Category Theory Lecture Notes

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Prologue

These notes, developed over a period of six years, were written for an eighteen lectures course in category theory. Although heavily based on Mac Lane's *Categories for the Working Mathematician*, the course was designed to be self-contained, drawing most of the examples from category theory itself.

The course was intended for post-graduate students in theoretical computer science at the Laboratory for Foundations of Computer Science, University of Edinburgh, but was attended by a varied audience. Most sections are a reasonable account of the material presented during the lectures, but some, most notably the sections on Lawvere theories, topoi and Kan extensions, are little more than a collection of definitions and facts.

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Lecture I

Introduction

Aim

- Learn category theory?

There is a lot to learn and we do not have much time. The crucial notion is that of **adjunction** and the course is geared towards getting there as quickly and as naturally as possible. Hard in the beginning, but it pays-off. Above all we aim to:

- Learn to *reason* categorically!

When one learns a foreign language it is often advised to listen to the language first, learning to understand the words before knowing how they are spelled. This is hard at first, but it pays back. Similarly, in this course we shall try and refrain from relating categories to other subjects and we shall try and work with examples and exercises from category theory itself. Surprisingly, one can go a long way without mentioning the traditional examples from mathematics and computer science: category theory is, by and large, a self-contained discipline.

Before starting, let us recall what the words *category* and *categorical* mean in English:

> *Category*: class or group of things in a complete system of grouping.
>
> *Categorical*: (of a statement) unconditional, absolute.

(Definitions from the Oxford dictionary.)

1 Universal Problems

In this first lecture we introduce universal problems. Following [Mac86, §II.3], we show that the *recursion theorem* is a categorical, compact way of expressing the Peano axioms for the natural numbers. This leads to Lawvere's notion of *natural number object*.

1.1 Natural Numbers in set theory and category theory

What are the natural numbers?

A1 Traditional, set-theoretic answer (Peano, one century ago):

The natural numbers form a set $\mathbb{N}$ such that:

1. $\exists\, \mathsf{zero} \in \mathbb{N}$
2. $\forall\, n \in \mathbb{N},\ \exists\, \mathsf{succ}\, n \in \mathbb{N}$
3. $\forall\, n \in \mathbb{N},\ \mathsf{succ}\, n \neq \mathsf{zero} \in \mathbb{N}$

4. $\forall\, n, m \in \mathbb{N},\ \mathsf{succ}\, n = \mathsf{succ}\, m \Longrightarrow n = m$ (*injectivity*)
5. $\forall\, A \subseteq \mathbb{N}\ (\mathsf{zero} \in A \wedge a \in A \Longrightarrow \mathsf{succ}\, a \in A) \Longrightarrow A = \mathbb{N}$

These axioms determine $\mathbb{N}$ uniquely *up to isomorphism.* (We shall prove this categorically in a moment.)

A2 Categorical answer (Lawvere, 60's):

A **natural number object**

$$0 \in N \xrightarrow{s} N$$

(in **Set**) consists of

- a set N
- with a distinguished element $0 \in N$
- and an endofunction $s : N \longrightarrow N$

which is **universal** in the sense that *for every* structure

$$e \in X \xrightarrow{g} X$$

there exists a unique function $f : N \longrightarrow X$ such that

- $f(0) = e$
- $f(s(n)) = g(f(n))$ for all n in N

The two characterisations are equivalent:

Theorem 1.1 A1 $\Longleftrightarrow$ A2

Proof.

(A1 $\Longrightarrow$ A2) $\equiv$ *Recursion Theorem* (see proof in [Mac86, page 45].)

(A2 $\Longrightarrow$ A1)

(1) and (2) by definition, setting $\mathbb{N} = N$, $\mathsf{zero} = 0$ and $\mathsf{succ}\, n = s(n)$.

(3) by contradiction: assume $s(n) = 0$ and consider a structure

$$e \in X \xrightarrow{g} X$$

with:

- $X = \{e, d\}$
- $g(e) = g(d) = d$

Then, by A2, there exists $f : N \longrightarrow X$ such that

$$f(0) = e \qquad f(s(n)) = g(f(n))$$

But if $s(n) = 0$ then $f(s(n)) = e \neq d = g(f(n))$.

(4) Exercise 1.2.

(5) We first express A2 in a really categorical way. For this we need to establish a language of **diagrams**.

Diagrams

Given functions $f : X \longrightarrow Y$, $g : Y \longrightarrow Z$, $h : X \longrightarrow Z$ we say that the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \underset{h}{\searrow} & \downarrow g \\ & & Z \end{array}$$

commutes if and only if $g(f(x)) = h(x)$ for all x in X. We write then:

$$g \circ f = h$$

Of course, we can add identity functions wherever we want without affecting commutation. Eg:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle id_X}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow[h]{} & Z \end{array}$$

Another example:

$$\begin{array}{ccccc} X & \xrightarrow{f} & Y & \xrightarrow{f'} & Y' \\ & \underset{h}{\searrow} & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle g'} \\ & & Z & \xrightarrow[k]{} & Z' \end{array}$$

commutes iff $k(g(f(x))) = k(h(x)) = g'(f'(f(x)))$ for all x in X.

A trivial yet important remark is that every element x of a set can be regarded as a function from a one-element (ie *singleton*) set $\{*\}$ to X. Moreover, this correspondence is a bijection. From now on we then write 1 for the generic singleton set and

$$x : 1 \longrightarrow X$$

as an alternative (very convenient!) notation for

$$x \in X$$

Finally, a dashed arrow

$$X \overset{f}{\dashrightarrow} Y$$

indicates that there is a *unique* map f from X to Y.

A2 with diagrams We can now rephrase the two equations of A2 (and the uniqueness condition) as follows:

$$\begin{array}{ccccc} 1 & \xrightarrow{0} & N & \xrightarrow{s} & N \\ {\scriptstyle id_1}\downarrow & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f} \\ 1 & \xrightarrow[e]{} & X & \xrightarrow[g]{} & X \end{array}$$

Proof of Theorem 1.1 (continued)

First note that everything in sight in the diagram

$$\begin{array}{ccccc} 1 & \xrightarrow{0} & N & \xrightarrow{s} & N \\ {\scriptstyle id_1}\downarrow & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f} \\ 1 & \xrightarrow[0]{} & A & \xrightarrow{s_{|A}} & A \\ {\scriptstyle id_1}\downarrow & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle i} \\ 1 & \xrightarrow[0]{} & N & \xrightarrow[s]{} & N \end{array}$$

commutes, where

$$i : A \longrightarrow N$$

is the evident inclusion function associated to $A \subseteq N$, and

$$s_{|A} : A \longrightarrow A$$

is the restriction of $s : N \longrightarrow N$ to A (which we assume it exists by A1.5). Trivially, also

$$\begin{array}{ccccc} 1 & \xrightarrow{0} & N & \xrightarrow{s} & N \\ {\scriptstyle id_1}\downarrow & & \downarrow{\scriptstyle id_N} & & \downarrow{\scriptstyle id_N} \\ 1 & \xrightarrow[0]{} & N & \xrightarrow[s]{} & N \end{array}$$

commutes, hence, by the uniqueness condition in A2, we have that

$$i \circ f = id_N$$

By the following lemma this implies that f is injective:

> **Lemma 1.2 (Right inverses are injective)** Given two composable functions f and g, if $g \circ f$ is the identity then f is injective.

We can then conclude that $N \subseteq A$, since the codomain A of the injective function f is $A \subseteq N$.

Theorem 1.3 A2 (hence A1) determines N uniquely up to isomorphism.

Proof. We need to prove that if there exists another structure

$$1 \xrightarrow{0'} N' \xrightarrow{s'} N'$$

satisfying A2 then there exists an isomorphisms between N and N'. That is, we would like to establish the existence of $f : N \longrightarrow N'$ and $f' : N' \longrightarrow N$ such that

$f' \circ f = id_N$ and $f \circ f' = id_{N'}$. We are going to find these two functions f and f' using the universal property of both structures given by A2:

$$\begin{array}{ccccc}
1 & \xrightarrow{0} & N & \xrightarrow{s} & N \\
{\scriptstyle id_1}\downarrow & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f} \\
1 & \xrightarrow[0']{} & N' & \xrightarrow[s']{} & N' \\
{\scriptstyle id_1}\downarrow & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f'} \\
1 & \xrightarrow[0]{} & N & \xrightarrow[s]{} & N
\end{array}$$

but also

$$\begin{array}{ccccc}
1 & \xrightarrow{0} & N & \xrightarrow{s} & N \\
{\scriptstyle id_1}\downarrow & & \downarrow{\scriptstyle id_N} & & \downarrow{\scriptstyle id_N} \\
1 & \xrightarrow[0]{} & N & \xrightarrow[s]{} & N
\end{array}$$

hence

$$f' \circ f = id_N$$

Similarly,

$$f \circ f' = id_{N'}$$

1.2 Universals

...*there exists* a *unique* function such that ...

- Existence: define entities
- Uniqueness: prove properties

Theorem 1.4 A universal construction defines an entity *uniquely* up to isomorphism.

Proof. Very much like for Theorem 1.3.

Exercises

E 1.1 Prove that A2 implies the following (where $X \times N$ is the cartesian product of the sets X and N):

> **Primitive Recursion.** For every set X with a distinguished element $e \in X$ and a function $h : X \times N \longrightarrow X$ there exists a unique function $f : N \longrightarrow X$ such that
>
> - $f(0) = e$
> - $f(s(n)) = h(f(n), n)$ for all n in N.

Hint: apply $A2$ to $g : X \times N \longrightarrow X \times N$, where $g(x, n) = (h(x, n), n)$.

E 1.2 Use the above primitive recursion to prove that A2 implies the fourth Peano axiom (injectivity). (*Hint.* You want to prove that the successor is injective. For this you can use Lemma 1.2 with respect to predecessor and successor. The predecessor function $p : N \longrightarrow N$ is definable by primitive recursion.)

Lecture II

2 Basic Notions

Category theory is the mathematical study of universal properties:

- it brings to light, makes explicit, and abstracts out the relevant structure, often hidden by traditional approaches;
- it looks for the universal properties holding in the categories of structures one is working with.

Category Theory vs Set Theory: primitive notions

Set Theory:

- *membership* and equality of those abstract collections called *sets*
  - an object is determined by its content.

Category Theory:

- *composition* and equality of those abstract functions called *arrows*
  - understand one object by placing it in a category and studying its relation with other objects of the same category (using arrows), or related categories (using *functors*, ie arrows between categories).

Informatics: we want to understand programs abstractly, independently from their implementation.

2.1 Categories

A *category* is a (partial) algebra of *abstract functions*:

- *arrows* with *identities* and a binary *composition* (partial) operation
  - obeying generalised monoid laws.

More formally, a **category** $\mathbb{C}$ consists of:

- A collection $\mathrm{Obj}_{\mathbb{C}}$ of **objects** $A, B, C, \ldots, X, Y, \ldots$.
- For each pair of objects A and B, a collection $\mathbb{C}(A, B)$ of **arrows** $f : A \longrightarrow B$ from A to B;
  - A is the **domain** and B is the **codomain** of $f : A \longrightarrow B$;

  - the collection of all arrows $f, g, h, k, \ldots$ of $\mathbb{C}$ is denoted by $\mathrm{Arr}_{\mathbb{C}}$;
  - arrows are also called *maps* or *morphisms*.

- For each object A, an **identity** arrow $id_A : A \longrightarrow A$.
- For each pair of arrows

  $$A \xrightarrow{f} B \xrightarrow{g} C$$

  a composite arrow

  $$g \circ f : A \longrightarrow C$$

These data have to satisfy the following generalised monoid laws.

1. *Identity*: if $A \xrightarrow{f} B$, then

   $$id_B \circ f = f = f \circ id_A$$

2. *Associativity*: if $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$, then

   $$(h \circ g) \circ f = h \circ (g \circ f)$$

Examples

1. Categories freely generated by directed graphs.
2. Degenerate categories such as:
   - Sets – ie categories where all arrows are identities.
   - Monoids – ie categories with *only one object*.
   - Preorders: categories with *at most one arrow* between every two objects.
3. $\mathbf{0}$, $\mathbf{1}$, $\cdot \longrightarrow \cdot$, $\boldsymbol{\omega}$.
4. Opposite categories $\mathbb{C}^{\mathrm{op}}$: obtained by reversing the arrows of given categories $\mathbb{C}$, while keeping the same objects.
5. **Set**: the category of (small) sets and functions; composition is the usual function composition (and so is in the remaining examples). Note that type matter: the identity on the natural numbers is a different function from the inclusion of the natural numbers into the integers.
6. $\mathbf{Set}_*$: pointed sets (ie sets with a selected base-point) and functions preserving the base point.
7. **N**: finite ordinals and functions.
8. **FinSet**: finite sets and functions.
9. **Preord**: preorders and monotone functions.
10. **Poset**: partial orders and monotone functions.

11. **Cpo**: complete partial orders and continuous functions.

12. **Mon**: monoids and monoid homomorphisms.

13. **Grp**: groups and group homomorphisms.

14. **SL**: semi-lattices and join-preserving functions.

15. **Top**: topological spaces and continuous functions.

16. **Met**: metric spaces and non-expansive functions.

17. **CMet**: complete metric spaces and non-expansive functions.

2.2 Functors

A functor is a homomorphism of categories

$$F : \mathbb{C} \longrightarrow \mathbb{D}$$

ie a morphism of categories preserving the structure, namely identities and composition.

Formally, a functor $F : \mathbb{C} \longrightarrow \mathbb{D}$ consists of a function $X \longmapsto F(X)$ from the objects of $\mathbb{C}$ to the objects of $\mathbb{D}$ and a function $f \longmapsto F(f)$ from the arrows of $\mathbb{C}$ to the arrows of $\mathbb{D}$ such that:

$$F(id_X) = id_{FX} \quad F(g \circ f) = F(g) \circ F(f)$$

Thus functors preserve all commuting diagrams, hence, in particular, isomorphisms.

Functors between sets are functions, between preorders are monotone functions, between monoids are monoid homomorphisms, between groups are isomorphisms (because a group is a category with one object and where every map has an inverse, and functors preserve isomorphisms).

For every category $\mathbb{C}$, there is an evident **identity functor**

$$Id_{\mathbb{C}} : \mathbb{C} \longrightarrow \mathbb{C}$$

Moreover, there is a composition $G \circ F : \mathbb{B} \longrightarrow \mathbb{D}$ of functors $F : \mathbb{B} \longrightarrow \mathbb{C}$ and $G : \mathbb{C} \longrightarrow \mathbb{D}$, namely $(G \circ F)(X) = G(F(X))$ and $(G \circ F)(f) = G(F(f))$.

A category $\mathbb{C}$ is **small** if its collection $\mathrm{Obj}_{\mathbb{C}}$ of objects and its collection $\mathrm{Arr}_{\mathbb{C}}$ of arrows are sets; it is **locally small** if the collection $\mathbb{C}(A, B)$ of arrows from A to B is a set for each pair of objects A and B. **N** is small, while the category **Finset** is only locally small, although every finite set is isomorphic to a finite ordinal. In the above examples, the categories 5, 6, and from 8 to 17 are not small, but only locally small.

Cat is the corresponding category of all (small) categories and functors between them.

E 2.1 Check that **Cat**, the category of all small categories, is indeed a category, ie check that the identity and the associative laws hold.

Lecture III

2.3 Initial and Final Objects

An object is **initial** in a category $\mathbb{C}$ if *for every* object X in $\mathbb{C}$ *there exists a unique* arrow in $\mathbb{C}$ from it to X. (It need not exist!)

