Math. Struct. in Comp. Science (1993), *vol.* 11, *pp.* 1–000

Order-Enrichment for Categories of Partial Maps

Marcelo P. Fiore†

Department of Computer Science,

Laboratory for Foundations of Computer Science,

University of Edinburgh, The King's Buildings,

Edinburgh EH9 3JZ, Scotland.

<mf@dcs.ed.ac.uk>

Received October 6, 1994

Synopsis

Motivated by the desire of treating non-termination directly in the semantics of computation, the notion of approximation between programs is studied in the context of categories of partial maps. In particular, contextual approximation and specialisation are considered and shown to coincide. Moreover, after exhibiting the approximation between total maps as a primitive notion, from an arbitrary (or axiomatic) approximation order on total maps a computationally natural approximation order on partial maps is derived. The main technical contribution is a characterisation of when this approximation order between partial maps is domain-theoretic (in the sense that the category of partial maps **Cpo**-enriches) provided that so is the approximation order between total maps.

1. Introduction

In (Plotkin 1985) a revitalised approach to domain theory was initiated. Roughly, the idea was to eliminate the bottom from the domains and to keep the functions partially defined. Thus replacing **Cppo** (the category of small cppos —posets with a least element and closed under lubs of ω-chains— and continuous functions) with **pCpo** (the category of small cpos —posets closed under lubs of ω-chains— and partial continuous functions — see Subsection 3.1). One important point in the reformulation is the recognition of **pCpo**

† Research partially supported by Fundación Antorchas and The British Council grant ARG 2281/14/6, and SERC grant RR30735.

as a category of partial maps as, for example, such presentation fits better with standard formulations of recursion theory and it allows a categorical description of data types (via partial cartesian closed categories (Longo and Moggi 1984) with finite coproducts) in the presence of fixed-point operators.

Following the main motivation of the reformulation (i.e. to provide a direct treatment of non-termination in the semantics of computation) we postulate categories of partial maps as the primitive universe in which to interpret possibly non-terminating programs. Thus we study the interaction between partiality and approximation, which constitutes the starting point of our programme, initiated in (Fiore 1994; Fiore and Plotkin 1994), towards an *axiomatic* theory of domains.

In Section 2 the basic notions of order-enriched category theory used throughout are reviewed. In Section 3, categorical structures suitable for describing partial maps, viz. *domain structures* $(\mathcal{K}, \mathcal{D})$, are introduced and their induced *categories of partial maps* $p(\mathcal{K}, \mathcal{D})$ are defined. Examples are included. (For a thorough survey on categories of partial maps consult (Robinson and Rosolini 1988)). In Section 4 we observe that the consideration of partiality provides notions of approximation between partial maps. In particular, *contextual approximation* and *specialisation* are defined and seen to coincide. In these cases, the approximation relation between partial maps can be characterised in terms of the approximation relation between total maps and testing for totality. This exhibits the notion of approximation for partial maps as derived and, in Section 5, is used to derive a computationally natural approximation order for partial maps ($\precsim$) from an arbitrary approximation order between total maps ($\sqsubseteq$). Sections 6 and 7 are devoted to the problem of when $\precsim$ is domain-theoretic provided that so is $\sqsubseteq$. In particular, our main technical result (Theorem 7.1) answers by characterising those categories of partial maps $p(\mathcal{D}, \mathcal{K})$ which are **Cpo**-enriched with respect to $\precsim$ provided that the category of total maps $\mathcal{K}$ is **Cpo**-enriched with respect to $\sqsubseteq$ and that the domain structure $(\mathcal{D}, \mathcal{K})$ is *uniform*. Intuitively, the uniformity axiom makes least upper bounds of ω-chains of admissible subobjects (i.e. domains of definition of partial maps) behave like unions. The conceptual import of the characterisation, see (Fiore 1994, Chapter 10, Section 4), is the insight that provides to identify families (of admissible monos) which, when $\mathcal{K}$ and $p(\mathcal{D}, \mathcal{K})$ are **Cpo**-enriched with respect to $\sqsubseteq$ and $\precsim$ respectively and $(\mathcal{D}, \mathcal{K})$ is uniform, generate a subcanonical Grothendieck topology on $\mathcal{K}$. Thus providing embeddings of small categories of domains in sheaf toposes.

2. Basic Order-Enriched Category Theory

We introduce the basic concepts of order-enrich category theory that will be needed in the sequel. (For a thorough treatment of enriched category theory consult (Kelly 1982).)

Let **Preo**/**Poset** (**Cpo**) be the category of small preorders/posets (cpos) and monotone (continuous) functions.

A **Preo**-*category*/**Poset**-*category* (**Cpo**-*category*) is a category whose hom-sets come equipped with a preorder/partial order (complete partial order) with respect to which composition of morphisms is a monotone (continuous) operation.

Let $\mathcal{V}$ denote either **Poset** or **Cpo**.

The immediate examples of **Poset**-categories are **Poset** and **Cpo** where each hom-set is ordered pointwise; even more, **Cpo** is a **Cpo**-category. More examples are obtained from the observation that every category $\mathcal{K}$ *freely* induces a $\mathcal{V}$-category $\mathcal{K}_{\mathcal{V}}$ by discretely ordering each hom-set.

For $\mathcal{V}$-categories $\mathcal{K}$ and $\mathcal{L}$, a $\mathcal{V}$-*functor* $F : \mathcal{K} \to \mathcal{L}$ is a functorial mapping such that for every $A, B \in |\mathcal{K}|$ the assignment $F_{A,B} : \mathcal{K}(A, B) \to \mathcal{L}(FA, FB)$ is a morphism in $\mathcal{V}$.

Convention. To ease the notation we generally contravene some conventions of the enriched-category theorist. Particularly, we write $\mathcal{K}$ both for a $\mathcal{V}$-category as well as for its underlying category $\mathcal{K}_0$ where the order structure has been forgotten. (This is in the vein of the usual convention of using the same notation for a poset or cpo as well as for its underlying set.) A similar convention is adopted for $\mathcal{V}$-functors.

A $\mathcal{V}$-*adjunction* $\chi : F \dashv U : \mathcal{K} \to \mathcal{L}$ is given by $\mathcal{V}$-categories $\mathcal{K}$ and $\mathcal{L}$, $\mathcal{V}$-functors $F : \mathcal{L} \to \mathcal{K}$ and $U : \mathcal{K} \to \mathcal{L}$, and a natural isomorphism

$$\chi : \mathcal{K}(F\_, =) \cong \mathcal{L}(\_, U=) : \mathcal{L}^{op} \times \mathcal{K} \to \mathcal{V}.$$

For a category $\mathcal{G}$ and a $\mathcal{V}$-category $\mathcal{K}$, a diagram Γ in $\mathcal{K}$ of type $\mathcal{G}$ is a $\mathcal{V}$-functor $\mathcal{G}_{\mathcal{V}} \to \mathcal{K}$. We have a "constant diagram" $\mathcal{V}$-functor $K : \mathcal{K} \to [\mathcal{G}_{\mathcal{V}}, \mathcal{K}]$ defined as the composite $\mathcal{K} \cong [\mathbf{1}, \mathcal{K}] \xrightarrow{[U,\mathrm{Id}]} [\mathcal{G}_{\mathcal{V}}, \mathcal{K}]$ where U is the unique $\mathcal{V}$-functor $\mathcal{G}_{\mathcal{V}} \to \mathbf{1}$.

