The Absurdity of Vastness
being a silver anniversary critique of
D. Terence Langendoen and Paul M. Postal's
The Vastness of Natural Languages

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This manuscript is made available for entertainment and comments. It doesn't yet know what it's trying to be, but the intended audience is people interested enough to read Vastness *but not with the technical knowledge to appreciate the very good reasons why they feel uneasy with it. Because I also used it as an opportunity to think about my own philosophical attitudes to set theory, it falls at times into a didactic style more appropriate for a student seminar than a research review. I would welcome comments on how much of this is worth retaining outside the seminar context.*

When Langendoen and Postal (1984) (henceforth LP84) came out, it received some reviews, the general tone of which was incredulity about the claim but a regretful inability to point out anything wrong with their argument. Unfortunately, although at least one reviewer was a knowledgeable user of logic, the book does not appear to have been reviewed by anyone working in set theory and logic, who might have addressed a fundamentally mathematico-philosophical flaw in the argument.

The book has since then been rarely cited, and even more rarely cited by anyone who understands the material it presupposes. A discussion in the early days of the Linguist list [ref needed] shows several instances of basic ignorance of even the fundamentals of the argument of LP84. This is not surprising, and does no discredit to the participants: LP84 relies on mathematics that even most maths undergraduates do not meet, not least because it is largely useless to the vast mass of mathematicians. One might as well expect set theorists to be au fait with pair network grammars.

Given LP84's lack of impact, why revisit it now? Because it was not refuted at the time, it is still there in the background – and recently, both authors have, in varying ways, repeated the claim. Postal (2004) explicitly asserts that no one has refuted the argument, either by finding a flaw in the proof, or by arguing against the premises. Langendoen, in his presentation at the workshop on recursion in April 2007, resiled infinitely far from the position of LP84, but still made a claim stronger than most linguists would accept.

In response to the recent debate around language and recursion, Pullum and Scholz (forthcoming), which I have had the advantage of seeing in draft, analyse the reasons for even the usual claim (vastly weaker than LP84) that languages are infinite, and argue that the claim is (usually) poorly justified and unnecessary. Part of the issue here is one of definition: is linguistics (and science in general) concerned with the actual physical universe (and its limitations), or is it permissible and desirable to abstract to infinite space, time and entities?

In this present article, I do not wish to address that issue, except in passing. Rather, here I examine the argument of LP84 (and the recent much weaker version of Langendoen) from the viewpoint of logic, foundations, and the philosophy of mathematics.

This article is intended for a general audience, and therefore I have tried to explain in simple terms the mathematical and mathematico-philosophical issues. Fortunately, LP84 requires no very technical material – although modern set theory is one of the most abstruse branches of mathematics, its foundations are relatively simple, and based on quite natural intuitions. However, because LP84 does rely entirely on (a poor understanding of) certain foundational aspects, it is necessary to spend some time discussing them, even though they would normally (and correctly) be of no interest to the working linguist *qua* linguist.

From time to time, I put paragraphs in smaller type. Sometimes this is because they are addressed to a mathematically knowledgeable reader, assuming background that is not worth the general reader's while to discuss; sometimes simply because although they amplify the argument, I fear I may be trying the general reader's patience. All paragraphs in smaller type may be omitted without detriment to the flow of the argument.

1 The core argument of LP84

The 180 pages of LP84 can be boiled down to the following claim, in which 'NL' means 'natural language', a term which is not defined in the book, but which appears to be intended to carry its naive interpretation of "language spoken (written, signed, etc.) by humans".

(1) An NL is a collection of 'grammatical' sentences.

(2) The appropriate mathematical framework for defining the term 'collection' in (1) is set theory, in particular (an inessential extension of) the standard set theory ZFC.

(3) NLs permit the coordination of arbitrary non-empty sets of sentences to form new sentences.

(4) It follows that the collection of sentences of an NL is arbitrarily large, and so not even a set.

In this argument, it is (4) which is described as the 'NL Vastness Theorem', a term which gives a spurious impression of solidity. The authors are well aware that (4) is a simple mathematical argument, though they devote quite a number of pages to it; for a set theorist, it is practically a one-liner. The content of the claim, therefore, is in the postulates (1)–(3) of the theorem.

Claim (1), of course, is the standard abstraction that grammarians and linguists have made for millennia, despite the now widespread acceptance that it is manifestly inadequate for any *real* account of language! For our purposes here, we accept it without question.

In the remainder of this article, we consider (2) and (3). We will argue that (2) is not justified, but may be accepted as a matter of convenience; and that (3) is simply false, even if (2) is accepted, provided that the implications of (2) are understood.

I use both 'we' and 'I', according as the willing reader is necessarily or not necessarily (respectively) included.

2 Claim (2): set theory and its role as a framework

2.1 Classical set theory

What is the 'set theory' referred to in claim (2)?

The traditional aim of set theory is to provide a sound mathematical framework in which the rest of mathematics can be done. It is, in the jargon, a meta-mathematical subject, as it makes statements about and places requirements upon actual mathematics; but also a mathematical subject in its own right.

Since in its beginnings set theory aimed to give a framework for *all* of mathematics, it took the most liberal view possible of the ontological questions of mathematics: objects exist if you can define them in some plausible way. Little other than a naive understanding was required or used at this point.

