf{C}})^{\mathscr{T}_1}.$$

Therefore, it is easy to see that

$$\left[B \vee B^{\mathsf{C}}\right]^{\mathscr{T}} \leq_W \left[B \vee B^{\mathsf{C}}\right]^{\mathscr{T}''}.$$

Hence,

$$A^{\mathscr{T}_A} \leq_W \left[B \vee B^{\complement}\right]^{\mathscr{T}} \leq_W \left[B \vee B^{\complement}\right]^{\mathscr{T}''} \leq_W A^{\mathscr{T}'}.$$

□

This claim completes the proof of the proposition. 24

Proof of Proposition 25. We need to show that the operation $X \longmapsto (X-\mathbf{1})$ preserves the Wadge ordering.

Let $A \leq_W B$. We let $\mathscr{T}_B$ be a -1-weak qtree on Σ that satisfies

$$B^{\mathscr{T}_B} \equiv_W (B-\mathbf{1}).$$

Proposition 23 yields some $\mathscr{T}_A$ a -1-weak qtree on Σ that satisfies

$$A^{\mathscr{T}_A} \leq_W B^{\mathscr{T}_B}.$$

By minimality, one obtains

$$(A-\mathbf{1}) \leq_W A^{\mathscr{T}_A} \leq_W B^{\mathscr{T}_B} \leq_W (B-\mathbf{1}).$$

□

Proof of Proposition 26. Finally, we need to show that the operation $X \longmapsto (X-\mathbf{1})$ decreases the Wadge rank by one in case the rank is a successor ordinal.

As in the previous proposition, the main work is to show the claim for non self-dual sets. Thus, let A be non self-dual, $d^\circ A = \alpha+1$ for some $\alpha > 0$. There is some non self-dual set B with $d^\circ B = \alpha$ such that either $A \equiv_W \varnothing \to \left[B \vee B^{\complement}\right]$, or $A \equiv_W \varnothing^{\complement} \to \left[B \vee B^{\complement}\right]$.

These cases are symmetric, so we only consider $A \equiv_W \varnothing \to \left[B \vee B^{\complement}\right]$.

Firstly, we have to show that $d^\circ (A-\mathbf{1}) \leq \alpha$. It suffices to find $\mathscr{T}_A$ a -1-weak qtree on Σ that satisfies

$$A^{\mathscr{T}_A} \leq_W \left[B \vee B^{\complement}\right].$$

Indeed, let $\mathscr{T}_A$ be the tree that asks on second move "will you stay inside 0^* or not?" That is, the closed set about which the player must answer is 0^ω. It is clear that a "no" is a best answer, but then, the player is like in charge of B or of $B^{\complement}$ depending on his answer.

Secondly, we have to show that $\alpha \leq d^\circ (A-\mathbf{1})$. Let $\mathscr{T}$ be any -1-weak qtree on Σ. We will show that it satisfies

$$B \leq_W A^{\mathscr{T}} \text{ or } B^{\complement} \leq_W A^{\mathscr{T}}.$$

Since B is non self-dual, we know that $B^{\complement} \equiv_W \varnothing^{\complement} \to B^{\complement}$ or $B \equiv_W \varnothing^{\complement} \to B$. By symmetry, we assume $B \equiv_W \varnothing^{\complement} \to B$. For this assumption it is enough to show that for any -1-weak qtree on Σ, $\mathscr{T}$ the following holds

$$B \leq_W (\varnothing \to \left[(\varnothing^{\complement} \to B) \vee B^{\complement}\right])^{\mathscr{T}}.$$

A winning strategy for player *II* in $\mathbf{W}(B, (\varnothing \to \left[(\varnothing^{\complement} \to B) \vee B^{\complement}\right])^{\mathscr{T}})$ consists in:

▷ playing $0000\ldots$ as long as no question is asked (hence remaining in the $\varnothing$ part).

▷ At some point, the question about a closed set C arises. Since $A \equiv_W \varnothing \to A$, we may assume $C \subseteq A$, because otherwise the rank would not be decreased by Claim 27. Thus, $C \subseteq \left(\varnothing \to \left[(\varnothing^{\complement} \to B) \vee B^{\complement}\right]\right)$, which by definition does not contain 0^ω. Hence, *II* answers "no" indicating how she will exit C by some sequence 0^k for some k long enough. Then player *II* will never have to answer any other question, and finds herself in charge of $\left(\varnothing \to \left[(\varnothing^{\complement} \to B) \vee B^{\complement}\right]\right)$ which easily reduces B.

This proves that $d^\circ (A - \mathbf{1}) = (d^\circ A) - 1$, in case $1 < d^\circ A$ is successor and A is non self-dual. We need to show that the same holds true for self-dual sets.

Note that, by combining proofs of previous items, we indeed showed that given any non self-dual set B,

$$\left(\left[B \vee B^{\complement}\right] - \mathbf{1}\right) \equiv_W \left[(B - \mathbf{1}) \vee \left(B^{\complement} - \mathbf{1}\right)\right].$$

Thus, we get

$$d^\circ \left(\left[B \vee B^{\complement}\right] - \mathbf{1}\right) = d^\circ \left[(B - \mathbf{1}) \vee \left(B^{\complement} - \mathbf{1}\right)\right]$$
$$= \max\left(d^\circ (B - \mathbf{1}),\ d^\circ \left(B^{\complement} - \mathbf{1}\right)\right) = d^\circ (B - \mathbf{1}) = (d^\circ B) - 1 = d^\circ \left[B \vee B^{\complement}\right] - 1.$$

□

12 Division by ω_1

We will now use the technique of question trees to reduce the Wadge rank of a set by the divisor ω_1. Of course, in general it is impossible to define a division for ordinal arithmetic, since, e.g., $5 \cdot \omega_1 = 3 \cdot \omega_1$, there is no inverse element. However, given an ordinal $\alpha = \beta \cdot \omega_1$, there is a smallest γ with $\alpha = \beta \cdot \omega_1 = \gamma \cdot \omega_1$. This smallest γ is what we are after. An easy inductive proof shows that γ itself is of the form $(\omega_1)^\delta$ for some ordinal δ. Thus, α is of the form $\alpha = (\omega_1)^{\delta+1}$. Sets of such a Wadge rank have a very nice representation, as it was proved in [4]. We will use this representation in a crucial way.

As previously, let us assume a single alphabet Σ to reduce notational complexity.