A $\mathcal{V}$-*colimit* of a diagram Γ is given by an object C of $\mathcal{K}$ and a natural transformation $\gamma : \Gamma \stackrel{.}{\to} KC : \mathcal{G}_{\mathcal{V}} \to \mathcal{K}$ such that for every $A \in |\mathcal{K}|$,

$$[\mathcal{G}_{\mathcal{V}}, \mathcal{K}](\gamma, \mathrm{id}) \circ K_{C,A} : \mathcal{K}(C, A) \to [\mathcal{G}_{\mathcal{V}}, \mathcal{K}](\Gamma, KA)$$

is an isomorphism in $\mathcal{V}$.

Remark. The treatment of $\mathcal{V}$-colimits offered above is far from been complete and just concerns a restricted class of *conical* colimits (Kelly 1982).

Proposition 2.1. Let $\mathcal{G}$ be a category, $\mathcal{K}$ be a **Cpo**-category, and Γ be a diagram in $\mathcal{K}$ of type $\mathcal{G}$. Then, the following are equivalent:

1 (C, γ) is a **Cpo**-colimit of Γ.

2 (C, γ) is a **Poset**-colimit of Γ.

3 (C, γ) is a colimit of Γ and for every $f, g : C \to A$, if $f \circ \gamma \le g \circ \gamma$ then $f \le g$. $\square$

3. Categories of Partial Maps

3.1. Categorical Setting

To motivate the definition of a partial map, observe that a partial function $u : A \rightharpoonup B$ is determined by its domain of definition $\mathrm{dom}(u) \subseteq A$ and the total function $\mathrm{dom}(u) \to B$ induced by the mapping $a \mapsto u(a)$. Thus, every partial function $A \rightharpoonup B$ can be described by a pair consisting of an injection $D \rightarrowtail A$ and a total function $D \to B$ with the same source. These *descriptions* are not unique, for (m, f) and (n, g) describe the same partial function if and only if

$$\text{for some isomorphism } i,\ m = n \circ i \text{ and } f = g \circ i. \tag{1}$$

Following the above reformulation, given a category whose morphisms are to be thought of as total maps, a pair $(m : D \rightarrowtail A, f : D \to B)$ will be a description of a partial map $[m, f] : A \rightharpoonup B$; and conversely, every partial map $A \rightharpoonup B$ will be described by some pair $(D \rightarrowtail A, D \to B)$. Formally, the descriptions (m, f) and (n, g) are regarded as equivalent whenever (1) holds; and a partial map is defined to be an equivalence class of descriptions.

When working with partial maps it is usual just to deal with their descriptions and observe that the arguments or constructions involved are independent of the chosen descriptions. For example, in a category with enough pullbacks, the composition of the partial map $v = [n, g] : B \rightharpoonup C$ with the partial map $u = [m, f] : A \rightharpoonup B$ is defined to be the partial map $v \circ u = [m \circ f^{-1}(n), g \circ (n^* f)] : A \rightharpoonup C$ where

$$\begin{tikzcd} {} \arrow[r, "n^*f"] \arrow[d, tail, "f^{-1}(n)"'] \arrow[dr, phantom, "\lrcorner", very near start] & {} \arrow[d, tail, "n"] \\ {} \arrow[r, "f"'] & {} \end{tikzcd} \tag{2}$$

is a pullback.

The suspicious reader should check that the above definition is independent of any choice of descriptions for the partial maps, as well as for the required pullback. It is also easy to see that composition is associative, and that the partial map $[\mathrm{id}, \mathrm{id}]$ is a unit for it.

Before defining categories of partial maps it will be necessary to refine the notion of partial map as the following examples show. Let **pPoset** be the category whose objects are posets and whose morphisms $u : P \rightharpoonup Q$ are partial monotone functions, in the sense that, for every $x, x' \in P$,

$$\text{if } x \sqsubseteq x' \text{ then } u(x) \precsim u(x') \tag{3}$$

where, for two mathematical expressions e and e', we write $e \precsim e'$ to mean that if e denotes some $y \in Q$ then e' denotes some $y' \in Q$ and $y \sqsubseteq y'$. We write $e \simeq e'$ whenever both $e \precsim e'$ and $e' \precsim e$. Further let **pCpo** (Plotkin 1985) be the subcategory of **pPoset** whose objects are cpos and whose morphisms $u : P \rightharpoonup Q$ are partial continuous functions, in the sense that, for every ω-chain $\langle x_k \rangle$ in P,

$$\bigsqcup u(x_k) \quad \simeq \quad u(\bigsqcup_{k \in |\, \omega\, |} x_k) \tag{4}$$

where the expression on the left is intended to be undefined if every $u(x_k)$ is; or, otherwise, denote $\bigsqcup_{k_0 \leq k \in |\, \omega\, |} u(x_k)$ where k_0 is any index for which $u(x_{k_0})$ is defined. In these two categories, the domain of definition of a morphism $P \rightharpoonup Q$ cannot be *any* subobject of P: in **pPoset**, by (3), the only possible domains of definitions are the *upper-closed* subsets of P (i.e. those $D \subseteq P$ such that for every $x \in D$ and $y \in P$, $x \leq y$ implies $y \in D$), whilst in **pCpo**, by (4), they are the *Scott-open* subsets of P (i.e. those $D \subseteq P$ which are upper-closed and inaccessible by lubs of ω-chains, in the sense that for every ω-chain $\langle x_k \rangle$ in P, $\bigsqcup x_k \in D$ implies $x_{k_0} \in D$ for some k_0).

To include such a possibility, descriptions of partial maps are to be pairs (m, f) where m is confined to a class of *admissible* monos. Such a class together with the category of total maps constitute a *domain structure*.

Definition 3.1. (Rosolini 1986; Moggi 1986) A *domain structure* is a pair $(\mathcal{K}, \mathcal{D})$ consisting of a well-powered category $\mathcal{D}$ and a category $\mathcal{K}$, such that $\mathcal{D}$ is a full-on-objects subcategory of $\mathcal{K}$ all of whose morphisms are monos in $\mathcal{K}$, with the following closure property: every diagram $\xrightarrow{f}\overset{n}{\leftarrowtail}$ with $f \in \mathcal{K}$ and $n \in \mathcal{D}$ has a pullback in $\mathcal{K}$ and whenever

$$\begin{tikzcd}
\cdot \arrow[r] \arrow[d, tail, "f^{-1}(n)"'] \arrow[dr, phantom, "\lrcorner", very near start] & \cdot \arrow[d, tail, "n"] \\
\cdot \arrow[r, "f"'] & \cdot
\end{tikzcd}$$

then $f^{-1}(n) \in \mathcal{D}$. $\square$

Convention. $\mathcal{K}$ is called the *category of total maps* and $\mathcal{D}$ is called the category of *admissible* monos. For every $A \in |\, \mathcal{D}\, |$, $\mathcal{D}(A)$ will denote the skeleton of the slice category $\mathcal{D}/A$. As $\mathcal{D}$ is well-powered, $|\, \mathcal{D}(A)\, |$ is a small set and hence $\mathcal{D}(A)$ is a small poset.