Set theory was kicked into active existence as a field of mathematics by the discovery that one can't be as liberal as one thinks at first. The paradoxes or antinomies of Cantor, Russell, etc. showed the limits: something as simple as 'the set of all sets' cannot exist (because, by a theorem of Cantor, it would have to be strictly bigger than itself). More generally, self-referential definitions are dangerous and easily lead to contradictions. (And in the classical logic on which most of mathematics is based, once you have a single contradiction, you can 'prove' anything.)

In response, set theory developed by trying to find simple constraints on what objects exist, so as to allow as many objects as possible to exist without actually generating any contradictions. The original hope was for a theory which was provably consistent (free of contradiction) and also sufficient for all of mathematics (including itself). Gödel, whose incompleteness theorems were perhaps even more shocking than the antinomies of three decades earlier, put paid to that hope by showing that no reasonably powerful theory could establish its own consistency; and Tarski (also, independently, Gödel) showed that no reasonably powerful language could explain its own meaning. Nonetheless, the standard set theory used today, known as ZFC, is powerful enough to represent all of mathematics (indeed, far more powerful than is required for most mathematics), has a simple intuitive meaning, and is not known to have any inconsistencies.

Simply in order to avoid some otherwise irrelevant additional notation and qualifications, I shall assume ZFC with the additional axiom GCH. This makes no difference to the argument. Indeed, LP84 make the same assumption, for the same reason.

Because classical set theory tries to be as liberal as possible, it is profoundly *non-constructive*. That is, it asserts the existence of classes of objects, without in any way exhibiting an example. This is mathematically risky, since if you don't have an example, how do you know that such objects really exist? Every maths Ph.D. student has heard the horror story of the student who defined an interesting class of woozles [1], proved nice theorems about them for three years, and then discovered in the viva that there are no woozles. These stories are not urban legends (I know of two examples). However, even in real mathematics, as opposed to set theory, there are things which most people believe

[1] Set theorists have a predilection for cutesy terminology. 'Mice' (and 'pre-mice') are well established objects in advanced set theory. Consequently, 'weasels' have recently been defined. To the best of my knowledge, no set theorist has yet called something a 'woozle'. If one has, my apologies to them.

exist, but which cannot (yet) be proved to exist, and so one studies their properties anyway. Perhaps the most accessible example is in computer science, where thousands of person-years of effort have been devoted to studying the complexity of algorithms under the assumption that P $\neq$ NP, when for all we know, somebody might prove tomorrow that indeed P = NP and cause all that work to collapse into vacuity.

Standard set theory has several non-constructive ways of asserting the existence of sets. A broad classification is Infinity, Comprehension, Choice. The Axiom of Infinity asserts the existence of the set of $\{0, 1, 2, \ldots\}$ of (non-negative) integers, or more generally of some infinite set: for it turns out that the only way to get an infinite set is to assert that one exists. Comprehension axioms, of which there are several, allow one, given a set S, to assert the existence of the set of subsets of S satisfying given properties (and in particular to assert the existence of the set of all subsets of S, the powerset). Finally, Choice (the C in ZFC) is a specific axiom that says: given a set S of sets, there is a set which contains one element of each of the sets in S (i.e. which chooses one element from each of the sets in S). This innocent-looking axiom turns out to be immensely powerful, and has far-reaching consequences: some of these are very nice mathematical theorems that the working mathematician would not want to lose, and some are profoundly anti-intuitive theorems that cause most people to recoil.

The standard example of the latter is the Banach–Tarski paradox. This theorem states that a solid sphere can be divided into a finite number of pieces (in fact, five pieces suffice), which can then be moved around and put back together to form two solid spheres the same size as the original. This is clearly repugnant to common sense; and we do not observe set theorists becoming indecently rich by reduplicating spheres of gold. The problem is that the cuts required are infinitely complex, so complex that the pieces don't even have a well-defined volume; not only that, but even with an infinitely fine and infinitely fast cutting machine, one wouldn't be able to make the cuts, as the theorem does not (and in general cannot) describe them, but merely asserts their existence. The pieces do not, therefore, correspond to any physical reality. Yet it is sets of this complexity that LP84 argue we should accept as part of human language.

The reader may suspect that firing the Banach–Tarski paradox at LP84 is a cheap shot – surely they don't really need Choice, and their arguments could go through without it? In a sense, this is true – the Cantor hierarchy of infinities exists without Choice, although cardinal arithmetic in general is messed up. However, LP84 explicitly assume properties, such as the Well-Ordering Theorem, which are equivalent to Choice. They could manage without the WOT, by changing their claim (3) to be the (obviously more correct) version referring to well-ordered sets of sentences, but they choose not to. Since they explicitly adopt ZFC, they deserve the cheap shots!

Another consequence of these non-constructive axioms is Cantor's hierarchy of infinities, which LP84 takes as its basic claim about NLs. It is easy to show that once one postulates the set of integers (which, recall, has to be simply asserted), one is committed to a never-ending hierarchy of increasingly large infinities, up to a certain level of infinitude ('inaccessibility'). (Much of modern set theory is concerned with how much further one can go in asserting the existence of even larger infinities.) The size of the set of integers is called $\aleph_0$, and then by generating powersets and taking unions one gets the larger infinities. However, in normal mathematical practice, one meets only $\aleph_1 = 2^{\aleph_0}$, the size of the real number line, and $\aleph_2 = 2^{\aleph_1}$, the size of the set of functions from reals to reals.