> Initial objects are unique up to isomorphism.

Notation for 'the' initial object: 0.

An object is **final** (or **terminal**) in a category $\mathbb{C}$ if *for every* object X in $\mathbb{C}$ *there exists a unique* arrow in $\mathbb{C}$ from X to it. (Again, a category may have no final object.) Note that:

> A final in $\mathbb{C}$ iff A initial in $\mathbb{C}^{\mathrm{op}}$

therefore,

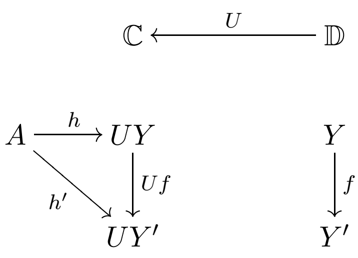
> final objects are unique up to isomorphism.

Notation for 'the' final object: 1.

The initial object in a preorder is the bottom element, if it exists; the final object is the top. In **Set** the initial object is the empty set, while the final object is the (unique up to isomorphism) singleton set. In **Cat** the initial object is the empty category **0**, with no objects nor arrows, and the final object is the category **1** with only one object and one arrow (the identity).

2.4 Comma Categories

Crucial for this course is Lawvere's notion of a **comma category**. Given a category $\mathbb{C}$, an object A of $\mathbb{C}$ and a functor $U : \mathbb{D} \longrightarrow \mathbb{C}$, the comma category $(A \downarrow U)$ is the category of *arrows from A to U*, with objects $\langle Y, h : A \longrightarrow UY\rangle$ given by arrows of $\mathbb{C}$ of type $A \longrightarrow UY$, where Y can range over the objects of $\mathbb{D}$. The homomorphisms $f : \langle Y, h : A \longrightarrow UY\rangle \longrightarrow \langle Y', h' : A \longrightarrow UY'\rangle$ are given by arrows $f : Y \longrightarrow Y'$ of $\mathbb{D}$ such that $Uf \circ h = h'$. I like to draw all this as follows.



Please try and conform to this notation (with U going from right to left) as we shall use it extensively throughout the course.

Exercises

E 2.2 Write down the proof that initial objects are unique up to isomorphism.

E 2.3 (We shall need this later on.) Just write down the more general notion of comma category $(T \downarrow U)$ involving two functors T and U with the same codomain

$$\mathbb{B} \xrightarrow{\quad T \quad} \mathbb{C} \xleftarrow{\quad U \quad} \mathbb{D}$$

that we have seen in the lecture. Also, write the two 'projection' functors $P : (T \downarrow U) \longrightarrow \mathbb{B}$ and $Q : (T \downarrow U) \longrightarrow \mathbb{D}$.

3 Universality

A **universal arrow** from an object A of $\mathbb{C}$ to a functor $U : \mathbb{D} \longrightarrow \mathbb{C}$ consists of an initial object in the comma category $(A \downarrow U)$, ie an object F_A of $\mathbb{D}$ and arrow $\eta_A : A \longrightarrow UF_A$ of $\mathbb{C}$ which are universal in the sense that for every Y of $\mathbb{D}$ and every $h : A \longrightarrow UY$ of $\mathbb{C}$ there exists a unique arrow $h^\sharp : F_A \longrightarrow Y$ in $\mathbb{D}$ such that $Uh^\sharp \circ \eta_A = h$. Diagrammatically:

$$\mathbb{C} \xleftarrow{\quad U \quad} \mathbb{D}$$

$$\begin{array}{ccc} A & \xrightarrow{\eta_A} & UF_A \\ & \searrow^{h} & \downarrow{\scriptstyle Uh^\sharp} \\ & & UY \end{array} \qquad\qquad \begin{array}{c} F_A \\ \downarrow{\scriptstyle h^\sharp} \\ Y \end{array}$$

It is at first hard to get accustomed to switching between the two categories $\mathbb{C}$ and $\mathbb{D}$ along the functor U and in getting the right order in the quantifications involved, but this notion is to category theory what the $\forall\epsilon\exists\delta$ formulation of continuity is to analysis. It is important to realize that the object F_A is by no means initial in $\mathbb{D}$: there might be more than one arrow from F_A to Y, but $h^\sharp$ is the unique arrow *such that* the triangle in the above diagram commutes.

Exercise

E 3.1 (*Important!*) Try and prove the following theorem.

> **Theorem 3.1** In the above situation, assume that, for every object A of $\mathbb{C}$ there is a universal arrow $\eta_A : A \longrightarrow UF_A$ from A to U. This defines a function F from the objects A of $\mathbb{C}$ to objects $FA \stackrel{\text{def}}{=} F_A$ of $\mathbb{D}$. Then, by universality, this extends to a functor $F : \mathbb{C} \longrightarrow \mathbb{D}$ in the opposite direction of U.

Lecture IV

Proof of Theorem 3.1 The action of F on arrows is defined as follows:

$$\mathbb{C} \xleftarrow{\quad U \quad} \mathbb{D}$$

$$\begin{array}{ccc} A & \xrightarrow{\eta_A} & UFA \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle UFf} \\ B & \xrightarrow[\eta_B]{} & UFB \end{array} \qquad\qquad \begin{array}{c} FA \\ \Big\downarrow {\scriptstyle Ff \stackrel{\text{def}}{=} (\eta_B \circ f)^\sharp} \\ FB \end{array}$$

Using universality one can prove that F is a functor $F : \mathbb{C} \longrightarrow \mathbb{D}$, that is:

$$Fid_A = id_{FA} \qquad Fg \circ Ff = F(g \circ f)$$

$\square$

Example 3.2 A monoid $M = \langle M, e, m\rangle$ consists of a carrier set M and an associative multiplication operation $m : M \times M \longrightarrow M$ with a unit $e \in M$:

$$m(x, m(y, z)) = m(m(x, y), z) \qquad m(x, e) = x = m(e, x)$$

for all x, y, z in M. A monoid homomorphism is a function between carriers which respects unit and multiplication. It is easy to see that monoids and their homomorphisms form a category **Mon**.

There is a trivial yet powerful *forgetful functor*

$$\mathbf{Set} \longleftarrow \mathbf{Mon} : U$$

which maps a monoid $\langle M, e, m\rangle$ to its carrier M and a homomorphism $f : \langle M, e, m\rangle \longrightarrow \langle M', e', m'\rangle$ to itself $f : M \longrightarrow M'$ by forgetting that it is a homomorphism.

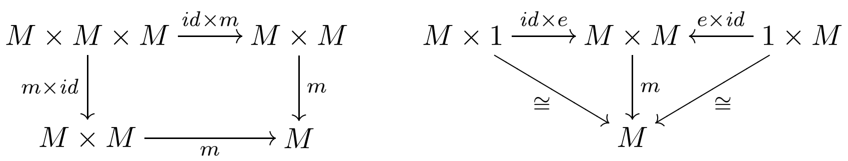
An important property of this forgetful functor is that for every set A there is a universal arrow from A to U, which gives rise, by the above theorem to a functor

$$F : \mathbf{Set} \longrightarrow \mathbf{Mon}$$

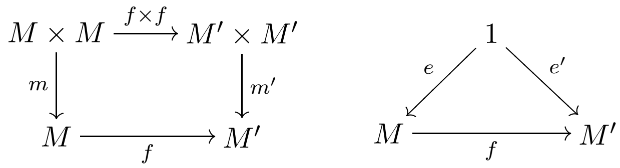
This F maps a set A to the set A^* of finite words over A, with concatenation as multiplication and with the empty word ϵ as unit. The universal arrow $\eta_A : A \longrightarrow UFA = A^*$ maps an element a of A to the one letter word $\langle a\rangle$. For every monoid $\langle M, e, m\rangle$ and every function $h : A \longrightarrow M$, there exists clearly only one monoid homomorphism $h^\sharp : FA \longrightarrow \langle M, e, m\rangle$ such that $h^\sharp\langle a\rangle = h(a)$.

Can you think of more similar examples?

Note that we can describe the category **Mon** without using elements as follows. A monoid consists of a set M together with functions $m : M \times M \longrightarrow M$ and $e : 1 \longrightarrow M$ such that



Monoid homomorphisms $f : \langle M, e, m\rangle \longrightarrow \langle M', e', m'\rangle$ are then functions $f : M \longrightarrow M'$ such that the diagrams



commute.

Exercises

E 3.2 (*Important.*) Consider the particular case of comma category when $(T \downarrow U)$ is with respect two parallel functors $T, U : \mathbb{B} \longrightarrow \mathbb{C}$ (ie, $\mathbb{B} = \mathbb{D}$). Assume there is a functor

$$\tau : \mathbb{B} \longrightarrow (T \downarrow U)$$

such that

$$P \circ \tau = Q \circ \tau = Id_{\mathbb{B}} \quad \text{(identity functor on } \mathbb{B})$$

(Recall: P and Q are the evident projection functors.) The question is: how would a generic such τ look like? Again, here spell out the details of the functor τ, looking at the type of $\tau(X)$ and of $\tau(f)$ for generic X and f in $\mathbb{B}$.

E 3.3 Given another parallel functor $V : \mathbb{B} \longrightarrow \mathbb{C}$ and a functor

$$\vartheta : \mathbb{B} \longrightarrow (U \downarrow V)$$

such that

$$P \circ \vartheta = Q \circ \vartheta = Id_{\mathbb{B}}$$

can you define a 'composite' functor

$$\vartheta\tau : \mathbb{B} \longrightarrow (T \downarrow V)$$

such that

$$P \circ (\vartheta\tau) = Q \circ (\vartheta\tau) = Id_{\mathbb{B}}?$$

Lecture V

4 Natural Transformations and Functor Categories

Given two parallel functors $T, U : \mathbb{B} \longrightarrow \mathbb{C}$, a **natural transformation** from T to U is a functor

$$\tau : \mathbb{B} \longrightarrow (T \downarrow U)$$

such that

$$P \circ \tau = Q \circ \tau = Id_{\mathbb{B}}$$

In other words, it is a mapping of each object A of $\mathbb{B}$ to an arrow

$$\tau_A : TA \longrightarrow UA$$

of $\mathbb{C}$ such that, for every arrow $f : A \longrightarrow B$ of $\mathbb{B}$, the diagram

$$\begin{array}{ccc} TA & \xrightarrow{\tau_A} & UA \\ {\scriptstyle Tf}\downarrow & & \downarrow{\scriptstyle Uf} \\ TB & \xrightarrow[\tau_B]{} & UB \end{array}$$

One usually writes either $\tau : T \stackrel{.}{\longrightarrow} U$ or $\tau : T \Longrightarrow U$ for such a natural transformation.

Theorem 3.1 (continued) Given a functor $U : \mathbb{D} \longrightarrow \mathbb{C}$, if for every object A of $\mathbb{C}$ there exists a universal arrow $\eta_A : A \longrightarrow UFA$ from A to U, we know that the function F from objects of $\mathbb{C}$ to objects of $\mathbb{D}$ extends, by universality, to a functor $F : \mathbb{C} \longrightarrow \mathbb{D}$ in the opposite direction of U. Moreover:

- The arrow η_A is natural in A

$$\eta : Id \Longrightarrow UF$$

Proof. By definition,

$$\begin{array}{ccc} A & \xrightarrow{\eta_A} & UFA \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle UFf} \\ B & \xrightarrow[\eta_B]{} & UFB \end{array}$$

commutes for every arrow $f : A \longrightarrow B$ of $\mathbb{C}$. □

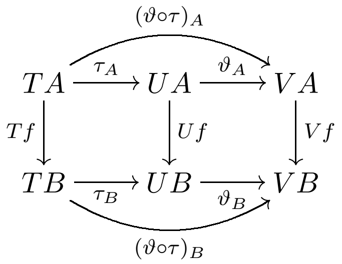
Thus, for instance, the function $A \longrightarrow A^*$ mapping an element a of A to the one letter word $\langle a \rangle$ is natural in the set A. Any more examples?

Given another parallel functor $V : \mathbb{B} \longrightarrow \mathbb{C}$ and a natural transformation $\vartheta : U \Longrightarrow V$, one can define a **composite**

$$\vartheta \circ \tau : T \Longrightarrow V$$

by putting

$$(\vartheta \circ \tau)_A \stackrel{\text{def}}{=} \vartheta_A \circ \tau_A$$

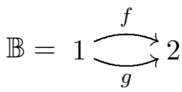


Note that this composition is associative because defined in terms of the composition of arrows in $\mathbb{C}$. Also note that, given any functor T from $\mathbb{B}$ to $\mathbb{C}$, there is also an evident identity natural transformation

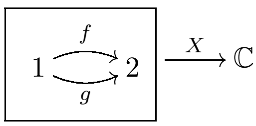
$$id_T : T \Longrightarrow T \qquad (id_T)_X \stackrel{\text{def}}{=} id_{TX}$$

For every two categories $\mathbb{B}$ and $\mathbb{C}$, one can then form the **functor category** $\mathbb{C}^{\mathbb{B}}$ having as object functors from $\mathbb{B}$ to $\mathbb{C}$ and as arrows natural transformations between them. Intuitively, objects of $\mathbb{C}^{\mathbb{B}}$ are diagrams of "shape" $\mathbb{B}$ and arrows are morphisms which preserve this shape.

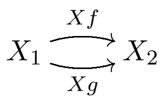
Example 4.1 Take



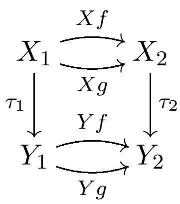
Then an object X of $\mathbb{C}^{\mathbb{B}}$ is a functor



that is, a diagram



in $\mathbb{C}$. An arrow τ between two such diagrams X and Y is a natural transformation $\tau : X \Longrightarrow Y$, that is a pair of arrows τ_1 and τ_2 in $\mathbb{C}$



such that

- $\tau_2 \circ Xf = Yf \circ \tau_1$
- $\tau_2 \circ Xg = Yg \circ \tau_1$

For every two categories $\mathbb{C}$ and $\mathbb{B}$ there exists a 'constant' functor

$$\Delta : \mathbb{C} \longrightarrow \mathbb{C}^{\mathbb{B}}$$

called the **diagonal functor** mapping an object C of $\mathbb{C}$ to the constant diagram of shape $\mathbb{B}$ in $\mathbb{C}$ where all objects are copies of C and all arrows are copies of $id_C : C \longrightarrow C$. Thus:

$$(\Delta C)(B) \stackrel{\text{def}}{=} C \qquad (\Delta C)(h) = id_C$$

for every object B and every arrow h of $\mathbb{B}$. Similarly, Δ maps an arrow $f : C \longrightarrow C'$ to the constant natural transformation

$$(\Delta f)_B \stackrel{\text{def}}{=} f : (\Delta C)(B) \longrightarrow (\Delta C')(B)$$

for every B in $\mathbb{B}$.