Categories of admissible monos are closed under isomorphisms, for every isomorphism i appears in the pullback

$$\begin{tikzcd} \cdot \arrow[r, "i"] \arrow[d, "i"'] \arrow[dr, phantom, "\lrcorner", very near start] & \cdot \arrow[d, "\mathrm{id}"] \\ \cdot \arrow[r, "\mathrm{id}"'] & \cdot \end{tikzcd} .$$

For monos m and n, we write $[m] \subseteq_i [n]$ (resp. $m \subseteq_i n$) and say that i realises the inequality $[m] \subseteq [n]$ (resp. $m \subseteq n$) whenever $m = n \circ i$. Realisers of inequalities between admissible subobjects are admissible, for consider the equivalence:

$$[m] \subseteq_i [n] \quad \Longleftrightarrow \quad \begin{tikzcd} \cdot \arrow[r, tail, "i"] \arrow[d, tail, "\mathrm{id}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \cdot \arrow[d, tail, "n"] \\ \cdot \arrow[r, tail, "m"'] & \cdot \end{tikzcd} .$$

Definition 3.2. The *category of partial maps* $p(\mathcal{K}, \mathcal{D})$ induced by a domain structure $(\mathcal{K}, \mathcal{D})$ has the same objects as $\mathcal{K}$, a morphism is an equivalence class of descriptions in $\mathcal{D} \times \mathcal{K}$, and composition and identities are given as detailed above. $\square$

Convention. Whenever we write $p(\mathcal{K}, \mathcal{D})$ it is assumed that $(\mathcal{K}, \mathcal{D})$ is a domain structure; when $\mathcal{D}$ is clear from the context we simply write $p\mathcal{K}$.

To distinguish total maps (i.e. morphisms in $\mathcal{K}$) from partial maps (i.e. morphisms in $p\mathcal{K}$) we denote the former with $\rightarrow$ and the latter with $\rightharpoonup$.

The category of total maps can be viewed as a full-on-objects subcategory of the category of partial maps by identifying $\mathcal{K}$ with its image under the faithful *inclusion* functor $J : \mathcal{K} \rightarrow p\mathcal{K}$ which acts as the identity on objects and sends a total map f to the partial map $[\mathrm{id}, f]$. It should then be clear what is meant by a *total* partial map and by the composition of a total map with a partial map. To indicate that a partial map u is total we write $u \downarrow$. Totality and composition interact in the desirable way: for partial maps u and v, $(v \circ u) \downarrow \Rightarrow u \downarrow$.

Definition 3.3. When the category of total maps has a terminal object 1, we say that totality is *extensional* if for all $u : A \rightharpoonup B$, $u \downarrow \Leftrightarrow \forall\, x : 1 \rightarrow A.\ (u \circ x) \downarrow$. $\square$

Every admissible mono $m : D \rightarrowtail A$ induces a *partial identity* on A, viz. the partial map $[m, m] : A \rightharpoonup A$. There is a *partial inclusion* functor $(\_)^R : \mathcal{D}^{op} \rightarrow p\mathcal{K}$ which acts as the identity on objects and sends $m : D \rightarrowtail A$ in $\mathcal{D}$ to $m^R = [m, \mathrm{id}] : A \rightharpoonup D$. Partial inclusions have nice equational properties, in particular

$$\begin{aligned} m^R = n^R &\iff [m] = [n], \\ m^R \circ m &= \mathrm{id}, && (5) \\ [m, f] &= f \circ m^R. && (6) \end{aligned}$$

While (5) justifies the terminology, (6) shows that every partial map can be factored as a partial inclusion followed by a total map. In fact, as observed by John Power, this determines a *factorisation system* (Freyd and Kelly 1972). Partial inclusions also provide a simple equational characterisation for the pullbacks we are interested in, as

$$\begin{tikzcd} \cdot \arrow[r, "g"] \arrow[d, tail, "m"'] \arrow[dr, phantom, "\lrcorner", very near start] & \cdot \arrow[d, tail, "n"] \\ \cdot \arrow[r, "f"'] & \cdot \end{tikzcd} \qquad \iff \qquad n^R \circ f = g \circ m^R.$$

The adopted description of partial maps allows the definition of the *inverse image*, $u^{-1}[n]$, of an admissible subobject $[n] \in \mathcal{D}(B)$ along a partial map $u = [m, f] : A \rightharpoonup B$, simply as the admissible subobject $[m \circ f^{-1}(n)] \in \mathcal{D}(A)$. In fact, we have an inverse image functor $\mathcal{D}(\_) : p(\mathcal{K}, \mathcal{D})^{op} \to \mathbf{Poset}$ sending $A \in |\, p\mathcal{K}\,|$ to $\mathcal{D}(A)$ and $u : A \rightharpoonup B$ to $u^{-1}(\_) : \mathcal{D}(B) \to \mathcal{D}(A)$. Its restriction to total maps, $\mathcal{K}^{op} \to \mathbf{Poset}$, is the composite $\mathcal{D}(\_) \circ J^{op} : \mathcal{K}^{op} \to \mathbf{Poset}$. As a convention, for $u : A \rightharpoonup B$ and $[m : D \rightarrowtail B] \in \mathcal{D}$ we write $u^{-1}(m) : u^{-1}(D) \to A$ for any representative of $u^{-1}[m] \in \mathcal{D}(A)$. Notice that, for $f : A \to B$, the convention agrees with the already adopted notation, $f^{-1}(m)$, for any mono obtained by pulling back m along f. Also, extending the n^*f notation of diagram (2) to partial maps, so that $n^*[m, f] = n^*f$, we have:

Proposition 3.1. The commuting square

$$\begin{tikzcd} u^{-1}(D) \arrow[r, "m^* u"] \arrow[d, tail, "u^{-1}(m)"'] & D \arrow[d, tail, "m"] \\ A \arrow[r, harpoon, "u"'] & B \end{tikzcd}$$

is a pullback for total maps. That is, for every $A \xleftarrow{f} C \xrightarrow{g} D$ satisfying $u \circ f = m \circ g$ there exists a unique $h : C \to u^{-1}(D)$ such that $f = u^{-1}(m) \circ h$ and $g = (m^*u) \circ h$. $\square$

From the inverse image of a partial map we can obtain its *domain of definition* by setting, for every $A, B \in |\, p\mathcal{K} \,|$,

$$\mathrm{dom}_{A,B}(\_) = (\_)^{-1}[\mathrm{id}_A] : p\mathcal{K}(A,B) \to \mathcal{D}(A).$$

It follows then that a partial map is total if and only if its domain of definition is the maximal subobject.

3.2. Examples

Every well-powered category $\mathcal{K}$ appears in two domain structures: a minimal one, $(\mathcal{K}, \mathbf{Isos}_{\mathcal{K}})$ where $\mathbf{Isos}_{\mathcal{K}}$ is the category of all isomorphisms in $\mathcal{K}$; and a maximal one, $(\mathcal{K}, \mathbf{Monos}_{\mathcal{K}})$ where $\mathbf{Monos}_{\mathcal{K}}$ is the category of all monomorphisms in $\mathcal{K}$.