In considering the arguments of LP84, it is important to emphasize that even within the classical setting, not all cardinalities (sizes of sets) are equal in philosophical importance. The finite cardinalities, the integers, all have the same status – if one accepts any reasonable principle of set construction at all, one will be able to construct sets of any finite size. The next cardinality is $\aleph_0$, the cardinality of the set of integers. The distinction between the finite and the infinite is the distinction between combinatorics and mathematics: from the *foundational* point of view, purely finite mathematics is almost trivial (though actually doing it may be very far from trivial).

The next cardinal is the first 'uncountable' cardinal, $\aleph_1$. The distinction between the countable ($\aleph_0$) and the uncountable (all the other infinities) is the most fundamental distinction in mathematics: in the words of Fraenkel, Bar-Hillel and Levy (1973), it characterizes the "abyss between discreteness and the continuum". It turns out that $\aleph_0$ has many properties that are not shared by any of the other alephs to which Cantor commits us – to get alephs that do share such properties, one has simply to assert their existence, and they are vastly larger than those we know already. (They are called 'large cardinals'.)

The simplest example of such a property is 'inaccessibility'. If you have only finite sets, there is no way of obtaining from them an $\aleph_0$-sized set using only finitely many operations – that is why we need the Axiom of Infinity. Now given all Cantor's infinities, we can postulate a cardinal θ_0 which is so big that it can't be obtained from smaller cardinals using fewer than θ_0 operations – that is, it can't be reached using the sets we've already defined. Consequently, θ_0 needs a new axiom to assert its existence, the Axiom of the First Inaccessible. In fact, nobody knows whether inaccessibles exist (i.e. are consistent with ZFC). This doesn't stop set theory going on, and on, and on . . . . On p. 69, LP84 say '[that finitely many operations on finite strings do not generate an infinite string] is really to say no more than that $\aleph_0$ is an inaccessible cardinal number,' a remark which displays a deep ignorance of the strength of the statement. Their sentence continues 'i.e. has no immediate predecessor.', which shows that they don't even know what 'inaccessible' means. 'Has no immediate predecessor' is a characterization of 'limit cardinals', which are perfectly normal and rather boring animals in ZFC.

There are differences among the smaller uncountable alephs – for example, there are a few theorems that are true of $\aleph_1$ but not $\aleph_2$ – but in general they are much of a muchness, until one reaches the 'large cardinals' we have just mentioned. The key points are the difference between finite and infinite, and between countable and uncountable – and it is these differences that LP84 dismiss in their argument.

2.2 The backlash against infinity in foundations.

We have described the situation in classical set theory, and it is fair to say both that the majority of workers in foundations, and the very large majority of working mathematicians (in so far as they care, which is usually not at all) are happy to take ZFC as their underlying meta-mathematical framework.

However, there developed, mainly in the early 20th century though with earlier roots, a movement which responded to the early paradoxes of naive set theory not by the minimal trimming of set existence assertions, but by the radical claim that the paradoxes demonstrate a deep foundational error of mathematics: that of treating the infinite as if were somehow similar to the finite, to be studied by the same methods, whereas in fact (they claimed) the infinite is fundamentally distinct. From an intuitive point of view, this is obvious: an infinite object cannot be perceived as a totality in finite time by any finite observer in any conceivable universe; whereas one can conceive a universe far larger than our own with intelligent beings who can apprehend an object of size 10^{100}, say.

The fundamental principle of this movement – or rather these movements – is that a mathematical object has no claim to existence unless it can be constructed in some more or less effective manner. Depending on the principles allowed in construction, one gets varying strengths of rejection of classical mathematics: constructivism, intuitionism, finitism, and even ultra-finitism, in which the number 10^{100} (say) has no good claim to existence.

Although the proponents of intuitionism etc. have been among the greatest logicians of the 20th century, and have included such foundational set theorists as Skolem, they remain in the minority. With sufficient effort, a surprising amount of classical mathematics, particularly in the discrete, countable domain, can be recovered using only constructivist or even intuitionist principles; however, it is often very hard work, and no working mathematician wishes to give up on the power tools of ZFC and return to arts-and-crafts proofs with the hand tools of intuitionism. However, the influence of intuitionism is immense; not only because now classically minded theorists appreciate the problems, but also because of the rise of importance of computability theory and actual computer science, in which only computable objects can be dealt with at all.

What has this to do with our topic? As well as the discovery that a fair amount of classical mathematics is doable intuitionistically, a large body of work has gone into working out just how much power one needs to do a given body of mathematics, even in a classical framework. Much mathematical work can be done in a far weaker foundational framework than ZFC. In particular, there is no need to work in a framework that admits Cantor's hierarchy of infinities; or indeed the existence of uncountable infinities at all. I would claim that all the mathematics required for all the grammatical analysis ever done in linguistics is included here. One might go further, and claim that even the existence of infinite sets need not be postulated. This claim would, I believe, be correct; but the loss of infinity is an inconvenience I would not personally wish to suffer. Uncountable infinities, on the other hand, may be dismissed without any significant penalties.