Example 4.2 Let $\mathbb{B}$ be the category $\boxed{\cdot\ \cdot}$ with two objects and no arrows (apart from the identities). Then

$$\mathbb{C}^{\boxed{\cdot\ \cdot}}$$

is the category of pairs (C_1, C_2) of objects of $\mathbb{C}$. The diagonal functor

$$\Delta : \mathbb{C} \longrightarrow \mathbb{C}^{\boxed{\cdot\ \cdot}}$$

acts as follows.

$$\begin{array}{c} C \\ {\scriptstyle f}\downarrow \\ C' \end{array} \quad \longmapsto \quad \begin{array}{cc} C & C \\ \downarrow{\scriptstyle f} & \downarrow{\scriptstyle f} \\ C' & C' \end{array}$$

Consider now a (not necessarily universal) arrow from a generic object (C_1, C_2) of $\mathbb{C}^{\boxed{\cdot\ \cdot}}$ to the above diagonal functor

$$\mathbb{C}^{\boxed{\cdot\ \cdot}} \longleftarrow \mathbb{C} : \Delta$$

That is, an object of $((C_1, C_2) \downarrow \Delta)$. By definition, this consists of two arrows $f_1 : C_1 \longrightarrow C$ and $f_2 : C_2 \longrightarrow C$ of $\mathbb{C}$. As a useful convention, let us write them as follows:

$$\begin{array}{ccc} C_1 & & C_2 \\ & {\scriptstyle f_1}\searrow \quad \swarrow{\scriptstyle f_2} & \\ & C & \end{array} \tag{1}$$

Exercises

E 4.1 Consider two parallel functors $T, U : \mathbb{C} \longrightarrow \mathbb{D}$, a functor $G : \mathbb{D} \longrightarrow \mathbb{E}$, a functor $F : \mathbb{B} \longrightarrow \mathbb{C}$, and a natural transformation $\tau : T \Longrightarrow U$.

a) Try and define a natural transformation, say $G\tau$, of type

$$GT \Longrightarrow GU$$

b) Try and define a natural transformation, say τ_F, of type

$$TF \Longrightarrow UF$$

E 4.2 Assume the universal arrow from a generic object (C_1, C_2) of $\mathbb{C}^{\boxed{\cdot\ \cdot}}$ to ΔC exists, where Δ is the diagonal functor from $\mathbb{C}$ to $\mathbb{C}^{\boxed{\cdot\ \cdot}}$. Describe it.

E 4.3 Same as the previous exercise but for the category of Example 4.1 instead of Example 4.2.

Lecture VI

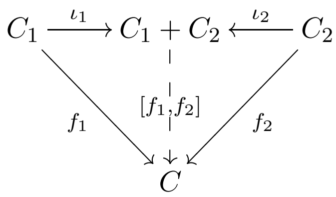
5 Colimits

5.1 Coproducts

A **coproduct** of two objects C_1 and C_2 of a category $\mathbb{C}$ is a universal arrow the object (C_1, C_2) of $\mathbb{C}^{\boxed{\cdot\ \cdot}}$ to ΔC exists, where Δ is the diagonal functor from $\mathbb{C}$ to $\mathbb{C}^{\boxed{\cdot\ \cdot}}$. A moment thought shows that such a universal (if it exists!) consists of an object of $\mathbb{C}$, which we denote by $C_1 + C_2$, together with two arrows

$$C_1 \xrightarrow{\iota_1} C_1 + C_2 \xleftarrow{\iota_2} C_2$$

such that for every pair of arrows as in (1) there exists a unique arrow (which we denote $[f_1, f_2]$) making the following diagram commute.



Equationally:

$$f_1 = [f_1, f_2] \circ \iota_1 \qquad [f_1, f_2] \circ \iota_2 = f_2$$

Terminology. The object $C_1 + C_2$ is called the **coproduct** (or **sum**) of C_1 and C_2; the arrows ι_1 and ι_2 are the first and second **injection** of the coproduct; the arrow $[f_1, f_2]$ is the **copairing** of f_1 and f_2. With respect to the general definition of universal arrow, we have that:

- $C_1 + C_2 = F_{(C_1, C_2)}$
- $(\iota_1, \iota_2) = \eta_{(C_1, C_2)}$
- $[f_1, f_2] = (f_1, f_2)^\sharp$

Clearly, by Theorem 3.1, if $\mathbb{C}$ has all coproducts then $+$ is a functor.

Remark. A coproduct puts two objects together while keeping them distinct, that is, keeping the ability to do a *case* analysis.

5.2 Coequalisers

For $\mathbb{B} = 1 \underset{v}{\overset{u}{\rightrightarrows}} 2$, we have already described the category $\mathbb{C}^{\mathbb{B}}$. Let us change notation though, and rather than explicitly writing the functor X from $\mathbb{B}$ to $\mathbb{C}$ we consider directly the objects of $\mathbb{C}^{\mathbb{B}}$ as parallel arrows

$$X \underset{g}{\overset{f}{\rightrightarrows}} X' \qquad (2)$$

in $\mathbb{C}$.

Note that before we used the symbols f and g for the arrows of $\mathbb{B}$, while now we use them for a generic object of $\mathbb{C}^{\mathbb{B}}$. You should now write the arrows of $\mathbb{C}^{\mathbb{B}}$ using this new notation. After having done this you should be able to see an arrow from (2) to the diagonal functor corresponding to this category as an arrow $h : X' \longrightarrow Z$ such that

$$h \circ f = h \circ g$$

The **coequaliser** of two parallel arrows $X \overset{f}{\underset{g}{\rightrightarrows}} X'$ in $\mathbb{C}$ is the universal arrow (if it exists) from them (regarded as an object of $\mathbb{C}^{\mathbb{B}}$) to the diagonal functor $\mathbb{C}^{\mathbb{B}} \longleftarrow \mathbb{C} : \Delta$. Spelling out the details, it consists of an object C and an arrow $q : X' \longrightarrow C$ of $\mathbb{C}$ such that

$$q \circ f = q \circ g$$

and, moreover, every $h : X' \longrightarrow Z$ such that $h \circ f = h \circ g$ factorises uniquely through q; formally, the latter means that there exists a unique arrow $h^\sharp : C \longrightarrow Z$ in $\mathbb{C}$ such that $h^\sharp \circ q = h$. Diagrammatically:

$$h \circ f = h \circ g \qquad \begin{array}{ccccc} X & \overset{f}{\underset{g}{\rightrightarrows}} & X' & \xrightarrow{q} & C \\ & & & \searrow^{h} & \downarrow{\scriptstyle h^\sharp} \\ & & & & Z \end{array}$$

5.3 Pushouts

For

$$\mathbb{B} = \begin{array}{ccc} & 1 & \\ {\scriptstyle u}\swarrow & & \searrow{\scriptstyle v} \\ 2 & & 3 \end{array}$$

we have that the objects of $\mathbb{C}^{\mathbb{B}}$ are diagrams in $\mathbb{C}$ of the form

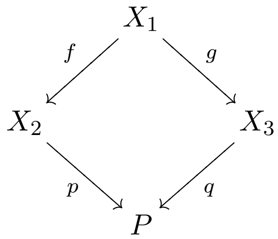
$$\begin{array}{ccc} & X_1 & \\ {\scriptstyle f}\swarrow & & \searrow{\scriptstyle g} \\ X_2 & & X_3 \end{array}$$

and arrows to the diagonal functor are pairs of arrows h and k making the diagram

$$\begin{array}{ccc} & X_1 & \\ {\scriptstyle f}\swarrow & & \searrow{\scriptstyle g} \\ X_2 & & X_3 \\ {\scriptstyle h}\searrow & & \swarrow{\scriptstyle k} \\ & Z & \end{array} \tag{3}$$

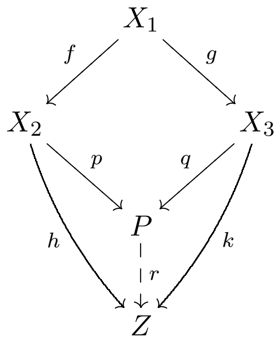
commute. Equationally: $h \circ f = k \circ g$.

The universal such an arrow (if it exists) is called the **pushout** of f and g. Spelling out the details, it consists of a pair of arrows p and q in $\mathbb{C}$ such that $p \circ f = q \circ g$



(4)

and with the universal property that for every commuting square as in (3) there exists a unique arrow r such that $r \circ p = h$ and $r \circ q = k$. Diagrammatically:



The diagram in (4) is then called a **pushout square**.

When $f = g$, the pushout of f and f is called the **cokernel pair** of f.

5.4 Initial objects as universal arrows

The initial object (if it exists) in a category $\mathbb{C}$ is easily seen to be the universal arrow from the unique object of the final category $\mathbf{1}$ (with one object and no arrow apart from the identity) to the unique functor from $\mathbb{C}$ to $\mathbf{1}$. But then note that

$$\mathbb{C}^{\mathbf{0}} \cong \mathbf{1}$$

where $\mathbf{0}$ is the initial category with no object, hence the initial object of $\mathbb{C}$ is the universal arrow from the unique object of $\mathbb{C}^{\mathbf{0}}$ to the diagonal functor $\Delta : \mathbb{C} \longrightarrow \mathbf{1}$.

Exercises

E 5.1 (*Important.*) Use universality to prove that in every category with coproducts the following holds:

a) $C_1 + C_2 \cong C_2 + C_1$

b) $(C_1 + C_2) + C_3 \cong C_1 + (C_2 + C_3)$

c) $C + 0 \cong C$ (where 0 is the initial object in $\mathbb{C}$)

E 5.2 Prove that the following law holds for the copairing operation.

For all arrows $f_1 : A_1 \longrightarrow C$, $f_2 : A_2 \longrightarrow C$, $h : C \longrightarrow D$,

$$h \circ [f_1, f_2] = [h \circ f_1, h \circ f_2]$$

E 5.3 Let $\mathbb{C}(A, C)$ be the set of arrows from A to C in $\mathbb{C}$.

Prove that in every category $\mathbb{C}$ with coproducts the following isomorphism (of sets!) holds:

$$\mathbb{C}(A_1, C) \times \mathbb{C}(A_2, C) \cong \mathbb{C}(A_1 + A_2, C)$$

Hint. Do not use universality but just note that one side of the isomorphism is:

$$(A_1 \xrightarrow{f_1} C,\ A_2 \xrightarrow{f_1} C) \longmapsto A_1 + A_2 \xrightarrow{[f_1, f_2]} C$$

E 5.4 Prove that the disjoint union

$$X + Y \cong X \mathbin{\dot{\cup}} Y \stackrel{\text{def}}{=} \{(0, x) \mid x \in X\} \cup \{(1, y) \mid y \in Y\}$$

of two sets X and Y is the coproduct of X and Y in **Set**, the category of sets and functions. (Hint: the left injection $X \longrightarrow X + Y$ maps x to $\langle 0, x \rangle$.)

E 5.5 Recall **N** is the category of finite ordinals

$$\underline{0} = \emptyset,\ \underline{1} = \{\underline{0}\},\ \underline{2} = \{\underline{0}, \underline{1}\},\ \underline{3} = \{\underline{0}, \underline{1}, \underline{2}\}, \ldots$$

and all functions. Prove that in **N** coproduct is addition:

$$\underline{n + m} \cong \underline{n} + \underline{m}$$

E 5.6 The coproduct in Mon, the category of monoids and monoid homomorphisms, of two free monoids A^* and B^* is simple:

$$A^* + B^* \cong (A + B)^*$$

Prove this.

Can you construct the coproduct $X + Y$ of two monoids X and Y that are not free? (It is not easy! You still need to use the coproduct in Set and free monoids, but also quotient by a suitable equivalence relation.)

Lecture VII

5.5 Generalised Coproducts

Let S be an arbitrary set (regarded as a category, ie objects are elements of S and all arrows are identities). An S-coproduct is a universal arrow from an object of $\mathbb{C}^S$ to the corresponding diagonal functor. This generalises the (binary) coproduct. One writes

$$\coprod_{s \in S} C_s$$

for the coproduct of an S-indexed family of objects C_s in $\mathbb{C}$ and

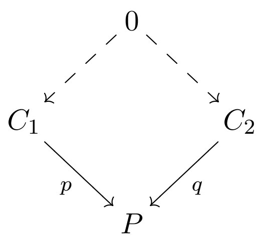
$$\iota_s : C_s \longrightarrow \coprod_{s \in S} C_s$$

for the s-th injection. One also has a generalised notion of copairing which we shall call cotupling.

5.6 Finite Colimits

Proposition 5.1 A category with pushouts and initial object has also coproducts.

Proof. We can use the pushout square



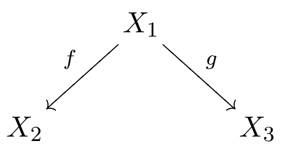
to define

$$C_1 + C_2 \stackrel{\text{def}}{=} P, \qquad \iota_1 \stackrel{\text{def}}{=} p, \qquad \iota_2 \stackrel{\text{def}}{=} q$$

One can check that this is a coproduct. □

Proposition 5.2 A category with coproducts and coequalisers has also pushouts.

Proof. The pushout of



is obtained by first taking the coproduct $X_2 + X_3$ and then the coequaliser of the two parallel arrows $\iota_1 \circ f$ and $\iota_2 \circ g$ of type $X \longrightarrow X_2 + X_3$. □

Remark 5.3 A coequaliser is a pushout of parallel arrows. An initial object is a $\emptyset$-coproduct.

5.7 Colimits

A **colimit** for a functor $J : \mathbb{B} \longrightarrow \mathbb{C}$ is nothing but a universal arrow from J to the diagonal functor $\Delta : \mathbb{C} \longrightarrow \mathbb{C}^{\mathbb{B}}$. Let us try and understand this concise definition in more elementary terms. First of all we have learnt to see a functor $J : \mathbb{B} \longrightarrow \mathbb{C}$ as a diagram of shape $\mathbb{B}$ in $\mathbb{C}$. A generic arrow $\tau : J \Longrightarrow \Delta Y$ from J to Δ is then a *cone* under J consisting of a vertex Y (which is an object of $\mathbb{C}$) and one arrow

$$\tau_B : JB \longrightarrow Y$$

(of $\mathbb{C}$) for each object B of $\mathbb{B}$ making the triangle

$$\begin{array}{ccc} JB & \xrightarrow{Ju} & JB' \\ & {\scriptstyle \tau_B} \searrow \quad \swarrow {\scriptstyle \tau_{B'}} & \\ & Y & \end{array}$$

for every arrow $u : B \longrightarrow B'$. In other words, all the subcones commute.

With a slight abuse of notation we shall also write $\tau : J \Longrightarrow Y$ for such a cone.

A colimit for J is then a universal such cone. That is, a cone $\mu : J \longrightarrow \mathit{Colim} J$ which is universal in the sense that every cone $\tau : J \Longrightarrow Y$ factorises uniquely through μ; that is, there exists a unique arrow $\tau^\sharp : \mathit{Colim} J \longrightarrow Y$ in $\mathbb{C}$ such that for every B in $\mathbb{B}$ the triangle

$$\begin{array}{ccc} JB & & \\ {\scriptstyle \mu_B} \downarrow & \searrow {\scriptstyle \tau_B} & \\ \mathit{Colim} J & \underset{\tau^\sharp}{\dashrightarrow} & Y \end{array}$$

commutes.

Please note that this triangle, in contrast with the previous one, does not involve the arrows of $\mathbb{B}$.

Terminology: μ is the **colimiting cone** of J, the vertex $\mathit{Colim} J$ of the colimit is the **colimiting object** of J, and $\tau^\sharp$ is **mediating arrow** corresponding to τ given by the universal property of the colimit.