In some of what follows, it will be helpful to have a way of dealing with pay-off sets that include finite sequences as well as infinite sequences, which then allows us to neglect self-dual sets in our proofs. Following [4], a **conciliatory** set is a set $A \subseteq \Sigma^* \cup \Sigma^\omega$. The Wadge hierarchy can be extended to conciliatory sets, by allowing *I* to skip as well as *II*. In this conciliatory Wadge hierarchy, it turns out that there are no self-dual sets; and if conciliatory sets are converted to usual Σ^ω sets by padding the finite sequences with 'blank' symbol, the conciliatory Wadge hierarchy coincides with the usual Wadge hierarchy. In fact, it can be shown (see [4, Theorem 3]) that a usual set $A \subseteq \Sigma^\omega$ is non-self-dual iff it is Wadge equivalent to the padding of a conciliatory set on the same alphabet. We shall assume these facts where helpful. For precision, the definition of padding is:

Definition 30 Let $A \subseteq \Sigma^* \cup \Sigma^\omega$. Let b be some symbol not in Σ. The *padding* A^b of A is defined by

$$A^b = \{x \in (A \cup \{ b \})^\omega \mid x{\restriction}\Sigma \in A\}$$

where $x{\restriction}\Sigma$ is the projection of x to $\Sigma^* \cup \Sigma^\omega$. ◁

Let $A \subseteq \Sigma^\omega$ be a Borel set, and let $\twoheadleftarrow$ be a new symbol called 'back space'. Let $\Sigma_{\twoheadleftarrow} = \Sigma \cup \{\twoheadleftarrow\}$; we will define a new set $A^\sim \subseteq \Sigma_{\twoheadleftarrow}^\omega$, simply being the set of all infinite sequences with the property that the result of executing all the back space symbols in the sequence yields an element of A. More formally, we define

Definition 31 The execution operation $\hookleftarrow$ is inductively defined by:
▷ $\langle\rangle^{\hookleftarrow} = \langle\rangle$
▷ for x finite with $\mathrm{lh}(x) = k > 0$:

$$(x^\frown\langle a\rangle)^{\hookleftarrow} = x^{\hookleftarrow\frown}\langle a\rangle \quad \text{if } a \neq \twoheadleftarrow$$

$$(x^\frown\langle\twoheadleftarrow\rangle)^{\hookleftarrow} = x^{\hookleftarrow} \restriction k-1$$

▷ for x infinite: $(x)^{\hookleftarrow} = \lim_{n<\omega}(x \restriction n)^{\hookleftarrow}$
For a set $A \subseteq \Sigma^\omega$ we define $A^\sim = \{x \in \Sigma_{\twoheadleftarrow}^\omega \mid x^{\hookleftarrow} \in A\}$.

◁

Essentially, the operation $A \mapsto A^\sim$ is the Borel counterpart of exponentiation by ω_1. From [4] we need Lemma 37:

Lemma 32 Let A be a Borel set, $d^\circ A = \lambda + n$. Then $d^\circ A^\sim = (\omega_1)^{\lambda+n}$.

The representation of a set of rank $(\omega_1)^{\delta+1}$ as a set with lower degree enriched with an eraser will be a powerful tool for the sequel.

Definition 33 Let $A \subseteq \Sigma^\omega$, $B \subseteq \Sigma^\omega$. $A \approx B$ stands for $A \equiv_W B$ or $A \equiv_W B^\complement$. Thus, $A \approx B$ iff $d^\circ A = d^\circ B$. ◁

Similar to the case of subtraction by 1, the main idea is as follows: There is a set S such that $A \equiv_W S \bullet \omega_1$. The question tree on $S \bullet \omega_1$ simply asks whether the player will never again play a_+ or a_-, the two symbols increasing the complexity of S by the factor ω_1. If the answer stays no, we'll know that the player has to play infinitely many a_+, a_-, thus missing the payoff set. If the player makes a commitment, he'll be in charge of either S or $S^\complement$. Overall, the rank is decreased in the desired way. Via continuous mappings we can argue that ω_1-weak question trees do the same job for A. However, depending on the structure of S we will need to look into different cases.

Let $A \subseteq \Sigma^\omega$, $B \subseteq \Sigma^\omega$, and $A \leq_W B$. We need to show the following propositions:

Proposition 34 Given $\mathscr{T}_B$ any ω_1-weak qtree on Σ, there exists a ω_1-weak qtree on Σ such that $A^{\mathscr{T}_A} \leq_W B^{\mathscr{T}_B}$.

Proposition 35 The operation $X \longmapsto X/_{\mathrm{w}}\omega_1$ preserves the Wadge ordering up to $\approx$:

$$A \leq_W B \Rightarrow A/_{\mathrm{w}}\omega_1 \leq_W B/_{\mathrm{w}}\omega_1 \text{ or } A/_{\mathrm{w}}\omega_1 \approx B/_{\mathrm{w}}\omega_1.$$

Proposition 36 Let α be some power of ω_1.

$$d^\circ A = \alpha \cdot \omega_1 \;\Rightarrow\; d^\circ A/_{\mathrm{w}}\omega_1 = \alpha + 1.$$

In contrast to the operation $X \mapsto (X-\mathbf{1})$ it is not the case that for any two trees $\mathscr{T}'$, $\mathscr{T}''$ reducing to the minimal degree we have $A^{\mathscr{T}'} \equiv_W A^{\mathscr{T}''}$; they can be incomparable. For instance, let B be any complete difference of two Σ^0_2 sets, and A be $\left[B \vee B^{\complement}\right]$, then $\{A^{\mathscr{T}} \mid \mathscr{T}$ a ω_1-weak qtree on $\Sigma\}$ contains two minimal elements C and C', where C is Σ^0_2-complete, and C' is Π^0_2-complete.

Proof of Proposition 34. Let $A \leq_W B$ and let $\mathscr{T}_B$ be some ω_1-weak qtree on Σ. We need to construct some $\mathscr{T}_A$, a ω_1-weak qtree on Σ, with $A^{\mathscr{T}_A} \leq_W B^{\mathscr{T}_B}$.

This is the same idea as in the proof of Proposition 23, where the question tree $\mathscr{T}_B$ for B was transformed into a question tree $\mathscr{T}_A$ for A with the aid of a continuous function σ^* given by the fact that $A \leq_W B$. Moreover, the winning strategy for player *II* in the game $\mathbf{W}(A, B)$ was enriched to a winning strategy for *II* in the game $\mathbf{W}(A^{\mathscr{T}_A}, B^{\mathscr{T}_B})$. It only remains to show that for every avoiding branch $x \in [\mathscr{T}_A]$ the set $A \cap \operatorname{Av} x$ is clopen.