There are several directions in which one could expand on this. One of the easiest would be to make the simple assertion that everything linguistics deals with should be computable. This is a claim that to me is self-evidently true, and which appears to be orthodoxy among generative grammarians at least. It is a claim that LP84 seem to be particularly offended by, and one of the main aims of their 'theorem' is to disprove it. This claim has the merit of ensuring that NLs are at most countably infinite without requiring any adjustment to foundations; its only demerit, as far as I can see, is adding the computability requirement to the properties of a grammar. If one is driven to accept infinite sentences, some revision is required, but this can be done.

Another approach is to change one's foundations to something analogous to the computable world. Here a natural contender is the admissible set theory of Barwise, which is in a sense a generalization of the notion of computable set to higher levels. Admissible sets are a model of Kripke–Platek set theory, which is much weaker than ZFC. KP does not even have the Infinity Axiom, though it is often useful to postulate the set of integers as a primitive. The Comprehension axioms are also drastically weakened, to allow only the analogue of recursive formulas. However, much can be done inside an admissible universe, certainly including the very small amount of mathematics required for any reasonable formalization of syntax. Even with the infinite set of integers postulated, admissible universes are countable objects, and may even be rather small countable objects (in a computability theoretic sense).

One could even adopt intuitionism, full stop. It seems to me that there is no good reason for linguistics to need anything beyond an intuitionistic framework. However, it is hard work to check and convert all one's arguments to be intuitionistically valid. As for other mathematicians, linguists can reasonably take the view that it is just too much work, and leave it to the logicians.

2.3 Summary: the case for claim (2)

As the reader will infer from the foregoing, I see no principled reason for admitting claim (2), as ZFC is certainly a far stronger system of mathematics than is required for NL syntax, and the principle of parsimony (on which more later) suggests that one should not build a factory full of steam-hammers to do a job requiring only a drawing-pin.

However, as I have remarked, as a matter of convenience (the intuitionists might say 'laziness'), ZFC is generally adopted as the default system for doing mathematics. It's unreasonable to expect linguists to be more worried about foundations than mathematicians. So perhaps we should concede claim (2). *But*, if we concede it, we should require recognition of the immense ontological complexity we have accepted by assuming ZFC, and in particular of the 'abyss' between the countable and uncountable. Even if we have a factory full of steam-hammers, we don't have to use them all to push in a drawing-pin.

3 Claim (3): Coordination in NLs

Let us repeat the claim:

(3) NLs permit the coordination of arbitrary non-empty sets of sentences to form new sentences.

3.1 Empirical considerations of claim(3).

Claims about natural languages should, one hopes, be supported empirically by examining what natural languages can do. The first problem with claim (3) is the following empirical fact:

(a) There is *no* natural language that has *any* construct for the coordination of *sets* of sentences.

For written or spoken languages, this is a simple consequence of the sequential nature of speech and writing systems representing speech – to coordinate a set of sentences, or a set of NPs, one must first impose an ordering on the set. Pragmatically, this changes the meaning of the sentence from the ideal – as all English speakers learn when told to 'say "you and I", not "I and you"!'. Sign languages could in principle coordinate simultaneously two sentences that each happened to be signable with few articulators, but in practice they do not. (They do, however, use concurrent articulation in other ways, including sometimes the coordination of NPs.) Even if they did, there would be a small limit (probably 2) on the number of concurrently signable sentences – and there might still be an implied ordering (e.g. dominant hand signing the more important member of the set).

This is not just a pedantic point. While LP84 do in fact address the issue (in section 7.5) to some extent, they don't appear to fully appreciate the import of the difference, critical in the analysis of foundations, between cardinals and ordinals, or between sets and sequences. Here we just outline the basics, for the non-mathematical reader who wishes to get a flavour.

We have talked about cardinalities, the size of sets. A *cardinal*, as in the linguistic use of the term, is a number that answers 'how many?'. One can also think about elements arranged in a sequence; then, as in the linguistic use, an *ordinal* is a number that answers 'how far along the sequence?'. As long as everything is finite, ordinals and cardinals can be identified, though conceptually different. In the transfinite world, this is no longer the case. The smallest infinite ordinal, called ω, is the length of the sequence $0, 1, 2, \ldots$; the size of this sequence is $\aleph_0$, and for reasons of technical convenience, we identify ω with $\aleph_0$. However, there are lots of longer sequences that still have cardinality $\aleph_0$. The sequence $0, 1, 2, \ldots, -1$ has length $\omega+1$. The sequence $0, 2, 4, 6, \ldots, 1, 3, 5, \ldots$ has length $\omega+\omega$ or $\omega \cdot 2$. The sequence $2, 4, 6, \ldots, 3, 6, 9, \ldots, 4, 8, 12, \ldots, \ldots$ has length $\omega \cdot \omega$ or ω^2. And so on, for a very long way – but all these sequences are still countable.