To obtain further examples, we show how to construct new domain structures from old ones. Let $(\mathcal{K}, \mathcal{D})$ be a domain structure.

1 *Convention.* For a category $\mathcal{C}$ and a set $O \subseteq |\, \mathcal{C} \,|$, the full subcategory of $\mathcal{C}$ with set of objects O is denoted as $\mathcal{C} \restriction O$.
   Let $\mathcal{L}$ be a full subcategory of $\mathcal{K}$ whose objects are closed under taking the source of admissible monos (i.e. for every $A \in |\, \mathcal{L} \,|$ and $D \in |\, \mathcal{D} \,|$, if $D \rightarrowtail A$ in $\mathcal{D}$ then $D \in |\, \mathcal{L} \,|$). Then $(\mathcal{L}, \mathcal{D} \restriction |\, \mathcal{L} \,|)$ is a domain structure.

2 *Convention.* For a category $\mathcal{C}$ and a small category $\mathcal{W}$, we write $[\mathcal{W}, \mathcal{C}]$ for the category of functors $\mathcal{W} \to \mathcal{C}$ and natural transformations.
   Let $\mathcal{W}$ be a small category. Then, $([\mathcal{W}, \mathcal{K}], \mathcal{D}\text{-}\mathbf{Nat}_{[\mathcal{W},\mathcal{K}]})$ where $\mathcal{D}\text{-}\mathbf{Nat}_{[\mathcal{W},\mathcal{K}]}$ is the subcategory of $[\mathcal{W}, \mathcal{K}]$ of all the pointwise-admissible natural transformations, and $([\mathcal{W}, \mathcal{K}], \mathcal{D}\text{-}\mathbf{Cart}_{[\mathcal{W},\mathcal{K}]})$ where $\mathcal{D}\text{-}\mathbf{Cart}_{[\mathcal{W},\mathcal{K}]}$ is the subcategory of $[\mathcal{W}, \mathcal{K}]$ of all the pointwise-admissible cartesian natural transformations are domain structures.

A more tangible example of domain structure is $(\mathbf{Top}, \mathbf{Open})$ where $\mathbf{Top}$ is the category of topological spaces and continuous functions, and $\mathbf{Open}$ is its subcategory of all those monos $m : (X, \mathcal{O}X) \rightarrowtail (Y, \mathcal{O}Y)$ such that $(m(X), \mathcal{P}(mX) \cap \mathcal{O}Y)$ is a topological space isomorphic to $(X, \mathcal{O}X)$. The construction (1) above applied to $\mathcal{K} = \mathbf{Top}$, $\mathcal{D} = \mathbf{Open}$, and $\mathcal{L} = \mathbf{Poset}$ (resp. $\mathbf{Cpo}$), viewed as the full subcategory of $\mathbf{Top}$ whose objects are posets (resp. cpos) with the Alexandrov (resp. Scott) topology, yields the domain structure $(\mathbf{Poset},\mathbf{Open})$ $\big($resp. $(\mathbf{Cpo},\mathbf{Open})$, usually denoted $(\mathbf{Cpo},\mathbf{\Sigma})\big)$.

Another domain structure on $\mathbf{Top}$ is $(\mathbf{Top}, \mathbf{Close})$ where $\mathbf{Close}$ consists of all those monos $m : (D, \mathcal{O}D) \rightarrowtail (A, \mathcal{O}A)$ such that $m(D)$ is closed in A and for every $U \subseteq D$, U is open in D if and only if $m(U)$ is open in A. Applying the construction (1) as before we obtain the domain structures $(\mathbf{Poset}, \mathbf{Close})$ and $(\mathbf{Cpo}, \mathbf{Close})$.

Outside the scope of topological spaces we may consider the domain structure

$(\mathbf{Cpo}_\wedge, \mathbf{\Sigma}_\wedge)$ where $\mathbf{Cpo}_\wedge$ is the category of small cpos (posets) with bounded meets and stable functions (viz. continuous functions preserving bounded meets) and $\mathbf{\Sigma}_\wedge$ is its subcategory of all those order-reflecting monos $m : D \rightarrowtail A$ such that $m(D)$ is Scott-open and closed under bounded meets in A.

The categories of partial maps induced by some of our examples of domain structures are: $p(\mathcal{K}, \mathbf{Isos}_\mathcal{K}) \cong \mathcal{K}$; $p(\mathbf{Set}, \mathbf{Monos}) \cong \mathbf{Pfn}$ (the category of sets and partial functions); $p(\mathbf{Poset}, \mathbf{Open}) \cong \mathbf{pPoset}$; $p(\mathbf{Cpo}, \mathbf{\Sigma}) \cong \mathbf{pCpo}$; $p(\mathbf{Cpo}_\wedge, \mathbf{\Sigma}_\wedge) \cong \mathbf{pCpo}_\wedge$ (the category of small cpos with bounded meets and partial functions defined on a Scott-open closed under bounded meets and such that their restriction to the domain of definition is stable); finally, $p([\mathcal{W}, \mathcal{K}], \mathcal{D}\text{-}\mathbf{Cart}_{[\mathcal{W},\mathcal{K}]}) \cong [\mathcal{W}, p(\mathcal{K}, \mathcal{D})] \restriction |\,[\mathcal{W}, \mathcal{K}]\,|$ (see (Fiore 1994, Proposition 5.3.5)).

4. Testing and Observing Partial Maps

The capability of observing that an input/output program terminates serves as the basis for discriminating between programs. For example, in the *contextual* approach, a program e is said to approximate a program e' (written $e \precsim^c e'$) if, in every program in which e occurs, e can be safely replaced by e' in the sense that new non-terminating behaviour will not happen.

The distinction between partial and total maps induces an approximation relation between maps, in which contexts are used to discriminate between maps based on their terminating behaviour. Following the informal description of a context as a "program with a hole"; in a category of partial maps, a context is described as a "partial map with a hole". Formally, contexts are *incomplete* composites $w \circ \_ \circ t$ where t and w are partial maps. The *contextual approximation* relation for partial maps is defined as follows,

Definition 4.1. (Plotkin) For every $u, v : A \rightharpoonup B$,

$$u \precsim^c v \text{ if and only if for every context } C[\_],\ C[u]\downarrow \Rightarrow C[v]\downarrow. \qquad \square$$

Remark. Observe that because of the adopted description of partial maps it is enough to consider contexts of the form $m^R \circ \_ \circ f$ for m an admissible mono and f a total map. But the definition is independent of the description of the category of partial maps, and thus, makes sense in any axiomatisation of such categories (e.g. the p-categories of (Robinson and Rosolini 1988)).

The restriction of $\precsim^c$ to total maps gives a contextual approximation relation on total maps which we denote $\sqsubseteq^c$. Both $\sqsubseteq^c$ and $\precsim^c$ are preorders and, in general, they are not partial orders.