Theorem 5.4 A category with coequalisers and generalised coproducts is **cocomplete** in the sense that it has all (small) colimits:

Given $J : \mathbb{B} \longrightarrow \mathbb{C}$, for $\mathbb{B}$ small, $\mathit{Colim} J$ is the coequaliser of the parallel arrows

$$\coprod_{u \in \mathrm{Arr}_{\mathbb{B}}} J(\mathrm{dom}(u)) \underset{g}{\overset{f}{\rightrightarrows}} \coprod_{B \in \mathrm{Obj}_{\mathbb{B}}} JB$$

where f is the copairing of the family of arrows

$$J(\mathrm{dom}(u)) \xrightarrow{\iota_{\mathrm{dom}(u)}} \coprod_{B \in \mathrm{Obj}_{\mathbb{B}}} JB$$

and g is the copairing of the family of arrows

$$J(\mathrm{dom}(u)) \xrightarrow{Ju} J(\mathrm{cod}(u)) \xrightarrow{\iota_{\mathrm{cod}(u)}} \coprod_{B \in \mathrm{Obj}_{\mathbb{B}}} JB$$

Moreover, the universal cone $\mu : J \Longrightarrow ColimJ$ at an object B is

$$JB \xrightarrow{\iota_B} \coprod_{B \in \mathrm{Obj}_{\mathbb{B}}} JB \xrightarrow{q} ColimJ$$

where q is the coequalising arrow.

Proof. We shall prove this theorem later on (using duality). □

Exercises

E 5.7 An arrow $X \xrightarrow{f} Y$ in $\mathbb{C}$ is **epi** if, for every pair of parallel arrows $Y \underset{h}{\overset{g}{\rightrightarrows}} Z$ in $\mathbb{C}$,

$$g \circ f = h \circ f$$

implies that

$$g = h$$

In **Set** the epi arrows are the surjective functions.

Prove that f is epi if and only if the following square is a pushout (more precisely, a cokernel pair).

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle id_Y} \\ Y & \xrightarrow[id_Y]{} & Y \end{array}$$

E 5.8 In **Set** coequalisers exist. Indeed, the coequaliser $q : Y \longrightarrow C$ of two parallel functions $f, g : X \longrightarrow Y$ is obtained by quotienting Y by the least equivalence relation $E \subseteq Y \times Y$ containing all pairs $\langle f(x), g(x) \rangle$ for x in X.

Check that this is really the case, proving that such a quotient enjoys the universal property of coequalisers.

E 5.9 We have constructed in two previous exercises coproducts and coequalisers in the category **Set** of sets and functions. Now use Proposition 5.2 to construct pushouts in **Set**.

Do the same in **Mon**.

Lecture VIII

6 Duality and Limits

Duality Principle

> Every categorical property, structure or theorem expressed in terms of diagrams of arrows has a *dual* such – obtained by reversing the arrows.

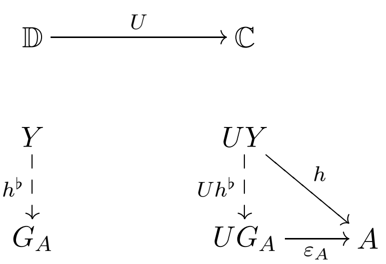
Note: you reverse arrows but not functors! But you do reverse natural transformations (because they are families of arrows).

Examples

Statement	Dual Statement
$X \xrightarrow{f} Y$	$X \xleftarrow{f} Y$
$X = \mathrm{dom}(f)$	$X = \mathrm{cod}(f)$
$X \xrightarrow{id} X$	$X \xleftarrow{id} X$
$h = g \circ f$	$h = f \circ g$
isomorphism	isomorphism
initial object 0	final object 1
initial objects are unique up to iso	final objects are unique up to iso
universal arrow from A to $U : \mathbb{D} \longrightarrow \mathbb{C}$	universal arrow from $U : \mathbb{D} \longrightarrow \mathbb{C}$ to A
coproducts	products
coequalisers	equalisers
pushouts	pullbacks
colimits	limits
$\vdots$	$\vdots$

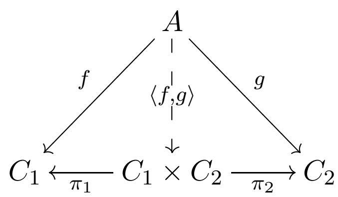
6.1 Universal arrows from a functor to an object

Given a functor $U : \mathbb{D} \longrightarrow \mathbb{C}$ and an object A of $\mathbb{C}$, a universal arrow from U to A consists of an object, say, G_A of $\mathbb{D}$ and an arrow $\varepsilon_A : UG_A \longrightarrow A$ of $\mathbb{C}$ such that for every object Y of $\mathbb{D}$ and every arrow $h : UY \longrightarrow A$ of $\mathbb{C}$ there exists a unique arrow $h^\flat : Y \longrightarrow G_A$ such that $\varepsilon_A \circ Uh^\flat = h$. Diagrammatically:



6.2 Products

The dual of a coproduct is called a product. Formally, given two objects C_1 and C_2 of a category $\mathbb{C}$, their **product** (if it exists) consists of an object $C_1 \times C_2$ of $\mathbb{C}$ and two arrows $\pi_1 : C_1 \times C_2 \longrightarrow C_1$ and $\pi_2 : C_1 \times C_2 \longrightarrow C_2$ of $\mathbb{C}$ such that for every object A of $\mathbb{C}$ and every pair of arrows $f : A \longrightarrow C_1$ and $g : A \longrightarrow C_2$ of $\mathbb{C}$ there exists a unique arrow $\langle f, g\rangle : A \longrightarrow C_1 \times C_2$ such that $f = \pi_1 \circ \langle f, g\rangle$ and $g = \pi_2 \circ \langle f, g\rangle$. Diagrammatically:



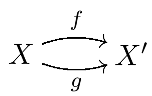
Terminology. π_1 and π_2 are the first and second **projection** of the product, respectively; $\langle f, g\rangle$ is the **pairing** of f with g.

$C_1 + C_2$ $=$ universal arrow from (C_1, C_2) to $\Delta : \mathbb{C} \longrightarrow \mathbb{C}$	$C_1 \times C_2$ $=$ universal arrow from $\Delta : \mathbb{C} \longrightarrow \mathbb{C}$ to (C_1, C_2)

E 6.1 Dualise $C_1 + C_2 \cong C_2 + C_1$, $(C_1 + C_2) + C_3 \cong C_1 + (C_2 + C_3)$, $C + 0 \cong C$;

6.3 Equalisers

The dual of a coequaliser is called an equaliser. Formally, given two parallel arrows

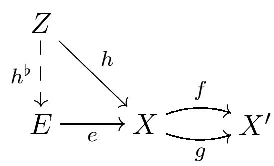


in $\mathbb{C}$, their **equaliser** consists of an object E and an arrow $e : E \longrightarrow X$ of $\mathbb{C}$ such that

$$f \circ e = g \circ e$$

and, moreover, every $h : Z \longrightarrow X$ such that $f \circ h = g \circ h$ factorises uniquely through e; formally, the latter means that there exists a unique arrow $h^\flat : Z \longrightarrow E$ in $\mathbb{C}$ such that $e \circ h^\flat = h$. Diagrammatically:

$f \circ h = g \circ h$

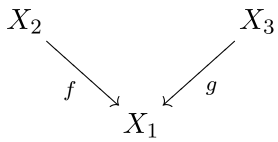


Terminology. E is called the **equalising object**.

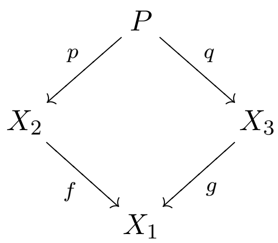
Coequaliser of $X \rightrightarrows X'$ $=$ universal arrow from $X \rightrightarrows X'$ to $\Delta : \mathbb{C} \longrightarrow \mathbb{C}^{\boxed{\cdot \rightrightarrows \cdot}}$	Equaliser of $X \rightrightarrows X'$ $=$ universal arrow from $\Delta : \mathbb{C} \longrightarrow \mathbb{C}^{\boxed{\cdot \rightrightarrows \cdot}}$ to $X \rightrightarrows X'$

6.4 Pullbacks

The dual of a pushout is called a pullback. Formally, given two arrows with a common codomain

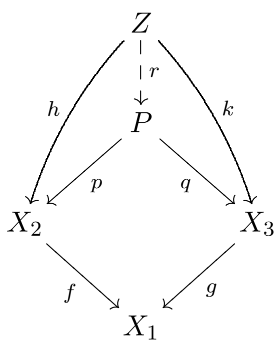


their **pullback** consists of a pair of arrows p and q in $\mathbb{C}$ such that $f \circ p = g \circ q$



(5)

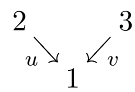
and with the universal property that for every pair of arrows h and k in $\mathbb{C}$ such that $f \circ h = g \circ k$ there exists a unique arrow r such that $p \circ r = h$ and $q \circ r = k$. This is expressed in the following diagram, where everything in sight commutes:



The diagram in (5) is then called a **pullback square**.

When $f = g$, the pullback of f and f is called the **kernel pair** of f.

Note that the category $\mathbb{B}$ such that a pullback is a universal arrow from the diagonal functor $\Delta : \mathbb{C} \longrightarrow \mathbb{C}^{\mathbb{B}}$, where

$\mathbb{B} =$ 

and not

$$\mathbb{B} = \begin{array}{ccc} & 1 & \\ {}^{u}\swarrow & & \searrow^{v} \\ 2 & & 3 \end{array}$$

as in the case of pushouts.

E 6.2 Dualise the following statements.

- a category with coproducts and coequalisers has pushouts;
- a category with initial object and pushouts has coproducts.

6.5 Monos

The notion of a **monic** arrow is dual to that of an epi arrow: an arrow $Y \xrightarrow{f} Z$ in $\mathbb{C}$ is **monic** if, for every pair of parallel arrows $X \underset{h}{\overset{g}{\rightrightarrows}} Y$ in $\mathbb{C}$,

$$f \circ g = f \circ h$$

implies that

$$g = h$$

In **Set** the monic arrows are the injective functions.

We can now generalise Lemma 1.2.

Proposition 6.1 (Right inverses are monic) Given two composable arrows f and g, if $g \circ f$ is the identity then f is monic.

Dually, **left inverses are epi**.

Right inverses are called **sections** and left inverses are called **retractions**.

E 6.3 Dualise the following statement.

- f is epi iff $\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle id_Y} \\ Y & \xrightarrow[id_Y]{} & Y \end{array}$ is a pushout square;

6.6 Limits

A **limit** for a functor $F : \mathbb{B} \longrightarrow \mathbb{C}$ is a universal arrow from F to the diagonal functor $\Delta : \mathbb{C} \longrightarrow \mathbb{C}^{\mathbb{B}}$. That is, a cone $\mu : LimF \longrightarrow F$ which is universal in the sense that every cone $\tau : Y \Longrightarrow F$ factories uniquely through μ; that is, there exists a unique arrow $\tau^\sharp : Y \longrightarrow LimF$ in $\mathbb{C}$ such that for every B in $\mathbb{B}$ the triangle

$$\begin{array}{ccc} & & Y \\ & \swarrow^{\tau_B} & \Big\downarrow{\scriptstyle \tau^\sharp} \\ FB & \xrightarrow{\mu_B} & LimF \end{array}$$

commutes.

Terminology: μ is the **limiting cone** of F, $LimF$ is the **limiting object** of F, and $\tau^\sharp$ is **mediating arrow** corresponding to τ given by the universal property of the limit.

Exercises

E 6.4 *Pasting Lemma.* [Mac97, page 72, exercise 8]

Assume everything in sight in the following diagram commutes.

$$\begin{array}{ccccc} \cdot & \longrightarrow & \cdot & \longrightarrow & \cdot \\ \Big\downarrow & & \Big\downarrow & & \Big\downarrow \\ \cdot & \longrightarrow & \cdot & \longrightarrow & \cdot \end{array}$$

(Assign your favourite names to the objects and arrows.)

- If both squares are pullbacks, prove that the outside rectangle (with top and bottom edges the evident composites) is a pullback.
- If the outside rectangle and the right-hand square are pullbacks, prove that so is the left-hand square.

E 6.5 *Pullbacks of monics are monic.*

Consider a square

$$\begin{array}{ccc} \cdot & \longrightarrow & \cdot \\ {\scriptstyle m'}\Big\downarrow & & \Big\downarrow{\scriptstyle m} \\ \cdot & \longrightarrow & \cdot \end{array}$$

which is a pullback.

First prove that if m is monic then also m' is monic; next dualise this property.

E 6.6 First prove that *equalisers are monic*; next dualise this property.

E 6.7 Dualise the notion of colimit and the theorem saying that a category with coequalisers and generalised coproducts has all (small) colimits (Thm 5.4). Try and prove the theorem.

E 6.8 The category **Set** of sets and functions has equalisers. The equalising object for two parallel functions $f, g : X \longrightarrow X'$ is the set

$$E \stackrel{\text{def}}{=} \{x \in X \mid f(x) = g(x)\}$$

The exercise consists in first checking that this is indeed the case and then use it to give an explicit description of pullbacks in **Set** in terms of cartesian products and equalisers.

Lecture IX

7 Adjunctions

"The multiple examples, here and elsewhere, of adjoint functors tend to show that adjoints occur almost everywhere in many branches of Mathematics. It is the thesis of this book that a systematic use of all these adjunctions illuminates and clarifies these subjects." [Mac97]

Category theory is the study of universal properties, formalised in terms of the notion of universal arrow. Constructions are canonical when they enjoy a universal property, but what does actually give rise to such constructions? The answer is – *adjunctions*.

We introduce adjunctions using, once more, universal arrows, and later show how, in turn, adjunctions define universal arrows. It is important to bear in mind that each of the three equivalent presentations of adjunctions that we study has its own relevance and that a full understanding of this deep notion requires an understanding of all three presentations.

7.1 From Universal Arrows to Adjunctions

The hypotheses of Theorem 3.1 suffice to define an adjunction. Indeed, we can extend that theorem as follows.

Theorem 7.1 Given a functor $U : \mathbb{D} \longrightarrow \mathbb{C}$, if for every object A of $\mathbb{C}$ there exists a universal arrow $\eta_A : A \longrightarrow UFA$ from A to U, then the following holds:

1. The function F from objects of $\mathbb{C}$ to objects of $\mathbb{D}$ extends, by universality, to a functor

$$F : \mathbb{C} \longrightarrow \mathbb{D}$$

   in the opposite direction of U.

2. The arrow η_A is natural in A

$$\eta : Id \Longrightarrow UF$$

3. For every object Y of $\mathbb{D}$ there is a universal arrow $\varepsilon_Y : FUY \longrightarrow Y$ from F to Y obtained by universality:

$$\mathbb{C} \xleftarrow{\quad U \quad} \mathbb{D}$$

$$\begin{array}{ccc} UY & \xrightarrow{\eta_{UY}} & UFUY \\ & \searrow^{id_{UY}} & \downarrow U\varepsilon_Y \\ & & UY \end{array} \qquad\qquad \begin{array}{c} FUY \\ \Big\downarrow \varepsilon_Y \stackrel{\text{def}}{=} (id_{UY})^\sharp \\ Y \end{array}$$

4. The arrow ε_Y is natural in Y

$$\varepsilon : FU \Longrightarrow Id$$

Proof. The proof of (1) is Theorem 3.1. The proofs of (2) and (4) are an immediate consequence of the way F and ε are defined. Before proving (3) let us restate the theorem in a more schematic way. Our hypothesis:

$$\frac{\forall A \in \mathbb{C}}{\exists \eta_A : A \longrightarrow UFA} \quad \text{such that} \quad \frac{\forall f : A \longrightarrow UY}{\exists ! f^\sharp : FA \longrightarrow Y \text{ s.t. } f = Uf^\sharp \circ \eta_A}$$

The first part of the theorem gives:

$$\frac{\forall h : A \longrightarrow B}{Fh \stackrel{\text{def}}{=} (\eta_B \circ h)^\sharp : FA \longrightarrow FB}$$

The third part consists of

$$\frac{\forall Y \in \mathbb{D}}{\exists ! \varepsilon_Y \stackrel{\text{def}}{=} (id_{UY})^\sharp : FUY \longrightarrow Y \text{ s.t. } id_{UY} = U\varepsilon_Y \circ \eta_{UY}}$$

plus the claim that the following holds:

$$\frac{\forall g : FA \longrightarrow Y}{\exists ! g^\flat : A \longrightarrow UY \text{ s.t. } g = \varepsilon_Y \circ Fg^\flat}$$

Existence of $g^\flat$.

$$g^\flat \stackrel{\text{def}}{=} A \xrightarrow{\eta_A} UFA \xrightarrow{Ug} UY$$

Note that, $g^\flat$ is defined in such a way that it makes

$$g = (g^\flat)^\sharp$$

hence it suffices to show that

$$(g^\flat)^\sharp = \varepsilon_Y \circ Fg^\flat$$

But this follows from the naturality of η.