Such sequences (*well-ordered* sequences, meaning one cannot go backwards for ever) are the foundations from which the entire set-theoretic universe is constructed. They are much easier to handle than arbitrary sets. Fortunately, in ZFC it can be shown that any set can be arranged into a well-ordered sequence; consequently, notions such as 'size' behave quite well. However, this *Well Ordering Theorem* requires (and is indeed equivalent to) the Axiom of Choice, which is violently non-constructive, as illustrated by the following. In ZFC we know that it is possible to arrange the (uncountable) set of real numbers into a well-ordered sequence. We also know, by a theorem of Feferman, that it may be impossible to describe this sequence, even using the entire apparatus of set theory. (The 'may be' is because this depends on axioms beyond those of ZFC; there is no reason to believe it possible in the usual model of set theory.) There is some discussion of a weak version of this issue in LP84 (around p. 171), and the get-out clause is essentially to adopt the claim (3′) we now introduce.

Fact (a) is an embarrassment in the presentation, rather than a significant flaw in the argument of LP84. The theorem (4) still goes through if we replace claim (3) by the more realistic version

(3′) NLs permit the coordination of arbitrary non-empty sequences of sentences to form new sentences.

I am finessing here the issue of what kind of sequences we are talking about: well-ordered, linearly ordered, etc. This is considered in LP84 p. 167ff. For the purposes of exposition, all my examples will be well-ordered.

Let us now consider the potential empirical basis for (3′). At the start of this article, we agreed to discount the possibility of finite bounds on sentence length, which would immediately falsify (3′). I make the following claims:

(b1) Almost all, if not all, languages permit the coordination of an arbitrarily long finite non-empty sequence of sentences to form a new sentence.

(b2) Infinite sentences cannot be used directly in NLs.

(b3) It is possible that infinite sentences of countable length are used indirectly by description, in NL explanations of infinite logical formulae and similar mathematical constructs. They may also, perhaps, be described in non-mathematical contexts.

(b4) Whether the infinite sentences used in (b3) are actually sentences of the NL is an empirical matter to be determined by native speaker judgment.

(b5) Sentences of uncountable length are not used even in the sense of (b3).

Points (b1) and (b2) are, I hope, unexceptionable.

(b3) considers one of the points LP84 raise when they are discussing objections to their argument: it may be impossible to utter a sentence, but the sentence may nevertheless be described in a perfectly comprehensible way, so that one can form a judgment of its grammaticality and meaning. By way of example, consider

(A) 'The numbers zero, two, four, ' ... 'are called *even*.'

where the ellipsis (outside quotation marks) indicates the indefinite continuation of the sentence in the obvious way; to be carefully distinguished from the finite sentence

(B) 'The numbers zero, two, four, ... are called *even*.'

in which the ellipsis (inside quotation marks) may be rendered as a pause, as 'et cetera', as 'and so on' or in other ways according to taste. Whether sentences like (A) are actually used and considered part of the language, is a difficult question. For one thing, the use/mention distinction is being somewhat blurred, and using language one can of course *mention* ungrammatical (but meaningful) sentences, as well as grammatical sentences. It happens that in my work I frequently have to explain concepts which correspond to infinitely long logical formulae, which I render in English so:

(C) '$\nu Z.[]Z \wedge \Phi$ means that we have Φ now and wherever you go next, we have Φ holds and wherever you go next, we have Φ holds and ... '

Clearly, when I write (or speak) (C), the actual linguistic construct is that of (B); but my intent is to describe an infinite sentence like (A). Does this commit me to accepting the real infinite version of (C) as a grammatical sentence of my native language, even in the doubtless perverse dialect used by mathematicians? My own view is that it does not.

This brings us to (b4). It is a *sine qua non* of modern linguistics that the grammaticality of a sentence, correctness of a sound, etc., is a matter to be determined by a consensus of native speakers of the language, dialect, sociolect or whatever in question. Yet LP84 treats the existence of infinite, and arbitrarily badly infinite, sentences, as a matter to be established by argument, and not once do they even consider the possibility of going out and asking the native speakers (other, presumably, than themselves). It might be interesting in several directions to explore the concept of infinite sentences among various types of speaker. A properly designed and executed study would require funding; but the following 'anecdata' raise some points of interest.

At a social meeting, I put to a group of friends the question of whether sentence (A) was a sentence of English. The group numbered seven (other than myself), among whom: four had substantial training or experience in pure mathematics or logic; one had mathematical/computer science training (but not in logic); two had no post-secondary training in mathematics (or other sciences). The initial responses may be surprising:

Of the four with logical training, two immediately and vehemently said that (A) was not an English sentence. One had difficulty believing that I really meant to ask about (A) rather than (B), and then judged (A) not a sentence. The last answered 'Say the whole sentence and I'll tell you whether it's grammatical.'

The non-'logician' computer scientist hesitated for a few seconds, and then decided it was not, 'as sentences must really be finite'.

The two non-mathematicians both responded 'I don't see anything wrong with it.'

After these initial responses, there was perhaps five minutes of discussion. In this discussion, the 'logicians' asserted that not only was (A) not an English sentence, but any sentence too long to be understood was not an English sentence. By the end of the discussion, all participants agreed that the only real sentences were those that could be understood within a normal human attention span. (The figure of three hours was mentioned as a marginal case.)

I then repeated the question (grammaticality of (A)) to a class of about 120 first-year computer science students. About 10% replied that (A) was grammatical. About 25% replied that it was definitely not grammatical. I then offered the option of 'it's a silly question', which attracted about another 25%. The remainder still declined any opinion. When asked whether a finite but very long (500 year) sentence was grammatical, 25% still said no, but now around 30% said yes. There was no time for discussion, so I don't know how views would have changed after argument.