Let $x \in \mathscr{T}_A$ be an avoiding branch, and let $y \in \mathscr{T}_B$ be an avoiding branch with $y_{\text{even}} = \sigma^* x_{\text{even}}$.

We know that

$$B \cap \operatorname{Av} y^{\complement} \in \Delta^0_1.$$

Thus,

$$\operatorname{Av} y = \bigcup_{n \in \mathbb{N}} C_{(y \restriction 2n)\langle \mathsf{yes} \rangle}$$

$$\operatorname{Av} x = \bigcup_{n \in \mathbb{N}} {\sigma^*}^{-1} C_{(y \restriction 2n)\langle \mathsf{yes} \rangle} = {\sigma^*}^{-1} \bigcup_{n \in \mathbb{N}} C_{(y \restriction 2n)\langle \mathsf{yes} \rangle} = {\sigma^*}^{-1} \operatorname{Av} y$$

Since $B \cap \operatorname{Av} y^{\complement}$ is clopen, we obtain ${\sigma^*}^{-1}(B \cap \operatorname{Av} y^{\complement})$ clopen, hence $A \cap \operatorname{Av} x^{\complement}$ clopen. □

Proof of Proposition 35. We need to show that the operation $X \longmapsto X/_{\mathrm{w}}\omega_1$ preserves the Wadge ordering up to $\approx$:

$$A \leq_W B \Rightarrow A/_{\mathrm{w}}\omega_1 \leq_W B/_{\mathrm{w}}\omega_1 \text{ or } A/_{\mathrm{w}}\omega_1 \approx B/_{\mathrm{w}}\omega_1$$

Moreover, in case $A/_{\mathrm{w}}\omega_1 \not\leq_W B/_{\mathrm{w}}\omega_1$, this is only because there is no minimal element up to $\equiv_W$ in $\{A^{\mathscr{T}} \mid \mathscr{T}$ a ω_1-weak qtree on $\Sigma\}$, but rather two non self-dual ones incomparable for $\leq_W$, namely $A/_{\mathrm{w}}\omega_1$ and some C such that $C \equiv_W B/_{\mathrm{w}}\omega_1$. Thus we can make a choice for $A/_{\mathrm{w}}\omega_1$ such that indeed $A \leq_W B \Rightarrow A/_{\mathrm{w}}\omega_1 \leq_W B/_{\mathrm{w}}\omega_1$.

The result follows easily from Proposition 34, because if $A \leq_W B$, then given $\mathscr{T}_B$ such that $B^{\mathscr{T}_B} = B/_{\mathrm{w}}\omega_1$, there exists $\mathscr{T}_A$ such that $A^{\mathscr{T}_A} \leq_W B^{\mathscr{T}_B} = B/_{\mathrm{w}}\omega_1$. So either $A/_{\mathrm{w}}\omega_1 \leq_W A^{\mathscr{T}_A}$, or $A/_{\mathrm{w}}\omega_1 \perp A^{\mathscr{T}_A}$. □

Proof of Proposition 36. Given A with $d^\circ A = \alpha \cdot \omega_1$ where $\alpha = \omega_1^\delta$, we need to show that $d^\circ A/_{\mathrm{w}}\omega_1 = \alpha + 1$. This will occupy us until the end of the article.

First, we need to do some preparatory work in form of several claims. Let B be any conciliatory set. We recall that $(B^\sim) \bullet \omega_1 = (\Sigma_{\twoheadleftarrow} \cup \{a_+, a_-\})^* a_+ B^\sim \;\cup\; (\Sigma_{\twoheadleftarrow} \cup$

$\{a_+, a_-\})^* a_-(B^\sim)^{\complement}$, i.e. that the last a played decides whether the player is in charge of the set $B^\sim$ or $(B^\sim)^{\complement}$. If infinitely many such a's are played, then the word is rejected.

Recall that an avoiding branch y is called $\varnothing\omega_1$-weak, if $B \subseteq \operatorname{Av} y$, and if a tree consists of only avoiding $\varnothing\omega_1$-weak branches, then the tree is called $\varnothing\omega_1$-weak. If $B^{\complement} \subseteq \operatorname{Av} y$, then the branch or tree respectively is called $\varnothing^{\complement}\omega_1$-weak.

Similar to the operation of subtraction by 1, we have

Claim 37 Given any ω_1-weak qtree $\mathscr{T}$ on $\Sigma_{B^\sim}$,

$$\text{If } \mathscr{T} \text{ is } \varnothing\omega_1\text{-weak, and } B \equiv_W \varnothing^{\complement} \to B, \text{ then } (B^\sim)^{\mathscr{T}} \equiv_W B^\sim.$$
$$\text{If } \mathscr{T} \text{ is } \varnothing^{\complement}\omega_1\text{-weak, and } B \equiv_W \varnothing \to B, \text{ then } (B^\sim)^{\mathscr{T}} \equiv_W B^\sim.$$

Proof. Both cases are symmetric, we only prove the first one. It is enough to show that $B^\sim \leq_W (B^\sim)^{\mathscr{T}}$. Since $(\varnothing^{\complement} \to B)^\sim \equiv_W B^\sim$, Proposition 34 yields some $\varnothing\omega_1$-weak qtree $\mathscr{T}'$ with $((\varnothing^{\complement} \to B)^\sim)^{\mathscr{T}'} \leq_W (B^\sim)^{\mathscr{T}}$. Thus, it is enough to show $B^\sim \leq_W ((\varnothing^{\complement} \to B)^\sim)^{\mathscr{T}'}$.

We will show that in $\mathbf{W}(B^\sim, ((\varnothing^{\complement} \to B)^\sim)^{\mathscr{T}'})$ player *II* has a strategy to reach a position such that no auxiliary questions within the tree are asked any more, but where the complexity of the remaining game is still of complexity $d^\circ B^\sim$ for *II*.

The strategy for player *II* is as follows, as long as no position is reached yet where no questions are asked any more:

For moves at position with even length (i.e. the moves dealing with the underlying Wadge game), *II* plays accordingly to her promises.

On the auxiliary moves, if the actual closed set C she is asked about satisfies there is a sequence v that exits this closed set, then *II* answers that she will, by playing this v. After fulfilling this promise she plays sufficiently many back space symbols $\twoheadleftarrow$ to initialize the whole play. If in the meantime, before reaching the end of the whole sequence of v and $\twoheadleftarrow$'s she is asked a question about another closed set, she answers as if she had already reached this position. In other words, although her formal promise is just to play v, she promises to play $v^\frown \twoheadleftarrow \ldots \twoheadleftarrow$.