Uniqueness of $g^\flat$. Assume given a $k : A \longrightarrow UY$ such that $g = \varepsilon_Y \circ Fk$. We have $k^\sharp : FA \longrightarrow Y$. If $k^\sharp = \varepsilon_Y \circ Fk$ then $k^\sharp = g = (g^\flat)^\sharp$, hence we would have the desired result that $k = g^\flat$. But this again follows from the naturality of η. □

We can now dualise:

$\mathbb{C} \xleftarrow{U} \mathbb{D}$ $\forall A \in \mathbb{C}:\ A \xrightarrow{\eta_A} UFA$ $\dfrac{\forall f{:}A\longrightarrow UY}{\exists! f^\sharp{:}FA\longrightarrow Y \text{ s.t. } f=Uf^\sharp\circ\eta_A}$	$\mathbb{C} \xrightarrow{F} \mathbb{D}$ $\forall Y \in \mathbb{D}:\ FUY \xrightarrow{\varepsilon_Y} Y$ $\dfrac{\forall g{:}FA\longrightarrow Y}{\exists! g^\flat A\longrightarrow UY \text{ s.t. } g=\varepsilon_Y\circ Fg^\flat}$
$\dfrac{\forall h{:}A\longrightarrow B}{Fh\overset{\text{def}}{=}(\eta_B\circ h)^\sharp{:}FA\longrightarrow FB}$ F extends to a functor opposite to U $\eta : Id \Longrightarrow UF$ $\dfrac{\forall Y\in\mathbb{D}}{\exists!\varepsilon_Y\overset{\text{def}}{=}(id_{UY})^\sharp{:}FUY\longrightarrow Y \text{ s.t. } id_{UY}=U\varepsilon_Y\circ\eta_{UY}}$ $\varepsilon : FU \Longrightarrow Id$ $\dfrac{\forall g{:}FA\longrightarrow Y}{\exists! g^\flat{:}A\longrightarrow UY \text{ s.t. } g=\varepsilon_Y\circ Fg^\flat}$ $g^\flat \overset{\text{def}}{=} Ug\circ\eta_A$ $g=(g^\flat)^\sharp$	$\dfrac{\forall k{:}X\longrightarrow Y}{Uk\overset{\text{def}}{=}(k\circ\varepsilon_X)^\flat{:}UX\longrightarrow UY}$ U extends to a functor opposite to F $\varepsilon : FU \Longrightarrow Id$ $\dfrac{\forall A\in\mathbb{C}}{\exists!\eta_A\overset{\text{def}}{=}(id_{FA})^\flat{:}A\longrightarrow UFA \text{ s.t. } id_{FA}=\varepsilon_{FA}\circ F\eta_A}$ $\eta : Id \Longrightarrow UF$ $\dfrac{\forall f{:}A\longrightarrow UY}{\exists! f^\sharp{:}FA\longrightarrow Y \text{ s.t. } f=Uf^\sharp\circ\eta_A}$ $f^\sharp \overset{\text{def}}{=} \varepsilon_Y\circ Ff$ $f=(f^\sharp)^\flat$

Note that $(\_)^\sharp$ and $(\_)^\flat$ are each other's inverse; that is, writing $\mathbb{C}(A,B)$ for the (possibly large) set of arrows from A to B in $\mathbb{C}$, we have the following bijection:

$$\begin{array}{ccc} \mathbb{D}(FA,Y) & \cong & \mathbb{C}(A,UY) \\ f^\sharp & \longleftarrow\!\!\dashv & f \\ g & \longmapsto & g^\flat \end{array}$$

Exercise

E 7.1 Prove that the operation $(\_)^\sharp$ associated to a universal arrow $\eta_A : A \longrightarrow UFA$ from A to U is natural in the sense that, for every $f : A \longrightarrow UY$,

$$\frac{\forall h : A' \longrightarrow A}{(f\circ h)^\sharp = f^\sharp \circ Fh} \quad\text{and}\quad \frac{\forall k : Y \longrightarrow Y'}{(Uk\circ f)^\sharp = k\circ f^\sharp}$$

(The first is by naturality of η and the second by definition.)

Dually, for every $g : FA \longrightarrow Y$,

$$\frac{\forall k : Y \longrightarrow Y'}{(k\circ g)^\flat = Uk\circ g^\flat} \quad\text{and}\quad \frac{\forall h : A' \longrightarrow A}{(g\circ Fh)^\flat = g^\flat \circ h}$$

7.2 Adjunctions

Definition 7.2 An **adjunction** from a category $\mathbb{C}$ to a category $\mathbb{D}$ is given by a pair of opposite functors

$$\mathbb{C} \underset{U}{\overset{F}{\rightleftarrows}} \mathbb{D}$$

and, for all A in $\mathbb{C}$ and Y in $\mathbb{D}$, a bijection

$$\varphi_{A,Y} : \mathbb{D}(FA, Y) \cong \mathbb{C}(A, UY)$$

which is natural in the sense that, for every $g : FA \longrightarrow Y$,

$$\frac{\forall k : Y \longrightarrow Y'}{\varphi(k \circ g) = Uk \circ \varphi(g)} \quad \text{and} \quad \frac{\forall h : A' \longrightarrow A}{\varphi(g \circ Fh) = \varphi(g) \circ h}$$

Notation and terminology. The above adjoint situation is denoted by

$$\mathbb{C} \underset{U}{\overset{F}{\rightleftarrows}}_{\bot} \mathbb{D}$$

or by $\langle F \dashv U, \varphi \rangle$ and F is the **left adjoint** of U, while U is the **right adjoint** of F; also, $\varphi^{-1}(f)$ and $\varphi(g)$ are the **left** and **right adjuncts** of f and g respectively.

The above shows that we already know two situations where we can 'construct' an adjunction, namely when we have a universal arrow from A to U for every object A of $\mathbb{C}$ or, equivalently, when we have a universal arrow from F to Y for every object Y of $\mathbb{D}$.

An example is given by a category $\mathbb{C}$ which has all colimits of shape $\mathbb{B}$:

$$\mathbb{C}^{\mathbb{B}} \underset{\Delta}{\overset{Colim}{\rightleftarrows}}_{\dashv} \mathbb{C}$$

The dual case is for a category $\mathbb{C}$ with all limits of shape $\mathbb{B}$:

$$\mathbb{C} \underset{Lim}{\overset{\Delta}{\rightleftarrows}}_{\dashv} \mathbb{C}^{\mathbb{B}}$$

An other example is the adjunction from **Set** to **Mon**, with the forgetful functor as right adjoint and the free monoid functor as left adjoint.

Lecture X

7.3 From Adjunctions to Universal Arrows

We have seen that universal arrows give rise to adjunctions. We now prove the converse.

First note that the naturality conditions for an adjunction $\langle F \dashv U, \varphi \rangle$ are equivalent to the following: for every $f : A \longrightarrow UY$,

$$\frac{\forall k : Y \longrightarrow Y'}{\varphi^{-1}(Uk \circ f) = k \circ \varphi^{-1}(f)} \quad \text{and} \quad \frac{\forall h : A' \longrightarrow A}{\varphi^{-1}(f \circ h) = \varphi^{-1}(f) \circ Fh}$$

Theorem 7.3 An adjunction

$$\mathbb{C} \underset{U}{\overset{F}{\underset{\perp}{\rightleftarrows}}} \mathbb{D} \qquad \varphi_{A,Y} : \mathbb{D}(FA, Y) \cong \mathbb{C}(A, UY)$$

determines:

i) a natural transformation

$$\eta \stackrel{\text{def}}{=} \varphi(id_F) : Id_{\mathbb{C}} \Longrightarrow UF$$

with $\eta_A : A \longrightarrow UFA$ universal from A to U for every object A of $\mathbb{C}$ and such that $\varphi(g) = Ug \circ \eta_A$. Such η is called the **unit of the adjunction**.

ii) a natural transformation

$$\varepsilon \stackrel{\text{def}}{=} \varphi^{-1}(id_U) : FU \Longrightarrow Id_{\mathbb{D}}$$

with $\varepsilon_Y : FUY \longrightarrow Y$ universal from F to Y for every object Y of $\mathbb{D}$ and such that $\varphi^{-1}(f) = \varepsilon_Y \circ Ff$. Such ε is called the **counit of the adjunction**.

Moreover the triangular identities

$$\begin{array}{ccc} U & \overset{\eta_U}{\Longrightarrow} & UFU \\ & \underset{id_U}{\searrow\!\!\searrow} & \Downarrow U\varepsilon \\ & & U \end{array} \qquad \begin{array}{ccc} F & & \\ F\eta \Downarrow & \overset{id_F}{\searrow\!\!\searrow} & \\ FUF & \underset{\varepsilon_F}{\Longrightarrow} & F \end{array}$$

hold for such η and ε.

Proof. (*Unit.*) For every $f : A \longrightarrow UY$ put

$$f^\sharp \stackrel{\text{def}}{=} \varphi^{-1}(f)$$

We have that $f = U\varphi^{-1}(f) \circ \varphi(id_{FA})$ because, by the naturality of φ, $U\varphi^{-1}(f) \circ \varphi(id_{FA}) = \varphi(\varphi^{-1}(f) \circ id_{FA})$. This shows existence. Uniqueness is similar, since if we have a g such that $f = Ug \circ \varphi(id_{FA})$ then the same reasoning as above shows that $\varphi(g) = f$, hence $g = \varphi^{-1}(f)$. This also gives:

$$\varphi(g) = Ug \circ \varphi(id_{FA})$$

(*Counit.*) By duality, this also prove the counit case.

(*Triangular identities.*) We prove the first triangular identity – the proof of the second is dual.

$$U\varepsilon_Y \circ \eta_{UY} = U\varepsilon_Y \circ \varphi(id_{FUY}) = \varphi(\varepsilon_Y \circ id_{FUY}) = \varphi(\varepsilon_Y) = id_{UY}$$

□

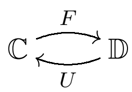
Corollary 7.4 φ is completely determined by its value at id_F. □

Sometimes adjunctions are denoted using unit and counit instead of the bijection φ:

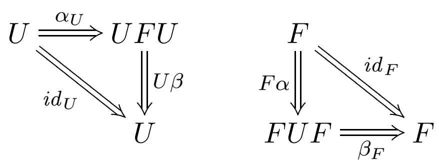
$$\langle F \dashv U, \eta, \varepsilon \rangle$$

Corollary 7.5 Adjoint functors determine each other uniquely up to isomorphism. □

Proposition 7.6 Let F and U be two opposite functors



Let $\alpha : Id_{\mathbb{C}} \Longrightarrow UF$ and $\beta : FU \Longrightarrow Id_{\mathbb{D}}$ be two natural transformations such that the following triangular identities hold:



Then F is left adjoint to U. □

Exercises

E 7.2 *Naturality of φ.* We have called a bijection of hom-sets

$$\varphi_{A,Y} : \mathbb{D}(FA, Y) \cong \mathbb{C}(A, UY)$$

natural if for every $g : FA \longrightarrow Y$,

$$\frac{\forall k : Y \longrightarrow Y'}{\varphi(k \circ g) = Uk \circ \varphi(g)} \quad \text{and} \quad \frac{\forall h : A' \longrightarrow A}{\varphi(g \circ Fh) = \varphi(g) \circ h}$$

The exercise consists in showing that this is naturality in a formal sense, namely with respect to the following functors associated to the (possibly large) sets $\mathbb{D}(FA, Y)$ and $\mathbb{C}(A, UY)$. Note that the exercise is easy once you understand how these (very important) functors act.

The claim is that φ is natural in Y with respect to

$$\mathbb{D} \xrightarrow{\mathbb{D}(FA,\_)} \mathbf{SET}$$

$$\begin{array}{cccc}
Y & \longmapsto & \mathbb{D}(FA,Y) & \ni g \\
{\scriptstyle k}\big\downarrow & \longmapsto & \big\downarrow{\scriptstyle k\circ(_)} & \\
Y' & \longmapsto & \mathbb{D}(FA,Y') & \ni k\circ g
\end{array}$$

and

$$\mathbb{D} \xrightarrow{\mathbb{C}(A,U(\_))} \mathbf{SET}$$

$$\begin{array}{cccc}
Y & \longmapsto & \mathbb{C}(A,UY) & \ni f \\
{\scriptstyle k}\big\downarrow & \longmapsto & \big\downarrow{\scriptstyle Uk\circ(_)} & \\
Y' & \longmapsto & \mathbb{C}(A,UY') & \ni Uk\circ f
\end{array}$$

and it is natural in A with respect to the *contravariant* functors

$$\mathbb{C}^{op} \xrightarrow{\mathbb{C}(\_,UY)} \mathbf{SET}$$

$$\begin{array}{cccc}
A & \longmapsto & \mathbb{C}(A,UY) & \ni f \\
{\scriptstyle h}\big\uparrow & \longmapsto & \big\downarrow{\scriptstyle (_)\circ h} & \\
A' & \longmapsto & \mathbb{C}(A',UY) & \ni f\circ h
\end{array}$$

and

$$\mathbb{C}^{op} \xrightarrow{\mathbb{D}(F(\_),Y)} \mathbf{SET}$$

$$\begin{array}{cccc}
A & \longmapsto & \mathbb{D}(FA,Y) & \ni g \\
{\scriptstyle h}\big\uparrow & \longmapsto & \big\downarrow{\scriptstyle (_)\circ Fh} & \\
A' & \longmapsto & \mathbb{D}(FA',Y) & \ni g\circ Fh
\end{array}$$

E 7.3 Do you see what an adjunction boils down to in case the two categories are preorders and the two functors are order-reversing functions?

E 7.4 *Composition of Adjoints.*

Show that if

$$\mathbb{C} \underset{U}{\overset{F}{\rightleftarrows}} \mathbb{D} \underset{\overline{U}}{\overset{\overline{F}}{\rightleftarrows}} \mathbb{E} \qquad (F \dashv U,\ \overline{F} \dashv \overline{U})$$

then

$$\mathbb{C} \underset{U\overline{U}}{\overset{\overline{F}F}{\rightleftarrows}} \mathbb{E} \quad (\bot)$$

What are the unit and counit of $\overline{F}F \dashv U\overline{U}$?

Lecture XI

7.4 Adjoints for Preorders

See [Mac97, §IV.5].

8 Preservation of Limits and Colimits

A functor $F : \mathbb{C} \longrightarrow \mathbb{D}$ **preserves colimits** of shape $\mathbb{B}$ if for every functor $H : \mathbb{B} \longrightarrow \mathbb{C}$ then:

1. if $\mathit{Colim} H$ exists then also $\mathit{Colim} FH$ exists and $\mathit{Colim} FH = F(\mathit{Colim} H)$, and
2. if $\mu : H \Longrightarrow \Delta \mathit{Colim} H$ is a colimiting cone for H then

$$FH \overset{F\mu}{\Longrightarrow} F\Delta \mathit{Colim} H \cong \Delta F \mathit{Colim} H \cong \Delta \mathit{Colim} FH$$

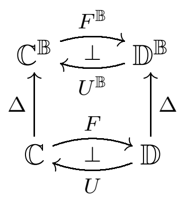
   is a colimiting cone for FH.