Finally, I repeated the experiment again at a lunch-time meeting of my institute, with an audience of about 22 people, almost all with at least graduate level maths and logic. Two people (9%) accepted (A), eighteen (80%) rejected it, and the remainder chose not to express an opinion. Discussion did not change anybody's mind.

The tentative conclusion one can draw is that there is a case for minority native-speaker acceptance of countably infinite sentences, at least among those who understand what 'infinite' means. However, the reliability is not good, since owing to the nature of such sentences and their non-appearance in any real situation, it is hard to do other than ask for an explicit judgment, rather than any more subtle method of determining acceptability, less open to conscious biases.

Finally, (b5).

— Are uncountably long sentences grammatical?

— Mu. [see Hofstadter (1979)]

The only fitting response to the question of whether uncountably long English sentences are grammatical according to native speakers, is to un-ask the question. Understanding, at all, the concept of an uncountably long sentence requires a degree of mathematical sophistication that only a small fraction of any speech community can obtain, and only a smaller still fraction do obtain. Those who have reached that level do not need the formulae explained in naive English translations, and I have never heard, nor heard of, anybody attempting to do so. Moreover, I have no reason to believe that those who routinely deal abstractly with uncountably long formulae have any way to visualize them in a way that allows intuitive application of NL grammar, and good reason to believe that it is impossible.

I would go further, and say that only very short countably long sentences can be visualized. On the basis of my own intuitions, and of many years of explaining transfinite numbers to others, I would judge that well-orderings of length ω^2 can be visualized by most, but that as n increases, ω^n becomes rapidly harder to visualize. I have a very vague intuition for ω^ω, but rarely manage to transfer this intuition to others.

3.2 The theoretical argument for claim (3).

Finally we reach the inner core of the argument, which has necessitated the long diversion into set theory and its ontology. Since LP84 do not consider empirical matters, their argument rests entirely on theoretical reasoning. Their argument in support of claim (3) goes thus:

(3.1) A grammar that includes the rule 'sentences are finite', or any other size constraint, is more complex than one that does not. Likewise, a coordination rule mentioning 'finite sets' is more complex than one that just says 'sets'.

(3.2) There is no need to impose a size constraint.

(3.3) Therefore, by Occam's Razor, the grammar without the size constraint is better.

The appeal to Occam's Razor is quite explicit, and repeated several times. Let us therefore recall what Occam's Razor says. The traditional formulation is *entia non sunt multiplicanda praeter necessitatem* (entities should not be multiplied beyond necessity). The nearest William himself got to this in his writings appears to be *numquam ponenda est pluralitas sine necessitate* (plurality is never to be posited without necessity), which for our purposes is even more appropriate.[2] In modern usage, 'Occam's Razor' is often used in the somewhat vaguer sense of 'keep things as simple as possible'.

Now let us consider the three steps in more detail.

(3.1) is perhaps true as stated. As we discussed in section 2, one could (and arguably should) adopt a foundational framework in which everything is in any case finite or countably infinite. However, we agreed to concede the use of ZFC, in which case a size constraint has to be explicitly stated in a grammar. Even then, the additional complexity is one rule per grammar, or better yet just one word in the definition of 'grammar'.

(3.2) has a little to be said for it, but a strong counter-argument. We argued in the previous section that there may perhaps be an empirical case for countably infinite sentences, in which case the grammar needs to describe them. We also proposed that one cannot make an empirical case either for or against larger sentences. LP84 themselves consider no empirical case at all. At this point, an intuitionist has no problems, but the rest of us need to decide whether this means there is a need to permit them to exist, a need to deny their existence, or no need to do either. My response would be to invoke Occam's Razor, as one might say[3] *sententiae non sunt multiplicandae praeter necessitatem*: a grammar should not overgenerate (or overaccept, if one is not a generativist) by admitting as grammatical sentences which cannot be put to native speaker intuition.

(3.3) now reveals the clash of competing razors. LP84's razor consists in removing a size constraint either from each grammar, or from the definition of grammar. The price they pay is to include the entire 'staggering ontology' (as Quine put it) of ZFC inside every natural language. As we discussed in section 2, this includes directly the entire transfinite Cantorian hierarchy, with the enormous ontological commitment to infinity and uncountable infinity; and even among the smaller of these sets we have sets of (literally) unimaginable and indescribable complexity. So every natural language becomes vastly more complex than our entire universe, or even than the wildest multiverse imagined by physicists. On the other hand, my razor shaves off this complexity, at the cost of leaving a single hair: the word 'finite' (or 'countable', if one admits the infinite sentences).

It is surely hard to doubt which razor gives the better shave.

[2] Slade (2006) says forcefully that William interpreted his Razor as a principle of *suspending judgement* on the existence of unnecessary objects, and not of *denying* their existence (because, in the end, only God is necessary). Almost a foreshadowing of intuitionism! However, most more modern interpretations of the principle take a more robust view.

[3] With apologies to Latinists. Although *sententia* is attested in classical times with the (very minor) meaning 'syntactic sentence', William would most likely have used *periodus*. But that doesn't sound so good.