Justifying our computational intuition, in the examples of interest the contextual approximation relation turns out to be a partial order which, further, coincides with the usual notion of approximation. In **Poset** and **Cpo**, $\sqsubseteq^c$ is the pointwise order $\sqsubseteq$; whilst in **pPoset** and **pCpo**, $\precsim^c$ is the appropriate generalisation of the pointwise order, namely for $u, v : P \rightharpoonup Q$,

$$u \precsim v \text{ if and only if for every } x \in P,\ u(x)\downarrow \Rightarrow \big(v(x)\downarrow \ \wedge\ u(x) \sqsubseteq v(x)\big). \tag{7}$$

Instead of checking these facts, which follow from general considerations (see the remark after Proposition 4.2), we exemplify the use of contexts to discriminate maps. Let $f, g : P \to Q$ in **Poset** be such that there exists $t \in P$ for which $ft \sqsubset gt$, and let $C[\_]$ be the context in **pPoset**, $W^R \circ \_ \circ t$, where $t : 1 \to P : * \mapsto t$ and $W = Q \setminus \{y \in Q \mid y \sqsubseteq ft\}$. Then, with respect to termination, $C[\_]$ distinguishes f and g, as $C[f]\uparrow$ and $C[g]\downarrow$.

In **pPoset** (**pCpo**), (7) reduces the order on partial monotone (continuous) functions to the test for definedness and the order on elements. As the next proposition shows this is a general phenomenon and sustains a claim that testing for totality and the approximation of total maps are primitive concepts, whilst the approximation of partial maps is a derived one.

Proposition 4.1. For every $u, v : A \rightharpoonup B$,

$$u \precsim^c v \text{ if and only if for every } x : X \to A,\ u \circ x\downarrow \Rightarrow (v \circ x\downarrow \ \wedge\ u \circ x \sqsubseteq^c v \circ x).$$

Moreover, when the category of total maps has terminal object 1 and totality is extensional (Definition 3.3) the above can be strengthened to

$$u \precsim^c v \text{ if and only if for every } x : 1 \to A,\ u \circ x\downarrow \Rightarrow (v \circ x\downarrow \ \wedge\ u \circ x \sqsubseteq^c v \circ x). \qquad \square$$

Remark. In the proposition, it would make no difference to allow x to range over partial maps, as $u \circ x\downarrow \Rightarrow x\downarrow$. Moreover, this property is central to the (omitted) proof.

Further evidence for the suitability of contextual approximation is that it interacts well with composition (see Proposition 4.2 and 8.1).

The description of partial maps based on domain structures provides another computationally natural notion of approximation. For this purpose, admissible monos are regarded as predicates describing *observable properties* (c.f. (Smyth 1992)).

A program e is said to *specialise* a program e' (written $e \precsim^s e'$) if the observable properties of the outputs of e are also satisfied by the outputs of e'. Put it in another way, for any observable property U, the weakest precondition ensuring that every execution of e satisfies U is a precondition ensuring that every execution of e' satisfies U. Thus, an observer content with the input/output behaviour of e should also be content with the input/output behaviour of e'. The *specialisation* relation for partial maps is defined as follows:

Definition 4.2. $u \precsim^s v : A \rightharpoonup B$ in $p\mathcal{K}$ if and only if $u^{-1}(\_) \subseteq v^{-1}(\_) : \mathcal{D}(B) \to \mathcal{D}(A)$ in **Poset**. $\square$

The specialisation relation generalises the one from topology (see (Smyth 1992)). Recall that for a T_0-space $\big(X, \mathcal{O}(X)\big)$, its specialisation order $\leq\, \subseteq\, X \times X$ is defined as

$$x \leq y \text{ if and only if for every } U \in \mathcal{O}(X),\ x \in U \Rightarrow y \in U.$$

Then, by identifying elements with functions out of the singleton, the above definition can be put in a form resembling Definition 4.2:

$$x \leq y \text{ if and only if for every } U \in \mathcal{O}(X),\ x^{-1}(U) \subseteq y^{-1}(U).$$

Pleasantly enough, the notions of approximation obtained by testing and by observing partial maps coincide.

Lemma 4.1. For $f \in \mathcal{K}(X, A)$, $u \in p\mathcal{K}(A, B)$ and $m \in \mathcal{D}(D, B)$,

$$m^R \circ u \circ f \downarrow \quad \Leftrightarrow \quad f \text{ factors through } u^{-1}(m).$$

Proof.
($\Rightarrow$) For every $n \in \mathcal{D}(E, Y)$ and $g \in \mathcal{K}(X, Y)$, $n^R \circ g \downarrow \Rightarrow\ g = n \circ n^R \circ g$.
Thus, assuming $m^R \circ u \circ f \downarrow$ we have that $u \circ f = m \circ (m^R \circ u \circ f)$, and since $A \xleftarrow{u^{-1}(m)} u^{-1}(D) \xrightarrow{m^* u} D$ is a pullback of $A \xrightarrow{u} B \xleftarrow{m} D$ for total maps (see Proposition 3.1) we are done.
($\Leftarrow$) Assume $f = u^{-1}(m) \circ g$. Then, g is total and so is

$$(m^* u) \circ g \;=\; m^R \circ m \circ (m^* u) \circ g \;=\; m^R \circ u \circ u^{-1}(m) \circ g \;=\; m^R \circ u \circ f. \qquad \Box$$

Proposition 4.2. $\sqsubseteq^{c}$ equals $\sqsubseteq^{s}$.

Proof.
($\sqsubseteq^{c}\ \subseteq\ \sqsubseteq^{s}$) Let $u \sqsubseteq^{c} v : A \rightharpoonup B$. Then,

$$\begin{aligned}
u \sqsubseteq^{c} v \quad & \Rightarrow \quad \forall\, [m] \in \mathcal{D}(B). \\
& \qquad\qquad (m^R \circ u \circ u^{-1}(m) \downarrow \Rightarrow\ m^R \circ v \circ u^{-1}(m) \downarrow) \\
& \Leftrightarrow \quad \forall\, [m] \in \mathcal{D}(B).\ m^R \circ v \circ u^{-1}(m) \downarrow \\
& \qquad\qquad\qquad\qquad\qquad\qquad , \text{ as } m^R \circ u \circ u^{-1}(m) = m^* u \downarrow \\
& \Leftrightarrow \quad \forall\, [m] \in \mathcal{D}(B).\ u^{-1}(m) \text{ factors through } v^{-1}(m) \\
& \qquad\qquad\qquad\qquad\qquad\qquad , \text{ by Lemma 4.1} \\
& \Leftrightarrow \quad u \sqsubseteq^{s} v.
\end{aligned}$$

($\sqsubseteq^{s}\ \subseteq\ \sqsubseteq^{c}$) Let $u \sqsubseteq^{s} v : A \rightharpoonup B$. Then,

$$
\begin{aligned}
u \precsim^s v \quad &\Leftrightarrow \quad \forall\,[m] \in \mathcal{D}(B).\ u^{-1}(m) \subseteq v^{-1}(m) \\
&\Leftrightarrow \quad \forall\,[m] \in \mathcal{D}(B).\ u^{-1}(m) \text{ factors through } v^{-1}(m) \\
&\Rightarrow \quad \forall\, f \in \mathcal{K}(X,A).\ \forall\,[m] \in \mathcal{D}(B). \\
&\qquad\qquad \big(f \text{ factors through } u^{-1}(m) \Rightarrow f \text{ factors through } v^{-1}(m)\big) \\
&\Rightarrow \quad \forall\, f \in \mathcal{K}(X,A). \forall\,[m] \in \mathcal{D}(B). \\
&\qquad\qquad (m^R \circ u \circ f \downarrow \Rightarrow\ m^R \circ v \circ f \downarrow) \qquad\qquad \text{, by Lemma 4.1} \\
&\Leftrightarrow \quad u \precsim^c v.
\end{aligned}
$$

□

Remark. In **Poset** (resp. **Cpo**), the contextual approximation relation, the specialisation relation, and the pointwise order are equal, by Proposition 4.2 and the well-known fact that, for $f, g : P \to Q$,

$f \sqsubseteq g$ if and only if for every upper-closed (resp. Scott-open) $U \subseteq Q$, $f^{-1}(U) \subseteq g^{-1}(U)$.