Theorem 8.1 Left adjoints are *cocontinuous* in the sense that they preserve colimits. Dually, right adjoints are *continuous*, ie they preserve limits.

This theorem is often used to prove that a functor is *not* an adjoint.

In order to prove the theorem we need the following lemma.

Lemma 8.2 Adjunctions lift to functor categories:



$$F^{\mathbb{B}}\Delta = \Delta F, \quad U^{\mathbb{B}}\Delta = \Delta U$$

where, for all $H : \mathbb{B} \longrightarrow \mathbb{C}$ and $K : \mathbb{B} \longrightarrow \mathbb{C}$,

$$F^{\mathbb{B}}(H) \overset{\text{def}}{=} FH, \quad U^{\mathbb{B}}(K) \overset{\text{def}}{=} UK$$

that is,

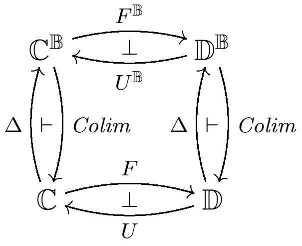
$$F^{\mathbb{B}} = F \circ (\_), \quad U^{\mathbb{B}} = U \circ (\_)$$

Proof. Unit and counit of $F^{\mathbb{B}} \dashv U^{\mathbb{B}}$ are defined using unit and counit of $F \dashv U$: for every H in $\mathbb{C}^{\mathbb{B}}$,

$$\eta^{\mathbb{B}}_H \overset{\text{def}}{=} \eta_H, \quad \varepsilon^{\mathbb{B}}_K \overset{\text{def}}{=} \varepsilon_K$$

Then the proof follows from the triangular identities for $F \dashv U$ instantiated at H and K. □

Proof of the Theorem. Use composition of adjoints, uniqueness of adjoints, the fact that adjunctions lift to functor categories, and the fact that (assuming that all colimits and limits of shape $\mathbb{B}$ exist) Δ is right adjoint to *Colim* and left adjoint to *Lim*. For colimits:



$$F^{\mathbb{B}}\Delta = \Delta F, \quad U^{\mathbb{B}}\Delta = \Delta U$$

$F\,Colim \dashv \Delta U$, $Colim F^{\mathbb{B}} \dashv U^{\mathbb{B}}\Delta$, and $\Delta U = U^{\mathbb{B}}\Delta$ imply that $F\,Colim \cong Colim F^{\mathbb{B}}$, hence

$$F\,Colim H \cong Colim F^{\mathbb{B}} H \cong Colim F H \quad .$$

This proves (1). Try and prove (2) as an exercise. □

Note that this implies, for instance, that the coproduct of two free monoids A^* and B^* over objects A and B of a category with coproducts $\mathbb{C}$ is $(A+B)^*$, since $(\_)^*$, together with its canonical monoid structure, is left adjoint to the forgetful functor $U : \mathbf{Mon} \longrightarrow \mathbf{Set}$.

Another application of the theorem gives us that if a functor is a left adjoint then it preserves epis, because epis are colimits, namely suitable pushouts as shown in an exercise. Dualising this, right adjoints preserve monos.

As mentioned above, the theorem is also useful to prove that a functor has no adjoint. For instance, let A be an object of a complete category $\mathbb{C}$: why, if A is not the final object 1 then the functor product with A

$$\_ \times A : \mathbb{C} \longrightarrow \mathbb{C}$$

cannot have a left adjoint?

Exercises

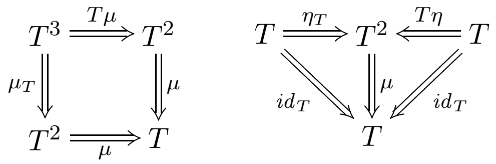
E 8.1 Study *Freyd's Adjoint Functor Theorem* [Mac97, §V.6].

E 8.2 *This is important: try and do it!*

Given an adjunction $\langle F \dashv U, \eta, \varepsilon \rangle$ from $\mathbb{C}$ to $\mathbb{D}$, define a functor T and a natural transformation μ as follows, where $T^2 \stackrel{\text{def}}{=} T \circ T$:

$$T \stackrel{\text{def}}{=} UF : \mathbb{C} \longrightarrow \mathbb{C} \qquad \mu \stackrel{\text{def}}{=} U\varepsilon_F : T^2 \Longrightarrow T$$

Prove that then the following diagrams (monoid laws in some sense!) commute.



Hint. For the triangles use the triangular identities of the adjunction.

Lecture XII

9 Monads

A *monad* $T = \langle T, \eta, \mu \rangle$ on a category $\mathbb{C}$ consists of

- an endofunctor $T : \mathbb{C} \longrightarrow \mathbb{C}$ on $\mathbb{C}$,
- a 'unit' $\eta : Id_{\mathbb{C}} \Longrightarrow T$ and
- a 'multiplication' $\mu : T^2 \Longrightarrow T$

such that the diagrams

$$\begin{array}{ccc} T^3 & \overset{T\mu}{\Longrightarrow} & T^2 \\ {\scriptstyle \mu_T}\Downarrow & & \Downarrow{\scriptstyle \mu} \\ T^2 & \underset{\mu}{\Longrightarrow} & T \end{array} \qquad \begin{array}{ccccc} T & \overset{\eta_T}{\Longrightarrow} & T^2 & \overset{T\eta}{\Longleftarrow} & T \\ & {\scriptstyle id_T}\searrow & \Downarrow{\scriptstyle \mu} & \swarrow{\scriptstyle id_T} & \\ & & T & & \end{array}$$

commute.

Exercise 8.2 shows that:

Theorem 9.1 Every adjunction gives rise to a monad. □

Thus, in particular, the adjunction we have seen from **Set** to **Mon** gives rise to a monad mapping an object X to the free monoid X^* over it (and forgetting the monoid structure).

9.1 Algebras of a Monad

We have seen how to get from adjunctions to monads. We now look at the converse.

Definition 9.2 Given a monad $T = \langle T, \eta, \mu \rangle$ on a category $\mathbb{C}$, the (*Eilenberg-Moore*) category T-**Alg** (or $\mathbb{C}^T$) of T-*algebras* has objects $X = \langle X, h \rangle$ given by an object X of $\mathbb{C}$ and a morphism $h : TX \longrightarrow X$ of $\mathbb{C}$ satisfying the following T-algebra laws:

$$\begin{array}{ccc} T^2X & \xrightarrow{Th} & TX \\ {\scriptstyle \mu_X}\downarrow & & \downarrow{\scriptstyle h} \\ TX & \xrightarrow[h]{} & X \end{array} \qquad \begin{array}{ccc} X & \xrightarrow{\eta_X} & TX \\ & {\scriptstyle id_X}\searrow & \downarrow{\scriptstyle h} \\ & & X \end{array}$$

The (homo)morphisms $f : \langle X, h \rangle \longrightarrow \langle X', h' \rangle$ of T-**Alg** are given by morphisms $f : X \longrightarrow X'$ such that

$$\begin{array}{ccc} TX & \xrightarrow{Tf} & TX' \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle h'} \\ X & \xrightarrow[f]{} & X' \end{array}$$

commutes. □

Example 9.3 For every object X of $\mathbb{C}$, the multiplication of the monad $\mu_X : TTX \longrightarrow TX$ is a T-algebra; note that μ_X is also T-algebra homomorphism from $\langle T^2X, \mu_{TX}\rangle$ to $\langle TX, \mu_X\rangle$.

There is an evident forgetful functor

$$U^T : T\text{-}\mathbf{Alg} \longrightarrow \mathbb{C}$$

mapping an algebra $\langle X, h\rangle$ to its carrier X, as in the forgetful functor of monoids. The natural question to ask then is whether this functor has an adjoint. We start by considering whether we can build a functor in the opposite direction at all. Well, we know that $\mu_X : TTX \longrightarrow TX$ is a T-algebra, for every object X; moreover, by naturality of μ, we have that

$$\begin{array}{ccc} T^2X & \xrightarrow{T^2 f} & T^2X' \\ {\scriptstyle \mu_X}\downarrow & & \downarrow{\scriptstyle \mu_{X'}} \\ TX & \xrightarrow[Tf]{} & TX' \end{array}$$

commutes for every map $f : X \longrightarrow X'$. Because T is a functor, this assignments give then rise to a functor

$$F^T : \mathbb{C} \longrightarrow T\text{-}\mathbf{Alg}$$

Theorem 9.4 (*Every monad is defined by its algebras.*)
For every monad $T = \langle T, \eta, \mu\rangle$ on a category $\mathbb{C}$, the above functor F^T is left adjoint to the forgetful functor U^T.

Proof. The unit η^T of the adjunction is simply the unit η of the monad; the value of the counit ε^T at a T-algebra $\langle X, h\rangle$ is simply the structure map $h : F^T U^T\langle X, h\rangle = TX \longrightarrow X$. For such η^T and ε^T the triangular identities are trivial (exercise). □

Exercises

E 9.1 Finish off the proof of Theorem 9.4.

E 9.2 Prove that for every monoid $M = \langle M, e, m\rangle$, the endofunctor

$$\_ \times M : \mathbf{Set} \longrightarrow \mathbf{Set}$$

mapping a set X to the set $X \times M$ is a monad. (*Hint.* Unit and multiplication of the monad are defined in terms of the two operations e and m of the monoid.)

E 9.3 What are the algebras of the monad of the previous exercise?

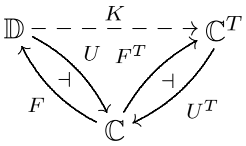
Lecture XIII

9.2 Comparison Functors

An adjunction $\langle F \dashv U, \nu, \varepsilon\rangle$ from $\mathbb{C}$ to $\mathbb{D}$ is a *resolution* for a monad $T = \langle T, \eta, \mu\rangle$ on $\mathbb{C}$ if $T = UF$, $\eta = \nu$, $\mu = U\varepsilon_F$. Clearly, the above adjunction $\langle F^T \dashv U^T, \eta^T, \varepsilon^T\rangle$ is a resolution for the monad T. The following theorem shows that actually it is the final resolution (in a suitable sense):

Theorem 9.5 *Comparison of adjunctions with algebras.*

For every resolution $\langle F \dashv U, \eta, \varepsilon\rangle$ from $\mathbb{C}$ to $\mathbb{D}$ of a monad T there exists a unique *comparison* functor $K : \mathbb{D} \longrightarrow \mathbb{C}^T$ such that $U^T K = U$ and $KF = F^T$:



Proof. The functor K maps an object Y of $\mathbb{D}$ to the morphism $U\varepsilon_Y : TUY \longrightarrow UY$, which is a T-algebra structure because of one triangular identity for the adjunction $F \dashv U$ and naturality of the counit ε. The action of K on arrows is $f \longmapsto Uf$ and this clearly gives a functor. One can easily check that this is the unique functor which satisfies the two equations above. □

The above comparison functor is an isomorphism if and only if the functor U 'creates' suitable coequalisers (*Beck's Theorem* [Mac97, Thm VI.7.1]). This is true for the forgetful functors associated to each category of algebras in the traditional sense (eg, monoids, groups, semilattices, etc), hence the ordinary notion of algebraic variety is encompassed by the notion of algebras of a monad [Mac97, §VI.8]. Try and see as an exercise how the algebras of the free monoid monad $(\_)^*$ look like and what is the isomorphism given by the corresponding comparison functor.

9.3 Free Algebras

The above theorem shows that the category $\mathbb{C}^T$ of T-algebras is final (in a suitable sense) among the resolutions of the monad T. The initial resolution also exists and is of interest. It is given by the category of free T-algebras. Equivalently, for every resolution $F \dashv U$ from $\mathbb{C}$ to $\mathbb{D}$ of T one can consider the category with objects of the form FA and maps simply all the maps $FA \longrightarrow FB$ in $\mathbb{D}$. There is a way to reduce every such representation to a canonical one which does not depend on the choice of the resolution. The trick is to use the bijection of the adjunction to have everything expressed in terms of $\mathbb{C}$ and T, rather than $\mathbb{D}$ and F. Indeed we have that, since $\mathbb{D}(FA, Y) \cong \mathbb{C}(A, UY)$, in particular

$$\mathbb{D}(FA, FB) \cong \mathbb{C}(A, UFB) = \mathbb{C}(A, TB)$$

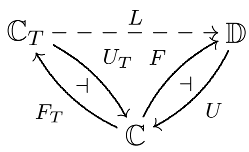
This leads to the following definition.

Definition 9.6 The *Kleisli category* $\mathbb{C}_T$ of a monad $T = \langle T, \eta, \mu \rangle$ on a category $\mathbb{C}$ has objects given by the very same objects of $\mathbb{C}$ and morphisms $A \longrightarrow_T B$ given by morphisms $A \longrightarrow TB$ in $\mathbb{C}$. The identity on an object A is given by the unit of the monad at A, ie $\eta_A : A \longrightarrow TA$, and composition of $f : A \longrightarrow TB$ with $g : B \longrightarrow TC$ is defined ad $\mu_C \circ Tg \circ f : A \longrightarrow TC$. The monad laws ensure that this is indeed a category. $\square$

The following two opposite functors

$$F_T : \mathbb{C} \longrightarrow \mathbb{C}_T \quad (A \xrightarrow{g} B) \longmapsto (A \xrightarrow{g} B \xrightarrow{\eta_B} TB)$$
$$U_T : \mathbb{C}_T \longrightarrow \mathbb{C} \quad (A \xrightarrow{f} TB) \longmapsto (TA \xrightarrow{Tf} T^2B \xrightarrow{\mu_B} TB)$$

give another resolution $F_T \dashv U_T$ of the monad T which is initial:



(Cf [Mac97, §VI.5].)

One can easily show that, in any category $\mathbb{C}$ with final object and coproducts, the endofunctor $1 + \_ : \mathbb{C} \longrightarrow \mathbb{C}$ is a monad (with evident unit and multiplication). For $\mathbb{C} = \mathbf{Set}$, its Kleisli category is the category **pSet** of sets and partial functions, and its category of algebras is the category $\mathbf{Set}_*$ of pointed sets and point-preserving functions; the (super-unique!) comparison functor between the two categories is an isomorphism.

Exercises

E 9.4 Consider the powerset endofunctor on **Set**

$$\mathcal{P}X = \{\alpha \subseteq X\} \qquad \mathcal{P}(f)(\alpha) = \{fx \mid x \in \alpha\}$$

This is a monad, with the singleton function as unit and the 'big union' operation as multiplication:

$$\eta_X : X \longrightarrow \mathcal{P}X \qquad x \longmapsto \{x\}$$
$$\mu_X : \mathcal{P}^2X \longrightarrow \mathcal{P}X \qquad A \longmapsto \bigcup A = \bigcup_{\alpha \in A} \alpha$$

Prove that its algebras are *complete semilattices* and its maps are join-preserving functions.

Next, prove that the Kleisli category of the powerset monad has sets as objects and binary relations as arrows. (*Hint.* Note that a function of type $X \longrightarrow \mathcal{P}Y$ is the same as a relation $R \subseteq X \times Y$.) What is composition?

E 9.5 Read the Introduction of [Mac97].