4 Objections dealt with in LP84

LP84 did of course anticipate a number of objections to their proposal. In this section, we review them briefly to convince the reader that our objection is not properly included.

Firstly, we have already discussed the simple objection 'but sentences are finite!', and agreed to pass over it. The more sophisticated objections are considered by LP84 in their sixth chapter 'Ontological Escape Hatches'.

The first section of the chapter considers the class of objections into which our objection falls: objections to the ontological assumptions required for the argument. However, the discussion is phrased in very general terms, as applicable to, say, set theory (the discussion owes somewhat to Fraenkel, Bar-Hillel and Levy (1973)). Three philosophical positions are defined: nominalism, conceptualism, and Platonism. Nominalism, which would deny the existence even of sentences too big to write out, is quickly dismissed as making any form of linguistic enquiry impossible; and as I am not a nominalist, I will not disagree, though I suspect a competent philosopher could criticize the dismissal. This leaves conceptualism, which is roughly the position that what 'exists' are the mental concepts with which we think about abstract objects; and Platonism, which is the position that abstract objects exist.

Section 6.1 argues firstly that there is no support for adopting conceptualism over Platonism (and in particular that Chomsky does not support his (said to be) conceptualist position), whereas Katz (1981) 'presents a well-developed body of arguments for the superiority of a platonist viewpoint over any possible conceptualism' (LP84 p. 112). With respect to the specific argument of Katz (1981) that platonism (in a rather extreme form) is right for linguistics, I may quote the review Dillinger (1984): 'His ontological "system" is so ill-defined and inexplicit that it could hardly constitute an improvement over the vague, tacit views now accepted. . . . He has offered an alternative to the nominalist and conceptualist interpretations of linguistic theories which is neither systematically developed nor viable as an alternative to either of the other views. Morever, it is wholly antiscientific. . . . In short, Platonism would make the philosophy of linguistics a convenience for the theoretician rather than a branch of metascience. Caveat lector!'.

This acerbic review crystallizes the question we are discussing: is linguistics a branch of science, or a branch of mathematics? If the latter, then almost everything currently done in linguistics departments becomes irrelevant; we may define languages to be whatever we like (as long as we don't introduce inconsistencies), and if our mathematical languages happen to agree with humans on which sentences are good, it doesn't matter what else they do. However, if linguistics is a science (or aspires to be a science), we have always to remember that abstract models are abstract models *of* something *real*, and it is a philosophical error to make the model vastly and unnecessarily more complex than the reality it tries to analyse.

Even in mathematics, platonism is by no means self-evidently correct. As Fraenkel, Bar-Hillel and Levy (1973) discuss at length, many of those involved in the foundations of mathematics have been and are avowed anti-platonists. The book Feferman (1998) to which I refer below is a modern example of the anti-platonist viewpoint in mathematics. Moreover, even if one is a platonist, one need not (contrary to LP84 p. 128 footnote 7) accept the existence of every object one is capable of defining – one might apply other reasonableness constraints to the question of existence.

The reader may perhaps be wondering what attitude other branches of science take to the existence of the outer reaches of set theory. They don't. There is one instance of large cardinal axioms being used to derive the first proofs of some algebraic theorems that have some application in physics, but there is no known instance of physics or any other hard science requiring anything of set theory – or, indeed, requiring non-trivial set theory (as opposed to arithmetic and analysis) at all.

The section continues to attack the constructivist, generative view said to be inherent in Chomsky's conceptualism. However, nothing we have discussed so far in this article relies on a specifically generative approach, so we may pass over this. It then continues to attack 'radical conceptualism', by which is meant Chomsky's position (as interpreted by LP84) as of 1980. Again, we have not taken a conceptualist position of any kind, so again we may pass over.

As far as I can determine, none of the many pages devoted to possible objections address our very simple argument that claim (3) introduces ridiculous complexity into what would otherwise be far simpler entities.

5 What remains?

What sentences *are* necessary? I hope the reader is convinced that there is no need for uncountably long sentences, but what of the countably infinite sentences, for the acceptability of which we have some empirical evidence? My answer would be, as remarked above, that in reality infinite sentences are described, not used or even properly mentioned, and so it is unnecessary to consider them as part of natural language. However, if we grant infinite sentences, what remains of LP84?

One of the main aims of the 'theorem' of LP84 is to attack the assumption of the generativist, and in particular Chomskyan, school that a language is a recursively enumerable set of sentences. (I presume that 'r.e.' is a reflection of the generativist viewpoint; surely most people would think the set is recursive, since given a sentence we should be able to decide whether it is or is not grammatical.) LP84 say that by demonstrating that NLs are vast, or even just uncountable, they have demolished this assumption. This is technically true, but substantially false. The term 'recursively enumerable' by definition applies only to countable sets of finite objects, and by admitting infinite sentences one goes outside the domain of application of the term. However, there are analogous classes of infinite sequences, in which the notion of computability still inheres. If one admits infinite sentences, this does not in any way prevent one from adhering to the Chomskyan principle that NLs should be describable by finite means.