Otherwise, *II* agrees to stay in C which means she cannot exit C anyway. In particular, after playing the finite sequence of v's and $\twoheadleftarrow$'s she may have promised previously she is free of any promise and can play what she wants. Thus, she has reached a position where no questions will be asked any more, and where the game is initialized. Thus, it is legal and winning from that position on to simply play 1 followed by copying all player *I*'s moves from the beginning.

It remains to show that player *II* cannot answer auxiliary questions with no infinitely many times. Assume she can, and let y be a witness. By following the outlined strategy in initializing the game after making promises we know that $y^{\hookleftarrow} = \varnothing \in (\varnothing^{\complement} \to B)^\sim$. However, $\mathscr{T}'$ is $\varnothing\omega_1$-weak, thus $(\varnothing^{\complement} \to B)^\sim \subseteq \operatorname{Av} y$, thus $y \notin (\varnothing^{\complement} \to B)^\sim$ which is impossible. $\square$

Claim 38 Let $\mathscr{T}$ be a ω_1-weak qtree on $\Sigma_{B^\sim}$.

(1) If $\mathscr{T}$ is $\varnothing\omega_1$-weak, and $B \equiv_W \varnothing \to B$, then $(B^\sim)^{\complement} \leq_W ((B^\sim) \bullet \omega_1)^{\mathscr{T}}$.

(2) If $\mathscr{T}$ is $\varnothing\omega_1$-weak, and $B \equiv_W \varnothing^{\complement} \to B$, then $B^\sim \leq_W ((B^\sim) \bullet \omega_1)^{\mathscr{T}}$.

(3) If $\mathscr{T}$ is $\varnothing^{\complement}\omega_1$-weak, and $B \equiv_W \varnothing^{\complement} \to B$, then $(B^{\sim})^{\complement} \leq_W \ ((B^{\sim}) \bullet \omega_1)^{\complement\mathscr{T}}$.

(4) If $\mathscr{T}$ is $\varnothing^{\complement}\omega_1$-weak, and $B \equiv_W \varnothing \to B$, then $B^{\sim} \leq_W \ ((B^{\sim}) \bullet \omega_1)^{\complement\mathscr{T}}$.

Proof. All four cases are symmetric, we only prove the first one. From $B \equiv_W \varnothing \to B$ we can easily derive

$$(\varnothing^{\complement} \to B^{\complement})^{\sim} \equiv_W (B^{\complement})^{\sim} \equiv_W (B^{\sim})^{\complement}.$$

Since $(B^{\sim})^{\complement} <_W (B^{\sim}) \bullet \omega_1$, by the above and Proposition 34 there exists some ω_1-weak qtree $\mathscr{T}'$ on $\Sigma_{B^{\sim}}$ such that

$$\left((\varnothing^{\complement} \to B^{\complement})^{\sim}\right)^{\mathscr{T}'} \leq_W ((B^{\sim}) \bullet \omega_1)^{\mathscr{T}}.$$

Moreover, in applying the construction used in the proof of Proposition 34, we can make sure that $\mathscr{T}'$ is $\varnothing\omega_1$-weak. By Claim 37, $\mathscr{T}'$ applied to $(\varnothing^{\complement} \to B^{\complement})^{\sim}$ does not decrease the Wadge rank. Thus,

$$(B^{\sim})^{\complement} \leq_W (B^{\complement})^{\sim} \leq_W (\varnothing^{\complement} \to B^{\complement})^{\sim} \leq_W \left((\varnothing^{\complement} \to B^{\complement})^{\sim}\right)^{\mathscr{T}'} \leq_W ((B^{\sim}) \bullet \omega_1)^{\mathscr{T}}.$$

This finishes the proof of the claim. □

Let us state

Claim 39 For any concilatory set B

$$(B^{\sim}) \bullet \omega_1 \equiv_W (\varnothing \to \left[B \vee B^{\complement}\right])^{\sim} \equiv_W (\varnothing \to (\varnothing \to \left[B \vee B^{\complement}\right]))^{\sim} \quad (1)$$

$$((B^{\sim}) \bullet \omega_1)^{\complement} \equiv_W (\varnothing^{\complement} \to \left[B \vee B^{\complement}\right])^{\sim} \equiv_W (\varnothing^{\complement} \to (\varnothing^{\complement} \to \left[B \vee B^{\complement}\right]))^{\sim} \quad (2)$$

Proof. It is easy to construct winning strategies for player *II* in the corresponding Wadge games to verify these equalities. We will omit the proof. □

After these preparations we are ready to show the proposition. First, we will only consider non self-dual sets. Thus, let A be non self-dual, $d^{\circ}A = \alpha \cdot \omega_1$ with $\alpha = \omega_1^{\delta}$. Therefore, for any non self dual set S of degree precisely ω_1^{δ} we know

$$A \equiv_W S \bullet \omega_1 \text{ or } A \equiv_W (S \bullet \omega_1)^{\complement}.$$

These two cases are symmetric, so we assume that $A \equiv_W S \bullet \omega_1$.

We discuss $A/_{\mathrm{w}}\omega_1$ depending on the cofinality of δ. First, suppose the cofinality of δ is 1.

Thus, $A \equiv_W (B^{\sim}) \bullet \omega_1 = (\Sigma_{\twoheadleftarrow} \cup \{a_+, a_-\})^* a_+ B^{\sim} \ \cup \ (\Sigma_{\twoheadleftarrow} \cup \{a_+, a_-\})^* a_- (B^{\sim})^{\complement}$. If infinitely many such a's are played, then the word is rejected.

By non self-duality, $B \equiv_W \varnothing \to B$ or $B \equiv_W \varnothing^{\complement} \to B$. The two cases are symmetric, so w.l.o.g. we assume $B \equiv_W \varnothing \to B$.

Moreover, since δ is successor, then $B^{\sim} \equiv_W C \bullet \omega_1$ or $B^{\sim} \equiv_W (C \bullet \omega_1)^{\complement}$ for any non self dual set C of degree $\omega^{\delta-1}$. If $B \equiv_W \varnothing \to B$, then it is easy to come up with a winning strategy for player *II* in the game $\mathbf{W}(C \bullet \omega_1, (\varnothing \to B)^{\sim})$: Whenever player *I* plays one of the letters a_+, a_-, then player *II* erases all her moves until she obtains a finite sequence of 0's, and she adds another 0 to that sequence, followed by 1. Otherwise, player *I* is in charge of C or $C^{\complement}$, and player *II* follows a winning strategy for these sets reducing to B. Thus, $C \bullet \omega_1 \leq_W B^{\sim}$, which means $B^{\sim} \equiv_W C \bullet \omega_1$.