Lecture XIV

10 Lawvere Theories

The traditional way of defining an algebraic theory is by giving a set of operators and some axioms. For instance, the algebraic theory of groups consists of a multiplication of arity 2, an inverse of arity 1, and a unit of arity 0 together with the following axioms:

$$(x \cdot y) \cdot z = x \cdot (y \cdot z), \qquad x \cdot x^{-1} = e = x^{-1} \cdot x, \qquad x \cdot e = x = e \cdot x$$

A group is then a model (in the sense of first-order equational logic) for this theory, that is, a set G and operations of type $G^2 \longrightarrow G$, $G \longrightarrow G$, $G^0 = 1 \longrightarrow G$ validating the axioms. This, however, is only one of the possible presentations of the theory of groups. Indeed, if we use just one operator t of arity 3 (think of $t(x, y, x)$ as $x \cdot y^{-1} \cdot z$) and the axioms

$$t(x, x, y) = y = t(y, x, x), \qquad t(t(x, u, z), y, v) = t(x, t(y, z, u), v) = t(x, u, t(z, y, v))$$

we are still defining the same theory, in the sense that its models are again exactly groups.

We have seen that monads (and their algebras) allow for a presentation-independent account of algebraic theories, but, as their name suggests, they do not reveal much of the structures giving rise to them. An alternative approach which, while being independent from the choice of presentation, still treats the operators as primitive data is in terms of Lawvere theories, a categorical formulation of the notion of *clones* (ie suitably closed sets of operations) used in universal algebra [Coh81].

Definition 10.1 A *Lawvere theory* is a category $\mathcal{L}$ with finite products and with a distinguished object A such that every other object of $\mathcal{L}$ is a finite power of A:

$$\forall X \in \mathrm{Obj}_{\mathcal{L}}.\ X \cong A^n \text{ (for some } n \in \mathbb{N})$$

The object A is called the *fundamental object* of $\mathcal{L}$. An arrow $\omega : A^n \longrightarrow A$ of $\mathcal{L}$ is called an n-ary operation (and, in particular, arrows of type $A^0 = 1 \longrightarrow A$ are called constants). One can regard operations of arity n as terms with By the universal property of products, we have

$$\mathcal{L}(A^n, A^m) \cong \mathcal{L}(A^n, A)^m \tag{6}$$

that is, every arrow $\alpha : A^n \longrightarrow A^m$ is the tupling of m operations $\alpha_i : A^n \longrightarrow A$ $(i = 1, \ldots, m)$.

Note that every Lawvere theory $\mathcal{L}$ has to contain at least the projections

$$\pi_i^{(n)} : A^n \longrightarrow A$$

since $\mathcal{L}$ has finite products.

Definition 10.2 A *model* (or algebra) of a Lawvere theory $\mathcal{L}$ consists of a product preserving functor

$$\mathcal{M} : \mathcal{L} \longrightarrow \mathbf{Set}$$

(More generally, one can consider models to be product preserving functors $\mathcal{M} : \mathcal{L} \longrightarrow \mathbb{C}$, where $\mathbb{C}$ is any category with finite products.)

The value $\mathcal{M}(A)$ of the model $\mathcal{M}$ at the fundamental object A is the *carrier* of the model. Let us put

$$|\mathcal{M}| \stackrel{\text{def}}{=} \mathcal{M}(A) \tag{7}$$

Since $\mathcal{M}$ preserves finite products, we have

$$\mathcal{M}(A^n) \cong (\mathcal{M}A)^n = |\mathcal{M}|^n$$

thus the image under $\mathcal{M}$ of an n-ary operation α of $\mathcal{L}$ is a function

$$\mathcal{M}\alpha : |\mathcal{M}|^n \longrightarrow |\mathcal{M}|$$

A natural transformation $\theta : \mathcal{M} \Longrightarrow \mathcal{N}$ between two models is easily seen as an algebra homomorphism. Indeed, if

$$\begin{array}{ccc} (\mathcal{M}A)^n & \xrightarrow{(\theta_A)^n} & (\mathcal{N}A)^n \\ {\scriptstyle \mathcal{M}\alpha}\downarrow & & \downarrow{\scriptstyle \mathcal{N}\alpha} \\ \mathcal{M}A & \xrightarrow[\theta_A]{} & \mathcal{N}A \end{array}$$

commutes for every operation $\alpha : A^n \longrightarrow A$ then θ is natural, because every map of $\mathcal{L}$ is given by tupling operations and the models preserve finite products (hence projections).

For every Lawvere theory $\mathcal{L}$ we have thus a category $\mathbf{Mod}(\mathcal{L})$ with $\mathcal{L}$-models as objects and natural transformations between them as arrows. The definition in (7) extends to a (forgetful) functor

$$|-| : \mathbf{Mod}(\mathcal{L}) \longrightarrow \mathbf{Set}$$

This has a (free model) left adjoint

$$F_{\mathcal{L}} : \mathbf{Set} \longrightarrow \mathbf{Mod}(\mathcal{L})$$

For every *finite* set S with n elements,

$$F_{\mathcal{L}}(S) = \mathcal{L}(A^S, \_) \cong \mathcal{L}(A^n, \_) : \mathcal{L} \longrightarrow \mathbf{Set}$$

Thus, the carrier of the free model over a set of n generators is the set $F_{\mathcal{L}}(S)(A) = \mathcal{L}(A^n, A)$ of all n-ary operations of $\mathcal{L}$.

Given a set of operators, one can easily generate the Lawvere theory associated to it: the n-ary operations correspond to terms with (possibly) n variables; in particular, projections correspond to variables. In general, Lawvere theories are in 1-1 correspondence with *finitary* monads on **Set**, for a suitable notion of 'finitary'. The latter are, in turn, in 1-1 correspondence with ordinary algebraic theories. These correspondences extend to models of algebraic theories, algebras of finitary monads on **Set**, and ordinary algebras.

Note. Lawvere theories are not treated in [Mac97]. I used material from [Law73b], [Wra75] and [Bor94]. You might want to have a look at [Coh81] for an account of universal algebra which also contains (in the 1981 edition) a chapter on Category Theory and Universal Algebra, including Lawvere theories.

Exercises

E 10.1 Let $\mathbb{N}^{\text{op}}$ be the opposite of the category of natural numbers and all functions. Show that Lawvere theories are equivalent to product preserving functors

$$\mathbb{N}^{\text{op}} \longrightarrow \mathbb{C}$$

that are bijective on objects.

E 10.2 Consider, for any *set* A, the functor (product with A)

$$\_ \times A : \mathbf{Set} \longrightarrow \mathbf{Set} \qquad X \longmapsto X \times A$$

Prove that it preserve colimits.

Hint. Show that it has a right adjoint, namely the exponential functor

$$(\_)^A : \mathbf{Set} \longrightarrow \mathbf{Set} \qquad X \longmapsto \mathbf{Set}(A, X) \qquad h \longmapsto h \circ (\_)$$

mapping a set X to the set of all *functions* from A to X. The counit $\varepsilon_X : X^A \times A \longrightarrow X$ of the adjunction maps a pair $\langle f, a \rangle$ to $f(a)$, ie it is evaluation.

Lecture XV

11 Cartesian Closed Categories

Recall that we have 'internalised' the notion of element using maps from the final object; that is, we have regarded maps $x : 1 \longrightarrow X$ from 1 to an object X as elements x of X. Now, intuitively, a cartesian closed category is a category which can 'think' of its own arrows as being functions, that is, a category with *internal* notions of function space, evaluation, composition, etc.

For functions, the key idea is that the evaluation of a function $f : A \longrightarrow X$ is a universal arrow from the functor product with A in **Set**

$$\_ \times A : \mathbf{Set} \longrightarrow \mathbf{Set}$$

to X.

Definition 11.1 Let A be an object of a category $\mathbb{C}$ with binary products. The right adjoint of $\_ \times A : \mathbb{C} \longrightarrow \mathbb{C}$, if it exists, is denoted by

$$(\_)^A : \mathbb{C} \longrightarrow \mathbb{C}$$

and is called the **exponential** functor; the corresponding counit is called the **evaluation** map and we denote it by

$$ev^A_X : X^A \times A \longrightarrow X \quad .$$

Note that **Cat** has exponentials, namely functor categories.

One way to denote exponentials X^A is as $A \Rightarrow X$. This stems from a logical reading of the adjunction. Indeed, in a preorder P with meets $\wedge$, if we interpret $\leq$ as logical entailment $\vdash$ (sorry for the overloading, but please do not read this as adjunction but as entailment here!) and $\wedge$ as conjunction, then the above adjunction is nothing but the well-known *deduction theorem*:

$$\frac{a \wedge b \vdash c}{\overline{a \vdash (b \Rightarrow c)}}$$

The exponential X^A gives the desired notion of internal function space. For instance, we can now define an internal notion of composition:

$$\frac{C^B \times B^A \longrightarrow C^A}{\overline{C^B \times B^A \times A \stackrel{id\times ev^A_B}{\longrightarrow} C^B \times B \stackrel{ev^B_C}{\longrightarrow} C}}$$

If a category $\mathbb{C}$ has final object, binary products and exponentials, then we have:

$$\frac{1 \times A \cong A \stackrel{f}{\longrightarrow} B}{\overline{1 \stackrel{\ulcorner f \urcorner}{\longrightarrow} B^A}}$$

The map $\ulcorner f \urcorner$, an internal element of B^A, is called the **name** of f.

Note that we can apply and compose names of functions. For instance:

$$\frac{1 \xrightarrow{\ulcorner f \urcorner} B^A \qquad 1 \xrightarrow{a} A}{1 \xrightarrow{\langle \ulcorner f \urcorner, a\rangle} B^A \times A \xrightarrow{ev_B^A} B}$$

Definition 11.2 A **cartesian closed category** (**CCC**) is a category $\mathbb{C}$ with binary products, exponentials, and terminal object, that is, the following functors have right adjoints:

$$\mathbb{C} \xrightarrow{!} \mathbf{1} \qquad \mathbb{C} \xrightarrow{\Delta} \mathbb{C}^{\cdot\cdot} \qquad \mathbb{C} \xrightarrow{-\times A} \mathbb{C}$$

for every A in $\mathbb{C}$.

11.1 Curry-Howard-Lawvere Isomorphism

I strongly recommend to read Dana Scott's seminal article on the relationship between cartesian closed categories and the λ-calculus [Sco80], both for its form and its content.

Exercises

E 11.1 Prove that in a CCC, just like in arithmetic, the following holds:

1. $1^A \cong 1$
2. $X^1 \cong X$
3. $(X \times Y)^A \cong X^A \times Y^A$
4. $X^{A \times B} \cong (X^A)^B$

Hint: 1 and 3 are trivial, while for 2 and 4 use transpositions and the uniqueness of adjoints.

E 11.2 Recall the contravariant functor

$$\mathbb{C}(\_, A) : \mathbb{C}^{op} \longrightarrow \mathbf{Set}$$

from the naturality exercise (Exercise 7.3.4) on adjunctions. Prove that, for all functors $K : \mathbb{C}^{op} \longrightarrow \mathbf{Set}$, any natural transformation

$$\tau : \mathbb{C}(\_, A) \Longrightarrow K$$

is completely determined by its value at the identity $id_A : A \longrightarrow A$ on A.

Hint. Use the naturality of τ and the fact that we start from a *set* of arrows to prove that

$$\tau_X(f) = (Kf)(\tau_A(id_A))$$

for every arrow $f : X \longrightarrow A$.

Lecture XVI

12 Variable Sets and Yoneda Lemma

First a definition we shall use later: a functor $H : \mathbb{C} \longrightarrow \mathbb{D}$ is *full* (resp. *faithful*) if the function

$$\mathbb{C}(X, Y) \longrightarrow \mathbb{D}(FX, FY) \qquad f \longmapsto Hf$$

is surjective (resp. injective).

We now consider the functor category $\mathbf{Set}^{\mathbb{C}^{op}}$ for a given small category $\mathbb{C}$. Functors $X : \mathbb{C}^{op} \longrightarrow \mathbf{Set}$ are called *presheaves over* $\mathbb{C}$ and can be regarded as sets varying over $\mathbb{C}$ or $\mathbb{C}$-sets: for each object c of $\mathbb{C}$ we have a set Xc – the (variable) set X at *stage* c. The functoriality of X corresponds to a 'right action' of $\mathbb{C}$ on X: for every map $f : c' \longrightarrow c$ in $\mathbb{C}$ we have a function (we are in **Set**!) $Xf : Xc \longrightarrow Xc'$ mapping the elements x of the set X at c to elements $x.f$ of X at stage c'. Using the notation $x.f = X(f)(x)$ we have just introduced we can write the functoriality condition of X as

$$x.(f \circ g) = (x.f).g \quad x.id_c = x$$

for every $g : c'' \longrightarrow c'$, which explains why we can speak of a right action of $\mathbb{C}$ over X. The elements x of Xc are also called elements of *sort* c.

A canonical example of variable set is given by the *representable functor*

$$\mathbf{y}_{\mathbb{C}}(c) \stackrel{\text{def}}{=} \mathbb{C}(\_, c) : \mathbb{C}^{op} \longrightarrow \mathbf{Set}$$

for every object c in $\mathbb{C}$. (Cf the naturality exercise (Exercise 7.3.4) on adjunctions.) The following proposition tells us that in fact the map

$$\mathbf{y}_{\mathbb{C}} : \mathbb{C} \longrightarrow \mathbf{Set}^{\mathbb{C}^{op}} \qquad c \longmapsto \mathbb{C}(\_, c)$$

is a *full and faithful* functor embedding $\mathbb{C}$ into $\mathbf{Set}^{\mathbb{C}^{op}}$.

Proposition 12.1 For all objects c and d in $\mathbb{C}$, every natural transformation from $\mathbb{C}(\_, c)$ to $\mathbb{C}(\_, d)$ is of the form

$$\mathbb{C}(\_, h) : \mathbb{C}(\_, c) \Longrightarrow \mathbb{C}(\_, d)$$

for some map $h : c \longrightarrow d$ in $\mathbb{C}$. □

The proof of this fact, in turn, is a corollary of the following lemma, due to the Japanese categorist Yoneda (and so the above embedding $\mathbf{y}_{\mathbb{C}}$ is called the *Yoneda embedding* of $\mathbb{C}$ into the category of presheaves over $\mathbb{C}$).