The analogues of the Chomsky hierarchy of formal languages for strings of length ω are very well understood, forming a mainstay of computer science; and they relate to various kinds of automaton in analogous ways. There are many introductions to such work. Standard modern references in computer science are several articles by Wolfgang Thomas, notably the handbook chapter Thomas (1997). However, a linguistically oriented presentation that is explicitly prompted by LP84 may be found in Zeitman (1993).

Analogues for strings longer than ω are much less studied, as they are less useful. I am not currently aware of work at this level, though at higher levels of complexity (beyond the 'computable'), there is a body of work in both the computability-theoretic and recursion-theoretic sides.

A point worth noting is that there is no need to describe a sentence completely in order to determine whether or not it is grammatical – it suffices to know that it has a particular structure, even if the words that occupy particular N or V slots in the structure are unknown. Thus, even though there are uncountably many sentences of the form

(D) 'John/Mary knows John/Mary, and John/Mary knows that John/Mary knows John/Mary, and . . . '

the set is easily finitely describable, and all such sentences are grammatical (if you buy infinite sentences).

6 Recent positions of Langendoen and Postal

6.1 Postal.

A recent published expression of Postal's opinion is in the book Postal (2004). He appears to hold by LP84, and (in a book the second part of which is mainly devoted to attacking Chomsky's work) expressly wonders how Chomsky can maintain the claim that languages are recursively enumerable, when LP84 has been out for twenty years.

The short answer is probably that Chomsky, like most people, can't be bothered to respond to obviously daft criticism. There are plenty of things one can criticize in Chomsky's work, and when Postal sticks to proper linguistics his criticism is presumably relevant; but the vastness of natural languages is not worth the time of anybody who understands it (which is probably why nobody has published an article such as this).

The long answer is this article.

6.2 Langendoen.

Langendoen's current position is curious. He presented at the workshop on recursion in language held in April 2007. A published version is not yet available, but he has made publicly available the manuscript Langendoen (2007), to which I refer.

Langendoen has completely withdrawn from the extreme 'vastness' claim of LP84; but he still claims the necessity for sentences of countably infinite length. His argument for the existence of such sentences is by the example of sentences such as (D) above (which has the smallest infinite length, ω). The explanation of why he now claims only $\aleph_0$ sized sentences, and not arbitrarily large transfinite sentences, comes in the final section. Unfortunately, the technical part is incomprehensible (that is, I am unable to assign it any meaning that is correct according to the definitions given in the paper); the non-technical part says essentially 'languages don't have uncountably long sentences', and suggests that there will never be an example of one (a position with which, of course, I agree).

One may note in passing that Langendoen seems confused (as was the case in LP84) about the difference between ordinals and cardinals. It is, in fact, unclear whether he is really arguing, as the paper actually says, for sentences of arbitrary $\aleph_0$ length (though this is necessary to maintain closure under finite compounding), or whether he really means to talk about sentences just of length ω (by taking his 'universal set' to be the set of ω-strings). Unfortunately, he uses a non-standard and confused notation, so one cannot infer from the notation what is really meant.

7 Further reading

This article is, like the book it criticizes, really not about linguistics at all, but about the foundations of mathematics and how they might impact upon or be used in a particular domain of real science, in this case linguistics. I believe that the attitude I have taken would be shared by most, if not all, those who work in or really use set theory and logic. If the reader wishes to explore further the various attitudes towards foundations, there are several possibilities.

For a casual coffee-break curiosity, Wikipedia has articles on many of the topics I have touched on, and most of them are at least not blatantly wrong, though few are detailed and coherent expositions.

For a more sustained curiosity, I recommend the main reference that LP84 use for set theory, namely Fraenkel, Bar-Hillel and Levy (1973). This is a remarkable book, perhaps the finest piece of mathematico-philosophical exposition I have ever read. It assumes almost nothing of its reader – even first-order logic is (albeit briefly) explained – and takes them, accompanied by masters of the subject (Fraenkel is the F in ZFC) through the historical and intellectual development of modern axiomatic set theory, concentrating on the axioms themselves and their mathematical and philosophical import. The book goes on in its second and third parts to discuss at length the objections to infinity I sketched above. It is curious that LP84, who do mention the existence of these later chapters, do not mention at all the philosophical arguments contained in them which we sketched above. (FBL are very fair: they are themselves firmly committed (unsurprisingly) to ZFC and set theory as the foundation, but they explain with great care and sympathy the opposing side.) The book is, sadly, out of print, and also hard to find second-hand in its second edition, but should be in most libraries.

A more recent work by a leading contemporary set theorist is a collection of essays by Solomon Feferman (1998). Feferman is a master technician, with many deep theorems to his credit, and also renowned for his philosophical and historical interests in foundations. This collection of essays contains several that address issues discussed here: notably chapters 2 and 12 (originally one essay) consider 'Infinity in mathematics: is Cantor necessary?'. However, even the first (more general) of these chapters will severely tax the non-mathematical reader – but the present reader can probably guess the tenor of Feferman's answer. Regrettably, the book is now only available via print-on-demand with a very poor production quality.

Fraenkel, Bar-Hillel and Levy discuss hardly any of the technical apparatus of set theory, such as well-founded sets, ordinals, cardinals etc. However, there are many elementary textbooks; and here Wikipedia is quite good for a quick introduction (see, e.g., the article on 'Ordinal number', as at 2008–02–06).

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