Similarly easily one can show that if $B \equiv_W \varnothing^{\complement} \to B$, we have $B^{\sim} \equiv_W (C \bullet \omega_1)^{\complement}$ instead.

So we have

$$A \equiv_W (B^{\sim}) \bullet \omega_1 \equiv_W ((C \bullet \omega_1) \bullet \omega_1).$$

We will show that we get $d^\circ A/_{\mathrm{w}}\omega_1 = \alpha = \omega_1^{\delta}$. We will start with $d^\circ A/_{\mathrm{w}}\omega_1 \leq \alpha$.

Claim 40 There exists a ω_1-weak qtree $\mathscr{T}$ on $\Sigma_{((C\bullet\omega_1)\bullet\omega_1)}$ such that

$$((C \bullet \omega_1) \bullet \omega_1)^{\mathscr{T}} \leq_W (C \bullet \omega_1)^{\complement}.$$

Proof. To be precise, we characterize $\Sigma_{((C\bullet\omega_1)\bullet\omega_1)}$:

$$\Sigma_{((C\bullet\omega_1)\bullet\omega_1)} = \Sigma_C \cup \{a_+, a_-, b_+, b_-\}$$
$$((C \bullet \omega_1) \bullet \omega_1) = (C \bullet \omega_1) \cup \Sigma^*_{((C\bullet\omega_1)\bullet\omega_1)} a_+ (C \bullet \omega_1) \cup \Sigma^*_{((C\bullet\omega_1)\bullet\omega_1)} a_- (C \bullet \omega_1)^{\complement}$$

where $(C \bullet \omega_1)$ is the subset of $\Sigma^{\omega}_{(C\bullet\omega_1)} = (\Sigma_C \cup \{b_+, b_-\})^{\omega}$ defined by

$$(C \bullet \omega_1) = C \cup \Sigma^*_{(C\bullet\omega_1)} b_+ C \cup \Sigma^*_{(C\bullet\omega_1)} b_- C^{\complement}.$$

We define $\mathscr{T}$ as the $\varnothing\omega_1$-weak qtree on $\Sigma_C \cup \{a_+, a_-, b_+., b_-\}$ that repeatedly asks the following questions:

(1) If the play is in a position that contains no a_+ nor a_-, or if the last letter in $\{a_+, a_-\}$ that was played is a_+, then

Will you never again play any letter in $\{a_+, a_-, b_+., b_-\}$ *?*

If the player answers yes to this question, then one easily sees that he agrees to be in charge of either C or $C^{\complement}$ depending on whether any b_+, b_- was played, and if so, depending on the last letter in $\{b_+., b_-\}$ that was played.

(2) If the play is in a position such that the last letter in $\{a_+, a_-\}$ that was played is a_-, then

Will you never again play any letter in $\{a_+, a_-\}$ *?*

If the player answers yes to this question, then one easily sees that he agrees to be in charge of $(C \bullet \omega_1)^{\complement}$.

Note that for any avoiding branch y we get $(C \bullet \omega_1) \bullet \omega_1 \subseteq \mathrm{Av}\, y$: only a commitment to stop playing $\{a_+, a_-, b_+., b_-\}$ can give a path in $(C \bullet \omega_1) \bullet \omega_1$. Thus, $\mathscr{T}$ is a $\varnothing\omega_1$-weak qtree on $\Sigma_C \cup \{a_+, a_-, b_+., b_-\}$, being a special case of a ω_1-weak qtree on $\Sigma_C \cup \{a_+, a_-, b_+., b_-\}$.

Finally, putting all this together, one gets:

$$((C \bullet \omega_1) \bullet \omega_1)^{\mathscr{T}} \leq_W \varnothing \to \left[\left[C \vee C^{\complement}\right] \vee (C \bullet \omega_1)^{\complement}\right] \leq_W \varnothing \to (C \bullet \omega_1)^{\complement} \leq_W (C \bullet \omega_1)^{\complement}.$$

Since $d^\circ C \bullet \omega_1 = d^\circ (C \bullet \omega_1)^{\complement}$, this gives $d^\circ A/_{\mathrm{w}}\omega_1 \leq \alpha$, concluding the claim. □

Continuing in the case $\mathrm{cf}(\delta) = 1$, it remains to show that $\alpha \leq d^\circ A/_{\mathrm{w}}\omega_1$. Let $\mathscr{T}$ be $\varnothing\omega_1$-weak. Recall that Claim 38 gives us

If $\mathscr{T}$ is $\varnothing\omega_1$-weak, and $B \equiv_W \varnothing \to B$, then $(B^\sim)^\complement \leq_W ((B^\sim) \bullet \omega_1)^{\mathscr{T}}$.

Thus, $(C \bullet \omega_1)^\complement \leq_W ((C \bullet \omega_1) \bullet \omega_1)^{\mathscr{T}}$, which shows $\alpha \leq d^\circ A/_{\mathrm{w}}\omega_1$.

Now let $\mathscr{T}$ be $\varnothing^\complement\omega_1$-weak. Recall that Claims 37 and 39.1 give us:

(1) If $\mathscr{T}$ is $\varnothing^\complement\omega_1$-weak, and $B \equiv_W \varnothing \to B$, then $(B^\sim)^{\mathscr{T}} \equiv_W B^\sim$.
(2) $(B^\sim) \bullet \omega_1 \equiv_W (\varnothing \to \left[B \vee B^\complement\right])^\sim \equiv_W (\varnothing \to (\varnothing \to \left[B \vee B^\complement\right]))^\sim$

Thus, we get $(C \bullet \omega_1)^\complement \equiv_W (B^\sim)^\complement <_W B^\sim \bullet \omega_1 \equiv_W (\varnothing \to (\varnothing \to \left[B \vee B^\complement\right]))^\sim \equiv_W [(\varnothing \to (\varnothing \to \left[B \vee B^\complement\right]))^\sim]^{\mathscr{T}} \equiv_W ((C \bullet \omega_1) \bullet \omega_1)^{\mathscr{T}}$. Again, this shows $\alpha \leq d^\circ A/_{\mathrm{w}}\omega_1$.