Moreover, the equality extends to **pPoset** and **pCpo**, by Propositions 4.1 and 4.2.

The following table describes the specialisation preorder for various domain structures:

Domain Structure	$f \sqsubseteq^s g : P \to Q$
$(\mathcal{K}, \mathbf{Isos}_{\mathcal{K}})$	for every f and g
$(\mathbf{Set}, \mathbf{Monos})$	iff $f = g$ (i.e. discrete)
$(\mathbf{Poset}, \mathbf{Close})$ $(\mathbf{Cpo}, \mathbf{Close})$	iff $\forall\, x \in P.\ f(x) \sqsupseteq g(x)$

Quite disapointingly the specialisation order induced by $(\mathbf{Cpo}_\wedge, \mathbf{\Sigma}_\wedge)$ is not the *stable* order $\sqsubseteq_{st}$ $\big($where $f \sqsubseteq_{st} g : P \to Q$ if and only if $\forall\, x, y \in P.\ x \sqsubseteq y \ \Rightarrow\ f(x) = f(y) \wedge g(x)\big)$ which is the one making $\mathbf{Cpo}_\wedge$ into a **Cpo**-cartesian-closed-category. In fact, as the following proposition shows, in order to observe stability a more subtle way of comparing weakest preconditions $\big($see (Zhang 1991, Theorem 8.2)$\big)$ is needed:

Proposition 4.3. The specialisation order induced by any domain structure $(\mathbf{Cpo}_\wedge, \mathcal{D})$ is not the stable order.

Proof. Let $\mathbf{2} = \{\bot \sqsubseteq \top\}$ be the Sierpinski space and write, respectively, $K_\bot$ and $K_\top$ for the constantly $\bot$ and $\top$ stable functions $\mathbf{2} \to \mathbf{2}$. Recall that the stable order on $\mathbf{2} \to \mathbf{2}$ is: $\mathrm{id}_{\mathbf{2}} \ {}_{st}\!\sqsupseteq\ K_\bot \ \sqsubseteq_{st}\ K_\top$.
If $\mathrm{id}_{\mathbf{2}} \not\sqsubseteq^s K_\top$ then there exists $D \rightarrowtail \mathbf{2}$ in $\mathcal{D}$ such that $D = \mathrm{id}_{\mathbf{2}}{}^{-1}(D) \not\subseteq {K_\top}^{-1}(D)$, so $D = \{\bot\}$ and hence $\{\bot\} \rightarrowtail \mathbf{2}$ is in $\mathcal{D}$. But then we have that $K_\bot \not\sqsubseteq^s \mathrm{id}_{\mathbf{2}}$ because $\mathbf{2} = {K_\bot}^{-1}\{\bot\} \supset \mathrm{id}_{\mathbf{2}}{}^{-1}\{\bot\} = \{\bot\}$. Hence, the result follows. □

In general, specialisation is a preorder and not a partial order (e.g. consider the domain structures $(\mathcal{K}, \mathbf{Isos}_{\mathcal{K}})$ and $(\mathbf{Top}, \mathbf{Open})$), but in every domain structure it is possible to identify a sub-domain-structure in which the specialisation preorder is antisymmetric. To this end we introduce a T_0-axiom:

Definition 4.3. Let $(\mathcal{K}, \mathcal{D})$ be a domain structure. We say that $A \in |\, \mathcal{K}\,|$ is a T_0-object if for all $f, g : X \to A$, $(\forall\, [m] \in \mathcal{D}(A).\ f^{-1}[m] = g^{-1}[m]) \;\Rightarrow\; f = g$. □

Admissible subobjects of T_0-objects are T_0:

Proposition 4.4. Let $(\mathcal{K}, \mathcal{D})$ be a domain structure. If $A' \in |\, \mathcal{K}\,|$ is T_0 and $n : A \rightarrowtail A'$ is in $\mathcal{D}$ then A is T_0.

Proof. Let $f, g : X \to A$ and assume $f^{-1}[m] = g^{-1}[m]$ for all $[m] \in \mathcal{D}(A)$.
Then, for all $[m'] \in \mathcal{D}(A')$,

$$(n \circ f)^{-1}[m'] = f^{-1}[n \cap m'] = g^{-1}[n \cap m'] = (n \circ g)^{-1}[m']$$

and, as A' is T_0, $n \circ f = n \circ g$; but then, as n is mono, $f = g$. □

For a domain structure $(\mathcal{K}, \mathcal{D})$ let $\mathcal{K}_0$ be the full subcategory of $\mathcal{K}$ of all T_0-objects. Then, $(\mathcal{K}_0, \mathcal{D} \restriction |\, \mathcal{K}_0\,|)$ is a domain structure in which the specialisation preorder is a partial order that coincides with that of $(\mathcal{K}, \mathcal{D})$.

5. Ordering Partial Maps

Motivated by the discussion in Section 4, the approximation order between total maps is adopted as primitive and is used to derive a notion of approximation between partial maps.

From now on and till Section 8, we will consider an arbitrary approximation order, $\sqsubseteq$, on total maps (instead of just restricting attention to the specialisation preorder $\sqsubseteq^s$), and define and study an induced approximation order $\precsim$ on partial maps. This generality is mandatory if, for example, a stable order on $\mathbf{pCpo}_\wedge$ is to be analysed.

Convention. For $\mathcal{V} = \mathbf{Poset}$ or $\mathbf{Cpo}$, a domain structure with a $\mathcal{V}$-category of total maps is called a $\mathcal{V}$-domain-structure and the order on total maps is denoted $\sqsubseteq$.

A **Poset**-domain-structure induces an order on partial maps whose definition is motivated by Proposition 4.1:

Definition 5.1. Given a **Poset**-domain-structure, for every $u, v : A \rightharpoonup B$,

$u \precsim v$ if and only if for every $x : X \to A$, $u \circ x \downarrow\; \Rightarrow (v \circ x \downarrow\; \wedge\, u \circ x \sqsubseteq v \circ x)$. □

Remark. $\precsim$ is clearly reflexive and transitive. That it is antisymmetric follows from Proposition 5.1 below.