Lemma 12.2 (*Yoneda.*) For all presheaves $X : \mathbb{C}^{op} \longrightarrow \mathbf{Set}$, there is a bijection (in **Set**!) between the set of natural transformations from $\mathbf{y}_{\mathbb{C}}(c) = \mathbb{C}(\_, c)$ to a variable set X and the elements of X at the stage c:

$$\mathbf{Set}^{\mathbb{C}^{op}}(\mathbf{y}_{\mathbb{C}}(c), X) \cong Xc$$

Proof. First note that any natural transformation

$$\tau : \mathbb{C}(\_, c) \Longrightarrow X$$

is completely determined by its value at the identity $id_c : c \longrightarrow c$ on c. Indeed, the naturality of τ implies that the diagram

$$\begin{array}{ccc} c & \qquad & \mathbb{C}(c,c) & \xrightarrow{\tau_c} & Xc \\ {\scriptstyle f}\big\uparrow & & {\scriptstyle (\_)\circ f}\big\downarrow & & \big\downarrow{\scriptstyle (\_).f} \\ c' & & \mathbb{C}(c',c) & \xrightarrow[\tau_{c'}]{} & Xc' \end{array}$$

commutes for every map $f : c' \longrightarrow c$. This is a diagram of functions, hence $\tau_{c'}(e \circ f) = \tau_c(e).f$ for all $e \in \mathbb{C}(c, c)$, ie for all endomaps $e : c \longrightarrow c$ on c. In particular, we can take $e = id_c$ and obtain

$$\tau_{c'}(f) = \tau_c(id_c).f$$

Now, clearly, the desired bijection can be obtained by mapping a natural transformation $\tau : \mathbf{y}_{\mathbb{C}}(c) \Longrightarrow X$ to $\tau_c(id_c) \in Xc$ and, conversely, an element x of Xc to the natural transformation $(f : c' \longrightarrow c) \longmapsto x.f$. □

I learned to look at presheaves as $\mathbb{C}$-sets from [Law89]. I would like now to quote the following passage from that paper which beautifully explains what the Yoneda lemma really tells us:

> For any $\mathbb{C}$-set X and any object a of $\mathbb{C}$, the set of elements of X of sort a is naturally identifiable with the set of $\mathbf{Set}^{\mathbb{C}^{op}}$-morphisms from $\mathbb{C}(\_, a)$ to X. It is thus justified, as well as extremely useful, to adjoin to the parenthetically-introduced abuse still a further abuse of notation and to henceforth regard the elements of X of sort a as morphisms $a \longrightarrow X$ in $\mathbf{Set}^{\mathbb{C}^{op}}$; thus the action of $\mathbb{C}$ on any $\mathbb{C}$-set becomes a special case of composition of morphisms
>
> $$\begin{array}{ccc} c' & \xrightarrow{f} & c \\ & {\scriptstyle xf}\searrow & \big\downarrow{\scriptstyle x} \\ & & X \end{array}$$
>
> now all in $\mathbf{Set}^{\mathbb{C}^{op}}$ as does the application of a morphism φ to an element, and the homogeneity (naturality) property of every $X \longrightarrow Y$ in $\mathbf{Set}^{\mathbb{C}^{op}}$ becomes a special case of the associativity of composition in $\mathbf{Set}^{\mathbb{C}^{op}}$:
>
> $$\begin{array}{cccc} c' & \xrightarrow{f} & c & \\ & {\scriptstyle xf}\searrow & \big\downarrow{\scriptstyle x} & \searrow{\scriptstyle \varphi x} \\ & & X & \xrightarrow[\varphi]{} & Y \end{array}$$

Proposition 12.3 For every small category $\mathbb{C}$, the category $\mathbf{Set}^{\mathbb{C}^{op}}$ is cartesian closed.

Proof. Assume the exponential X^Y exists for two presheaves X and Y over $\mathbb{C}$. Then, by Yoneda,

$$X^Y(c) \cong \mathbf{Set}^{\mathbb{C}^{op}}(\mathbf{y}_{\mathbb{C}}(c) \times Y, X) \tag{8}$$

for every object c in $\mathbb{C}$. For the rest of the proof see, eg, [MM92, Prop. I.6.1, pg 46]. □

Note that there is also a dual Yoneda embedding, namely

$$\mathbb{C}^{op} \longrightarrow \mathbf{Set}^{\mathbb{C}} \qquad c \longmapsto \mathbb{C}(c, \_)$$

and a corresponding lemma.

Not surprisingly, the Yoneda embedding $\mathbf{y}_{\mathbb{C}}$ has a universal property. First note that $\mathbf{Set}^{\mathbb{C}^{op}}$ pointwisely inherits all limits and colimits from **Set** thus, in particular it is cocomplete.

Proposition 12.4 For every functor H from $\mathbb{C}$ to any *cocomplete* category $\mathbb{E}$ there exists a unique cocontinuous (ie colimit preserving) functor $H^\sharp : \mathbf{Set}^{\mathbb{C}^{op}} \longrightarrow \mathbf{Set}$ such that $H^\sharp \circ \mathbf{y}_{\mathbb{C}} = H$.

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{\mathbf{y}_{\mathbb{C}}} & \mathbf{Set}^{\mathbb{C}^{op}} \\ & \searrow^{H} & \downarrow H^\sharp \quad \text{cocontinuous} \\ & & \mathbb{E} \end{array}$$

Proof. It is a consequence of the fact that every $\mathbb{C}$-set X is a (canonic) colimit of a diagram of representable functors. (See, eg, [Mac97, §III.7].) □

Thus $\mathbf{Set}^{\mathbb{C}^{op}}$ is the free cocomplete category over $\mathbb{C}$ and the fact that $\mathbf{y}_{\mathbb{C}}$ is full and faithful implies that even if $\mathbb{C}$ may miss some or all colimits, we can always regard it as part of the larger category $\mathbf{Set}^{\mathbb{C}^{op}}$ which does have all colimits. Note that this almost gives us a left adjoint to the forgetful functor from the category of cocomplete categories and cocontinuous functors to the category of all categories, except from the fact that we would need a Yoneda embedding not only for small but also for large categories (and in that case $\mathbf{y}_{\mathbb{C}}$ would be the universal arrow from $\mathbb{C}$ to such forgetful functor).

Exercise

E 12.1 (*Important.*) Let 2 be the two-elements set $\{\mathsf{true}, \mathsf{false}\}$. For every subset S of a given set X we have a corresponding *characteristic function*:

$$\phi_S : X \longrightarrow 2 \qquad \phi_S(x) = \begin{cases} \mathsf{true} & \text{if } x \in S \\ \mathsf{false} & \text{otherwise} \end{cases}$$

Prove that

$$\begin{array}{ccc} S & \longrightarrow & 1 \\ \downarrow & & \downarrow \mathsf{true} \\ X & \xrightarrow[\phi_S]{} & 2 \end{array}$$

is a pullback square (where $S \longrightarrow X$ is the evident inclusion function and $S \longrightarrow 1$ is the unique arrow from S into the terminal object 1).

Lecture XVII

13 Set theory without sets

13.1 Subobject Classifiers

Two monic arrows $m : S \rightarrowtail X$ and $m' : S' \rightarrowtail X$ with a common codomain are *equivalent* if there is an isomorphism $f : S \cong S'$ such that $m' \circ f = m$. A *subobject of* X is an equivalence class of monic arrows into X. Write $\mathrm{Sub}(X)$ for the set of all subobjects of X. If the category has pullbacks then this extends to a functor $\mathrm{Sub} : \mathbb{C}^{op} \longrightarrow \mathbf{Set}$. In the sequel we shall identify subobjects with the monic arrows representing them.

Recall that a category with terminal object and pullbacks has finite limits.

Definition 13.1 A *subobject classifier* for a category $\mathbb{C}$ with finite limits consists of an object Ω (of $\mathbb{C}$) and a monic arrow $\mathsf{true} : 1 \rightarrowtail \Omega$ universal in the sense that for every monic $S \rightarrowtail X$ there exists a unique arrow $\phi_S : X \longrightarrow \Omega$ such that

$$\begin{array}{ccc} S & \longrightarrow & 1 \\ \downarrow & & \downarrow \mathsf{true} \\ X & \underset{\phi_S}{\dashrightarrow} & \Omega \end{array}$$

is a pullback square.

In other words there is an isomorphism

$$\mathrm{Sub}(X) \cong \mathbb{C}(X, \Omega)$$

$$\frac{S \rightarrowtail X}{X \longrightarrow \Omega}$$

natural in X.

An example is clearly 2 in **Set**. But also sets over time $\boldsymbol{X : \omega} \longrightarrow \mathbf{Set}$ have a subobject classifier which gives "time till truth": it is the constant presheaf

$$\mathbb{N}_\infty \xrightarrow{p} \mathbb{N}_\infty \xrightarrow{p} \mathbb{N}_\infty \xrightarrow{p} \cdots$$

where $\mathbb{N}_\infty$ is the set of natural numbers with infinity and p is the predecessor function (mapping $n + 1$ to n, while leaving 0 and ∞ unchanged). Then 0 is true, n is 'n steps till true', and ∞ is 'never true'. In general:

Proposition 13.2 The category $\mathbf{Set}^{\mathbb{C}^{op}}$ of presheaves over a small category $\mathbb{C}$ has a subobject classifier.

Proof. See, eg, [MM92, §I.4]. □

13.2 Topoi

If a category is cartesian closed and has a subobject classifier then one can define a *power object*

$$PX \stackrel{\text{def}}{=} \Omega^X$$

for every object X of $\mathbb{C}$. Note that then:

$$\dfrac{\dfrac{S \rightarrowtail X}{X \longrightarrow \Omega}}{1 \longrightarrow PX}$$

In the case $\mathbb{C} = \mathbf{Set}$, PX is the powerset of X and the above correspondences boil down to the fact that subsets are the same as predicates and the same as elements of powerset. Note also that the evaluation map $X \times \Omega^X \longrightarrow \Omega$ is nothing but the element predicate $X \times PX \longrightarrow \Omega$.

Definition 13.3 An **elementary topos** is a cartesian closed category $\mathbb{E}$ with all finite limits and a subobject classifier.

Examples: the category **Set** of ordinary sets, but also all categories $\mathbf{Set}^{\mathbb{C}^{op}}$ of *variable* sets.

Exercises

E 13.1 We know from Proposition 13.2 that every category of presheaves has a subobject classifier Ω. Use Yoneda to show that

$$\Omega(c) \cong \mathrm{Sub}(\mathbf{y}_{\mathbb{C}}(c))$$

for every object c of $\mathbb{C}$.

E 13.2 *Nice – it gives a very powerful notion!*

For any category $\mathbb{E}$, a functor

$$\mathbb{C} \xrightarrow{J} \mathbb{D}$$

induces a functor

$$\mathbb{E}^{\mathbb{C}} \xleftarrow{(\_)\circ J} \mathbb{E}^{\mathbb{D}}$$

between the functor categories $\mathbb{E}^{\mathbb{D}}$ and $\mathbb{E}^{\mathbb{C}}$; its action on an object H of $\mathbb{E}^{\mathbb{D}}$, ie on a functor $H : \mathbb{D} \longrightarrow \mathbb{E}$, is simply precomposition with J:

$$\mathbb{C} \xrightarrow{J} \mathbb{D} \xrightarrow{H} \mathbb{E}$$

The question is: how does a universal arrow from a generic object G of $\mathbb{E}^{\mathbb{C}}$ to the functor $(\_) \circ J : \mathbb{E}^{\mathbb{D}} \longrightarrow \mathbb{E}^{\mathbb{C}}$ look like? It is a pair (L, η_G) with $L : \mathbb{D} \longrightarrow \mathbb{E}$ and $\eta_G : G \Longrightarrow LJ$ such that ...?

Hint. Beware that the difficulty is in the fact that our objects are functors and our maps are natural transformations. The diagram I would like to see should have the functors written as arrows rather than as objects; the fact that composition with J is a simple (meta) functor makes it possible to express the universal property of the universal arrow in not too complicated a diagram, although you need three dimensions.

Lecture XVIII

14 Kan Extensions

The first edition of [Mac71] ends with a section entitled '*All concepts are Kan extensions*'. Here we introduce Kan extensions using, once more, universal arrows. In fact, (left) Kan extensions are the answer to Exercise 13.2:

Definition 14.1 A *left Kan extension* of a functor $G : \mathbb{C} \longrightarrow \mathbb{E}$ along a functor $J : \mathbb{C} \longrightarrow \mathbb{D}$ is a universal arrow $\langle \mathrm{Lan}_J G, \alpha \rangle$ from G (regarded as an object of the functor category $\mathbb{E}^{\mathbb{C}}$) to

$$\mathbb{E}^{\mathbb{C}} \xleftarrow{(\_)\circ J} \mathbb{E}^{\mathbb{D}}$$

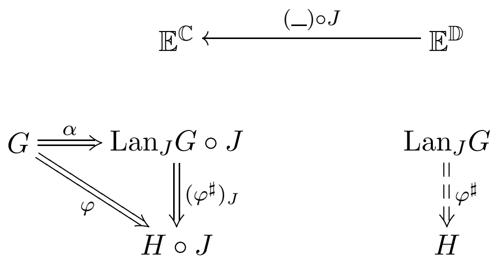
Thus it consists of a functor

$$\mathrm{Lan}_J G : \mathbb{D} \longrightarrow \mathbb{E}$$

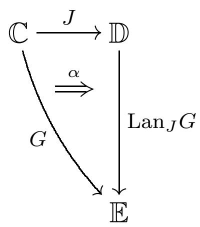
and a natural transformation

$$\alpha : G \Longrightarrow \mathrm{Lan}_J G \circ J$$

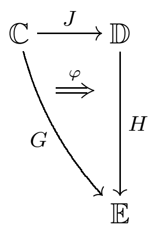
which are universal in the sense that for every functor $H : \mathbb{D} \longrightarrow \mathbb{E}$ and every natural transformation $\varphi : G \Longrightarrow H \circ J$ there exists a unique natural transformation $\varphi^\sharp : \mathrm{Lan}_J G \Longrightarrow H$ such that $\varphi = (\varphi^\sharp)_J \circ \alpha$.



More elementary, the left Kan extension of G along J consists of a diagram



such that for all diagrams



there exists a unique $\varphi^\sharp : \mathrm{Lan}_J G \Longrightarrow H$ such that $\varphi = (\varphi^\sharp)_J \circ \alpha$.

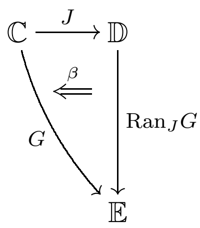
Alternatively, we can state this in terms of the following bijection natural in H:

$$\frac{\mathrm{Lan}_J G \Longrightarrow H}{G \Longrightarrow HJ}$$

Dually, a *right Kan extension* of G along J is a universal arrow $\langle \mathrm{Ran}_J G, \beta \rangle$ from

$$\mathbb{E}^{\mathbb{C}} \xleftarrow{(\_)\circ J} \mathbb{E}^{\mathbb{D}}$$

to G:



$$\frac{HJ \Longrightarrow G}{H \Longrightarrow \mathrm{Ran}_J G}$$

Proposition 14.2 Colimits are left Kan extensions.

Proof. First note that objects and arrows of a category $\mathbb{C}$ are in 1-1 correspondence with functors of type $\mathbf{1} \longrightarrow \mathbb{C}$ and natural transformations between them, respectively. Next, note that a cone from a functor $F : \mathbb{B} \longrightarrow \mathbb{C}$ to an object $\boldsymbol{X : 1} \longrightarrow \mathbb{C}$ is a natural transformation from F to $X \circ !$, where ! is the unique functor from $\mathbb{B}$ to the terminal category 1. It is then clear that:

$$\mathit{Colim} F \cong \mathrm{Lan}_! F$$

and the colimiting cone is the natural transformation of the Kan extension. □

Dually, limits are right Kan extensions.

Proposition 14.3 A functor has a right adjoint if and only if the left Kan extension of the identity along F exists and is preserved by F.

Proof. See [Mac97, Thm X.7.2]. □

Further examples: essential geometric morphisms of presheaves and of monoid actions in particular.

15 2-Categories

See [Mac97, §XII.3].

Further Reading

The aim of this course was to learn to reason in a categorical way. I hope we have achieved this. Now you should be in the position to easily access Mac Lane's book [Mac71]. In particular, Chapters IV, V, and VI will strengthen your understanding of adjunctions, limits, and monads, Chapter VII will introduce you to monoidal categories, and Chapter X to Kan extensions. The new edition [Mac97] also contains important material on topos theory, 2-categories, bicategories, and presheaves. Beyond that, I would recommend categorical logic and fibrations [Jac99, Pho92], enriched category theory [Law73a], and any further writing by Lawvere such as [Law70, Law69, Law91].

Links

- <http://www.mta.ca/∼cat-dist/categories.html>
- <http://www.acsu.buffalo.edu/∼wlawvere/>

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