Finally, let us turn to an arbitrary ω_1-weak tree. By definition, for any avoiding path y we know that $A \cap (\mathrm{Av}\, y)^\complement$ is clopen. In other words, after finitely many steps on an avoiding path the cone of $\mathscr{T}$ looks like $\varnothing\omega_1$-weak or like $\varnothing^\complement\omega_1$-weak. Thus, an arbitrary ω_1-weak tree cannot decrease the Wadge rank more than a $\varnothing\omega_1$-weak tree or a $\varnothing^\complement\omega_1$-weak tree. Therefore, for any ω_1-weak tree $\mathscr{T}$ we have $(C \bullet \omega_1)^\complement \leq_W ((C \bullet \omega_1) \bullet \omega_1)^{\mathscr{T}}$, hence $\alpha \leq d^\circ A/_{\mathrm{w}}\omega_1$, in the case $\mathrm{cf}(\delta) = 1$.

Now suppose the cofinality of δ is ω_1. This case means (see [5]) that $A \equiv_W (B^\sim \bullet \omega_1)$ where

$$\varnothing \to B^\sim \equiv_W \varnothing^\complement \to B^\sim \equiv_W B^\sim.$$

Thus, by the previous claims we get for any $\varnothing\omega_1$-weak qtree on $\Sigma_{B^\sim}$ that $(B^\sim)^{\mathscr{T}} \equiv_W B^\sim$. Moreover, for any $\varnothing^\complement\omega_1$-weak qtree on $\Sigma_{B^\sim}$ we get $(B^\sim)^{\mathscr{T}} \equiv_W B^\sim$. Therefore, since any ω_1-weak qtree on $\Sigma_{B^\sim}$ will look like some $\varnothing\omega_1$-weak qtree or some $\varnothing^\complement\omega_1$-weak qtree in any cone whose stem is long enough, we get $(B^\sim)^{\mathscr{T}} \equiv_W B^\sim$.

In other words, there is no ω_1-weak qtree on $\Sigma_{B^\sim}$ that can lower the degree of $B^\sim$. By symmetry, the same holds true for $(B^\sim)^\complement$. We need to keep that fact in mind while discussing this case.

Claim 41 There exists a ω_1-weak qtree $\mathscr{T}$ on $\Sigma_{(B^\sim \bullet \omega_1)} = \Sigma_{B^\sim} \cup \{a_+, a_-\}$ such that

$$(B^\sim \bullet \omega_1)^{\mathscr{T}} \leq_W \varnothing \to \left[B^\sim \vee (B^\sim)^\complement\right].$$

Thus, $d^\circ A/_{\mathrm{w}}\omega_1 \leq \alpha + 1$.

Proof. This is the $\varnothing\omega_1$-weak qtree $\mathscr{T}$ on $\Sigma_{(B^\sim \bullet \omega_1)}$ that repeatedly asks

Will you never again play any letter in $\{a_+, a_-\}$?

If the player keeps on rejecting the offer, then he will necessarily produce a rejected sequence. If he agrees to never play a_+ nor a_-, then he agrees to be in charge of $B^\sim$ or of $(B^\sim)^\complement$, which proves the claim. □

Claim 42 For any ω_1-weak qtree $\mathscr{T}$ on $\Sigma_{(B^\sim \bullet \omega_1)}$ we have:

$$\emptyset \to \left[B^{\sim} \vee (B^{\sim})^{\complement}\right] \leq_W (B^{\sim} \bullet \omega_1)^{\mathscr{T}}.$$

Thus, $\alpha + 1 \leq d^{\circ}A/_{\mathrm{w}}\omega_1$.

Proof. Firstly, if $\mathscr{T}$ is a $\emptyset^{\complement}\omega_1$-weak qtree, the result is obvious since

$$\emptyset \to \left[B^{\sim} \vee (B^{\sim})^{\complement}\right] \leq_W (B^{\sim} \bullet \omega_1) \equiv_W (B^{\sim} \bullet \omega_1)^{\mathscr{T}}.$$

Secondly, if $\mathscr{T}$ is a $\emptyset\omega_1$-weak qtree, we describe a winning strategy for player *II* in the game

$$\mathbf{W}(\emptyset \to \left[B^{\sim} \vee (B^{\sim})^{\complement}\right], (B^{\sim} \bullet \omega_1)^{\mathscr{T}}).$$

As long as player *I* plays only 0's (that is, it is as if he is in charge of $\emptyset$), then

- ▷ either the question asked is about a closed set that player *II* cannot exit (it is the whole space from the actual position), *II* answers yes and easily wins the whole game, because every position is allowed, so it's like she's really in charge of $(B^{\sim} \bullet \omega_1)$.
- ▷ or there is a sequence v that *II* can play so she exits the actual closed set she is asked about. In this case, *II* exits the closed set with $v\ a_+$.

If player *I* decides to stop playing 0's and decides to switch from being in charge of $\emptyset \to \left[B^{\sim} \vee (B^{\sim})^{\complement}\right]$, then *II* waits until *I* decides if he wants to be in charge of $B^{\sim}$ or of $(B^{\sim})^{\complement}$. But in both case, *II* has a winning strategy since

- ▷ if *I* decides to be in charge of $B^{\sim}$, then since $B^{\sim} \leq_W (B^{\sim} \bullet \omega_1)$, we know that for any $\emptyset\omega_1$-weak qtree $\mathscr{T}'$ there exists a $\emptyset\omega_1$-weak qtree $\mathscr{T}''$ such that

  $$(B^{\sim})^{\mathscr{T}''} \leq_W (B^{\sim} \bullet \omega_1)^{\mathscr{T}'}.$$

  But since $B^{\sim} \equiv_W \emptyset^{\complement} \to B^{\sim}$ we know that

  $$B^{\sim} \equiv_W (B^{\sim})^{\mathscr{T}''} \leq_W (B^{\sim} \bullet \omega_1)^{\mathscr{T}'}.$$

  So, we let $\mathscr{T}'$ be the trace of $\mathscr{T}$ from the actual position. This gives a winning stratey for *II*.
- ▷ if *I* decides to be in charge of $(B^{\sim})^{\complement}$, then since the Wadge rank of $(B^{\sim})^{\complement}$ cannot be lowered by any question tree neither, *II* has a winning strategy by the same arguments as in the previous case.