As an example consider the **Poset**-domain-structure $(\mathbf{Cpo}_\wedge, \boldsymbol{\Sigma}_\wedge)$ with the stable order $\sqsubseteq_{st}$, then $u \underset{\sim}{\sqsubset}_{st} v : P \rightharpoonup Q$ in $\mathbf{pCpo}_\wedge$ if and only if

$$\forall\, x \in P.\ u(x)\downarrow \Rightarrow \big(v(x)\downarrow\ \wedge \forall\, y \in P.\ x \sqsubseteq y \ \Rightarrow\ u(x) = u(y) \wedge v(x)\big).$$

The intuition behind Definition 5.1 is that a possibly non-terminating input/output program e approximates another one e' provided that under all possible inputs, the outputs of e approximate the outputs of e'. Thus, in particular, whenever e outputs, so does e'. In symbols, $\mathrm{dom}(e) \subseteq \mathrm{dom}(e')\ \wedge\ e \restriction \mathrm{dom}(e) \sqsubseteq e' \restriction \mathrm{dom}(e)$. This intuition is technically correct:

Proposition 5.1. Given a **Poset**-domain-structure, for every $[m,f],[n,g] : A \rightharpoonup B$,

$$[m,f] \underset{\sim}{\sqsubset} [n,g] \quad\Leftrightarrow\quad m \subseteq_i n\ \wedge\ f \sqsubseteq g \circ i \text{ for some (unique) } i.$$

(In diagram form:

$$\begin{tikzcd}
 & D \arrow[dl, tail, "m"'] \arrow[dr, "f"] \arrow[dd, dashed, tail, "i"] & \\
A & \sqsupseteq & B \\
 & E \arrow[ul, tail, "n"] \arrow[ur, "g"'] &
\end{tikzcd}$$
.)

Proof.
$(\Rightarrow)$ Assume $[m,f] \underset{\sim}{\sqsubset} [n,g]$. Since $[m,f] \circ m = f$ is total then so is $[n,g] \circ m$ and therefore there exists a unique i for which

$$\begin{tikzcd}
{} \arrow[r, tail, "i"] \arrow[d, "\mathrm{id}"'] \arrow[dr, phantom, "\lrcorner", very near start] & {} \arrow[d, tail, "n"] \\
{} \arrow[r, tail, "m"'] & {}
\end{tikzcd}
\quad \text{and} \quad f \ =\ [m,f] \circ m \ \sqsubseteq\ [n,g] \circ m \ =\ g \circ i.$$

$(\Leftarrow)$ Let $m \subseteq_i n$ and $f \sqsubseteq g \circ i$ for some i. For $x : X \to A$, if $[m,f] \circ x\downarrow$ then

$$[\mathrm{id}] \ =\ \mathrm{dom}([m,f] \circ x) \ =\ x^{-1}[m] \ \subseteq\ x^{-1}[n] \ =\ \mathrm{dom}([n,g] \circ x)$$

and therefore $[n,g] \circ x\downarrow$.
Moreover, we have

$$
\begin{tikzcd}
{} \arrow[r, "m^*x"] \arrow[d, tail, "\mathrm{id}"'] \arrow[dr, phantom, "\lrcorner", very near start] & {} \arrow[d, tail, "i"] \arrow[dr, "f"] \arrow[dr, phantom, "\sqsupseteq"'] & \\
{} \arrow[r, "n^*x"] \arrow[d, tail, "\mathrm{id}"'] \arrow[dr, phantom, "\lrcorner", very near start] & {} \arrow[d, tail, "n"] \arrow[r, "g"'] & {} \\
{} \arrow[r, "x"'] & {} &
\end{tikzcd}
$$

and hence, $[m,f]\circ x \;=\; f\circ(m^*x) \;\sqsubseteq\; g\circ i\circ(m^*x) \;=\; [n,g]\circ x$. $\square$

Convention. We introduce the notation $[m,f]\, \underset{\sim}{\sqsubset}_i\, [n,g]$ to indicate that i realises the inequality between the representatives (i.e. $m \subseteq_i n$ and $f \sqsubseteq g\circ i$). Clearly, if $[m,f]\, \underset{\sim}{\sqsubset}\, [n,g]$ and $m\subseteq_i n$ then $[m,f]\, \underset{\sim}{\sqsubset}_i\, [n,g]$.

An example of the order induced by a **Poset**-domain-structure is a generalisation of the *graph inclusion order*; denoted $\subseteq$, and defined to be the order induced by $(\mathcal{K}_{\mathbf{Poset}}, \mathcal{D})$ where $(\mathcal{K},\mathcal{D})$ is a domain structure and $\mathcal{K}_{\mathbf{Poset}}$ is the **Poset**-category obtained from $\mathcal{K}$ by discretely ordering each hom-set. Explicitly,

$$[m,f]\subseteq_i [n,g] \text{ if and only if } m\subseteq_i n \text{ and } f = g\circ i.$$

Justifying the terminology, when $(\mathcal{K},\mathcal{D}) = (\mathbf{Set},\mathbf{Monos})$,

$$u\subseteq v \text{ if and only if } \mathrm{graph}(u)\subseteq \mathrm{graph}(v).$$

The characterisation of $\underset{\sim}{\sqsubset}$, provided by Proposition 5.1, is more convenient to work with than its definition; for example, using the characterisation the following is straightforward:

Proposition 5.2. Let $(\mathcal{K},\mathcal{D})$ be a **Poset**-domain-structure. Then, for $A, B \in |\,\mathcal{K}\,|$,

1 $(p\mathcal{K}(A,B),\ \underset{\sim}{\sqsubset}\ )$ is a poset.
2 $J_{A,B} : (\mathcal{K}(A,B), \sqsubseteq) \hookrightarrow (p\mathcal{K}(A,B),\ \underset{\sim}{\sqsubset}\ )$ is a full functor.
3 $\mathcal{K}(A,B)$ is upper-closed in $(p\mathcal{K}(A,B),\ \underset{\sim}{\sqsubset}\ )$, i.e. for every $f : A\to B$ and $u : A \rightharpoonup B$, if $f\ \underset{\sim}{\sqsubset}\ u$ then $u\downarrow$.
4 $J_{A,B}$ preserves lubs.
5 $\mathrm{dom}_{A,B} : (p\mathcal{K}(A,B),\ \underset{\sim}{\sqsubset}\ ) \to (\mathcal{D}(A), \subseteq)$ is monotone. $\square$

Remark. From (2) and (3) above, it follows that $J_{A,B}$ is a *full* and *upper-closed* morphism in **Poset**, in the sense that for every $f : A\to B$ and $u : A\rightharpoonup B$, if $Jf\ \underset{\sim}{\sqsubset}\ u$ then there exists a unique $g : A\to B$ such that $f\sqsubseteq g$ and $Jg = u$. This property together with monotonicity imply (4).

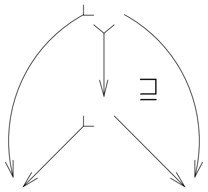
Full and upper-closed morphisms play a central role in the next section.

6. Poset-Categories of Partial Maps

Now our main goal is to understand when $\sqsubseteq$ is domain theoretic, in the technical sense that the category of partial maps **Cpo**-enriches with respect to it. For simplicity we undertake this study in two steps: in this section, we characterise **Poset**-categories of partial maps induced by domain structures with a **Poset**-category of total maps; in Section 7, we restrict our attention to *uniform* domain structures (Definition 7.2) with a **Cpo**-category of total maps, and provide necessary and sufficient conditions so that the induced category of partial maps **Cpo**-enriches. As an example, in Section 8, these characterisations are used when the approximation order is specialisation.