□

So, combining the two results we obtain for any ω_1-weak tree $\mathscr{T}$:

$$(B^{\sim} \bullet \omega_1)^{\mathscr{T}} \leq_W \emptyset \to \left[B^{\sim} \vee (B^{\sim})^{\complement}\right].$$

Thus, the two claims give us $d^{\circ}A/_{\mathrm{w}}\omega_1 = \alpha + 1$, concluding the case $\mathrm{cf}(\delta) = \omega_1$.

Lastly, suppose the cofinality of δ is ω. Then

$$A \equiv_W S \bullet \omega_1 \text{ or } A \equiv_W (S \bullet \omega_1)^{\complement}$$

with S of Wadge degree precisely ω_1^{δ}. Moreover, from [5] we know that such a set is up to Wadge equivalence of the form $\sup_{n\in\mathbb{N}} B_n{}^{\sim} = \bigcup_{n\in\mathbb{N}} 0^n(\Sigma \setminus \{0\})B_n{}^{\sim}$, or of the form $(\sup_{n\in\mathbb{N}} B_n{}^{\sim})^{\complement}$. Here $(B_n)_{n\in\mathbb{N}}$ is strictly increasing for the Wadge ordering.

So, it is immediate to see that $(\sup_{n\in\mathbb{N}} B_n{}^{\sim})^{\complement} \equiv_W (\sup_{n\in\mathbb{N}} B_n{}^{\sim}) \cup \{0^\omega\}$.

Claim 43 Given any ω_1-weak qtree $\mathscr{T}$ on $\Sigma_{((\sup_{n\in\mathbb{N}} B_n{}^{\sim})\bullet\omega_1)}$:

$$\sup_{n\in\mathbb{N}} B_n{}^{\sim} \ \leq_W \left(\left(\sup_{n\in\mathbb{N}} B_n{}^{\sim}\right) \bullet \omega_1\right)^{\mathscr{T}}$$

Proof. The proof goes in two steps:
▷ If $\mathscr{T}$ is a $\emptyset^{\complement}\omega_1$-weak qtree, the result is obvious since

$$\sup_{n\in\mathbb{N}} B_n{}^{\sim} \ \leq_W \left(\left(\sup_{n\in\mathbb{N}} B_n{}^{\sim}\right) \bullet \omega_1\right) \equiv_W \left(\left(\sup_{n\in\mathbb{N}} B_n{}^{\sim}\right) \bullet \omega_1\right)^{\mathscr{T}}$$

▷ if $\mathscr{T}$ is a $\emptyset\omega_1$-weak qtree, we describe a winning strategy for player *II* in the game

$$\mathbf{W}(sup_{n\in\mathbb{N}} B_n{}^{\sim}, \left(\left(\sup_{n\in\mathbb{N}} B_n{}^{\sim}\right) \bullet \omega_1\right)^{\mathscr{T}})$$

- ◇ As long as *I* keeps on playing only 0's, then

  either the question asked is about a closed set that player *II* cannot exit (it is the whole space from the position we're in), *II* answers yes and easily wins the whole game, because every position is allowed, so it's like he's really in charge of $((\sup_{n\in\mathbb{N}} B_n{}^{\sim}) \bullet \omega_1)$.

  or there is a sequence v that *II* can play so he exits the actual closed set he is asked about. In this case, *II* exits the closed set with $v\ a_+$.

- ◇ If player *I* decides to stop playing 0's and plays a letter different from 0, then depending on the number n of zeros that he's been repeatedly playing since the beginning, he finds himself in charge of $B_n{}^{\sim}$. At that point, *II* has been initializing his payoff set again and again, exiting the many closed set he's been asked about. So he is in a position that is equivalent to the initial one.

  To obtain the result it is enough to remark that given any $\emptyset\omega_1$-weak qtree, $\mathscr{T}'$, *II* has a w.s. in the game

$$\mathbf{W}(B_n{}^{\sim}, \left(\left(\sup_{n\in\mathbb{N}} B_n{}^{\sim}\right) \bullet \omega_1\right)^{\mathscr{T}'})$$

  Because, setting, $\mathscr{T}'$ is the trace of $\mathscr{T}$ at the acual position of *II* in the game

$$\mathbf{W}(sup_{n\in\mathbb{N}} B_n{}^{\sim}, \left(\left(\sup_{n\in\mathbb{N}} B_n{}^{\sim}\right) \bullet \omega_1\right)^{\mathscr{T}})$$

  will bring the result.

  But this comes from the fact $B_n{}^{\sim} <_W B_{n+1}{}^{\sim}$ which implies both $B_n{}^{\sim} \leq_W B_{n+1}{}^{\sim}$ and $(B_n{}^{\sim})^{\complement} \leq_W B_{n+1}{}^{\sim}$.

  By definition, $(B_n{}^{\complement})^{\sim} \equiv_W (B_n{}^{\sim})^{\complement}$. So now the argument goes :
  either $\emptyset^{\complement} \to B_n \leq_W B_n$,

or $\emptyset^{\complement} \to (B_n)^{\complement} \leq_W (B_n)^{\complement}$

These two cases are symmetric, so w.l.o.g. we assume $\emptyset^{\complement} \to B_n \leq_W B_n$. (If $\emptyset^{\complement} \to B_n \not\leq_W B_n$, consider B_{n+2} instead of B_{n+1}).Since $\mathscr{T}'$, is a $\emptyset\omega_1$-weak qtree, it follows that:

there exists some $\emptyset\omega_1$-weak qtrees $\mathscr{T}''$, $\mathscr{T}'''$ $\mathscr{T}''''$ such that

$$B_n^{\sim} \leq_W \left(\emptyset^{\complement} \to B_n\right)^{\sim} \leq_W \left(\left(\emptyset^{\complement} \to B_n\right)^{\sim}\right)^{\mathscr{T}'''} \leq_W ({B_{n+1}}^{\sim})^{\mathscr{T}''} \leq_W \left(\left(\sup_{n\in\mathbb{N}} {B_n}^{\sim}\right) \bullet \omega_1\right)^{\mathscr{T}'}$$

So, this implies

$$B_n^{\sim} \leq_W \left(\left(\sup_{n\in\mathbb{N}} {B_n}^{\sim}\right) \bullet \omega_1\right)^{\mathscr{T}'}$$

which gives a w.s. for *II* in the underlying Wadge game.

Thus we have proved the claim in both cases.

□

Claim 44 There exists a $\emptyset\omega_1$-weak qtree $\mathscr{T}$ on $\Sigma_{\left(\left(\sup_{n\in\mathbb{N}} {B_n}^{\sim}\right)\bullet\omega_1\right)}$ such that:

$$\left(\left(\sup_{n\in\mathbb{N}} {B_n}^{\sim}\right) \bullet \omega_1\right)^{\mathscr{T}} \leq_W \sup_{n\in\mathbb{N}} {B_n}^{\sim}$$

Proof. We simultaneously describe the $\emptyset\omega_1$-weak qtree $\mathscr{T}$ and a w.s. for *II* in the underlying Wadge game.

The $\emptyset\omega_1$-weak qtree $\mathscr{T}$ repeatedly asks the question

Are you going to never play again a_+ nor a_- ?

As long as the answer is "no", *II* keeps on playing 0's. If there is a positive answer, then it means that *I* indicates the set ${B_n}^{\sim}$ he wants to be in charge of (in the main run). But then *II* chooses to be in charge of the same and applies a winning strategy. □

This concludes our discussion of the case $\mathrm{cf}(\delta) = \omega$.

Finally, let A be self dual. Since we work in a compact space, we know that $A \equiv_W \left[B \vee B^{\complement}\right]$ for some suitable non-selfdual B.

Here we finally use the fact that a ω_1-weak qtree $\mathscr{T}$ on Σ has the property

$$A \cap \mathrm{Av}\, y^{\complement} \in \Sigma^0_1 \cap \Pi^0_1.$$

Note that for selfdual sets we only needed $\emptyset\omega_1$-weak trees and $\emptyset^{\complement}\omega_1$-weak trees.

Let $\mathscr{T}_B$ and $\mathscr{T}_{B^{\complement}}$ be two ω_1-weak qtrees with $B/_{\mathrm{w}}\omega_1 = B^{\mathscr{T}_B}$ and $B^{\complement}/_{\mathrm{w}}\omega_1 = (B^{\complement})^{\mathscr{T}_{B^{\complement}}}$. We define

$$\mathscr{T}_A = 0\,\langle - \rangle\, \mathscr{T}_B \cup 1\,\langle - \rangle\, \mathscr{T}_{B^{\complement}}.$$

It is easy to see that $\mathscr{T}_A$ is a ω_1-weak qtree. In particular, $A \cap \mathrm{Av}\, y^{\complement} \in \Sigma^0_1 \cap \Pi^0_1$. Moreover, one easily verifies that $A^{\mathscr{T}_A} \equiv_W \left[B^{\mathscr{T}_B} \vee {B^{\complement}}^{\mathscr{T}_{B^{\complement}}}\right]$. Thus, $A/_{\mathrm{w}}\omega_1 = A^{\mathscr{T}_A}$, thereby at last establishing the proposition. [36]

To finish this section, let us sum up our results about non self-dual sets:

If $A \equiv_W S \bullet \omega_1$ with $d^{\circ}A = \omega_1^{\delta} \cdot \omega_1$:

(1) if δ is successor, then $d^\circ A/_{\mathrm{w}}\omega_1 = {\omega_1}^\delta$ and $A/_{\mathrm{w}}\omega_1$ is unique (up to Wadge equivalence) and it is non self-dual

(2) if δ is limit of cofinality ω_1, then $d^\circ A/_{\mathrm{w}}\omega_1 = {\omega_1}^\delta + 1$ and $A/_{\mathrm{w}}\omega_1$ is unique (up to Wadge equivalence) and it is non self-dual and satisfies

$$\emptyset \rightarrow A/_{\mathrm{w}}\omega_1 \equiv_W A/_{\mathrm{w}}\omega_1$$

(3) if δ is limit of cofinality ω, then $d^\circ A/_{\mathrm{w}}\omega_1 = {\omega_1}^\delta$ and $A/_{\mathrm{w}}\omega_1$ is unique (up to Wadge equivalence) and it is non self-dual and satisfies

$$\emptyset \rightarrow A/_{\mathrm{w}}\omega_1 \equiv_W A/_{\mathrm{w}}\omega_1$$

References

[1] R. Barua The Hausdorff–Kuratowski hierarchy of ω-regular languages and a hierarchy of Muller automata *Theoretical Computer Science*, 96:345–360, 1993.

[2] J. Bradfield. Fixpoints, Games and the Difference Hierarchy. *Theoretical Informatics and Applications*, 37:1–15, 2003.

[3] J. Bradfield and C. Stirling. Modal Logics and mu-calculi: An Introduction. In J. Bergstra, A. Ponse, and S. Smolka, editors, *Handbook of Process Algebra*, pages 293–329. Elsevier Science B.V., 2001.

[4] J. Duparc. Wadge Hierarchy and Veblen Hierarchy, Part I: Borel sets of finite rank. *Journal of Symbolic Logic*, 66:56–86, 2001.

[5] J. Duparc. Wadge Hierarchy and Veblen Hierarchy, Part II: Borel sets of infinite rank. . Submitted.

[6] E. Grädel and I. Walukiewicz. Positional determinacy of games with infinitely many priorities. *Logical Methods in Computer Science* **2**(4:6), 2006. DOI: 10.2168/LMCS-2(4:6)2006

[7] A. Kechris. *Classical descriptive set theory.* Springer Verlag, 1995.

[8] N. Klarlund. *Progress Measures and Finite Arguments for Infinite Computations.* Ph.D. dissertation. Cornell University, 1990.

[9] A. Louveau. Some results in the Wadge hierarchy of Borel sets. In A. Kechris, D. Martin, and Y. Moschovakis, editors, *Cabal Seminar 79-81*, Lecture notes in Mathematics, pages 28–55. Springer, 1983.

[10] R. S. Lubarski. μ-definable sets of integers. *Journal of Symbolic Logic*, 58:291–313, 1993.

[11] D. Martin. Borel determinacy. *Annals of Mathematics*, 102:336–371, 1975.

[12] M. O. MedSalem, K. Tanaka. Δ^0_3-determinacy, comprehension and induction. *J. Symbolic Logic*, 72(2):452–462, 2007.

[13] D. Niwiński. Fixed point characterisation of infinite behavior of finite state systems. *Theoretical Computer Science*, 189:1–69, 1997.

[14] V. Selivanov, Wadge degrees of ω-languages of deterministic Turing machines *Theoretical Informatics and Applications*, 37:67–83, 2003.

[15] W. W. Wadge, *Reducibility and Determinateness on the Baire Space.* Ph.D. dissertation. University of California at Berkeley, 1984.

Version Control: main2.tex:1.61.