We have seen that the category of partial maps induced by a **Poset**-domain-structure has hom-posets with respect to the induced order (Proposition 5.2 (1)). Thus, it would be **Poset**-enriched if composition (or equivalently, composition on the right and composition on the left) was monotone.

Checking for monotonicity, and also continuity (as will be seen in the next section), of composition on the right is easier than for composition on the left due to the asymmetry in the diagram



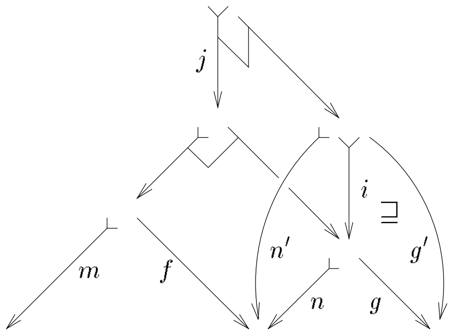
(8)

expressing the inequality between two partial maps. But when the order on partial maps is the graph inclusion order, the situation in (8) is symmetric (as both triangles commute) and *potentially* more manageable.

Monotonicity of composition on the right is immediate:

Lemma 6.1. For a **Poset**-domain-structure, $p\mathcal{K}(\_, A)$ is a functor $p\mathcal{K}^{op} \to$ **Poset** for every $A \in |\, p\mathcal{K}\,|$.

Proof. Let $[m, f] : X \rightharpoonup Y$. Assume $[n', g'] \sqsubseteq_i [n, g] : Y \rightharpoonup A$, then the diagram



states that j realises the inequality $[n', g'] \circ [m, f] \sqsubseteq [n, g] \circ [m, f]$. $\square$

Aiming at characterising monotonicity of composition on the left, some domain theoretic concepts in the context of **Poset**-categories, due to Barry Jay, are introduced.

Definition 6.1. (Jay) In a **Poset**-category $\mathcal{C}$, a morphism $m : A \to B$ is said to be

1 *full* if for every $C \in |\, \mathcal{C}\,|$, the monotone map $m \circ \_ : \mathcal{C}(C, A) \to \mathcal{C}(C, B)$ is full as a functor (i.e. $m \circ x \leq m \circ y \;\Rightarrow\; x \leq y$).
2 *upper-closed* if for every $x : C \to A$ and $y : C \to B$, whenever $m \circ x \leq y$ there exists a $z : C \to A$ such that $m \circ z = y$. $\square$

In **Poset** (**Cpo**), a morphism $m : P \to Q$ is full if and only if establishes an isomorphism between P and $m(P)$ ordered as in Q. Further, if $f : P \to Q$ is upper-closed then $f(P)$ is an upper-closed subset of Q. The converse need not hold, but whenever $m : P \to Q$ is full and such that $m(P)$ is an upper-closed subset of Q then m is an upper-closed morphism. Therefore, the full and upper-closed subobjects of a poset (cpo) are exactly its upper-closed subsets with the induced order.

Full and upper-closed morphisms can be characterised along the lines of the remark after Proposition 5.2:

Proposition 6.1. In a **Poset**-category, a morphism $m : A \to B$ is full and upper-closed if and only if it is a mono and for every $x : C \to A$ and $y : C \to B$, if $m \circ x \leq y$ then there exists a (necessarily) unique $z : C \to A$ such that $x \leq z$ and $m \circ z = y$. $\square$

In **pPoset** every admissible mono is full and upper-closed. This is not accidental:

Theorem 6.1. Given a **Poset**-domain-structure $(\mathcal{K}, \mathcal{D})$, the following are equivalent

1 $p(\mathcal{K}, \mathcal{D})$ **Poset**-enriches with respect to $\sqsubseteq$.
2 for every $A \in |\, p\mathcal{K}\,|$, $p\mathcal{K}(A, \_)$ is a functor $p\mathcal{K} \to$ **Poset**.
3 For every $A \in |\, \mathcal{K}\,|$ and $m : D \rightarrowtail B$ in $\mathcal{D}$, the map $m^R \circ \_ : \mathcal{K}(A, B) \to p\mathcal{K}(A, D)$ is monotone.
4 Every admissible mono is full and upper-closed.

Proof.
(1) $\Rightarrow$ (2) Vacuously.
(2) $\Rightarrow$ (1) By Lemma 6.1.
(2) $\Rightarrow$ (3) Obvious.
(3) $\Rightarrow$ (4) Let $m \in \mathcal{D}$, and assume that $m \circ x \sqsubseteq y$. Then, $x \;=\; m^R \circ m \circ x \;\lesssim\; m^R \circ y$. In diagram form,

$$\begin{tikzcd}
& {} \arrow[d, "i"] \arrow[ddl, bend right, "\mathrm{id}"'] \arrow[ddr, bend left, "x"] \arrow[ddr, phantom, "\sqsupseteq"] & \\
& {} \arrow[dl, "y^{-1}(m)"'] \arrow[dr, "m^*y"] & \\
{} \arrow[dr, "y"'] & & {} \arrow[dl, "m"] \\
& {} &
\end{tikzcd}$$

for some i.
Therefore, $x \sqsubseteq (m^*y) \circ i$ and $m \circ (m^*y) \circ i = y$.
(4) $\Rightarrow$ (2) Let $[n, g] : X \rightharpoonup Y$. Assume $[m', f'] \lesssim_i [m, f] : A \rightharpoonup X$, then

$$n \circ (n^* f') \;=\; f' \circ f'^{-1}(n) \;\sqsubseteq\; f \circ i \circ f'^{-1}(n).$$

Since n is full and upper-closed, it follows that there exists a unique z such that

$$\begin{tikzcd}
& {} \arrow[dl, "f'^{-1}(n)"'] \arrow[ddr, "z"'] \arrow[ddr, bend left, "n^*f'"] & \\
{} \arrow[d, "i"'] & & \\
{} \arrow[dr, "f"'] & & {} \arrow[dl, "n"] \\
& {} &
\end{tikzcd}$$
.

Then, by the universal property of pullbacks, there exists a unique j such that

$$
\begin{tikzcd}
 & & \cdot \arrow[dll, "f'^{-1}(n)"'] \arrow[d, tail, "j"] \arrow[ddrr, "z"] & & \\
\cdot \arrow[d, tail, "i"'] & & \cdot \arrow[dll, "f^{-1}(n)"] \arrow[drr, "n^* f"] & & \\
\cdot \arrow[drr, "f"'] & & & & \cdot \arrow[dll, "n"] \\
 & & \cdot & &
\end{tikzcd}
.
$$

And finally, it follows that j realises the inequality $[n, g] \circ [m', f'] \precsim [n, g] \circ [m, f]$. □

Corollary 6.1. For every domain structure $(\mathcal{K}, \mathcal{D})$, $p(\mathcal{K}_{\mathbf{Poset}}, \mathcal{D})$ **Poset**-enriches with respect to $\subseteq$. □

As a consequence of the theorem we have that the categories of partial maps induced by (**Poset**,**Open**) and (**Cpo**,$\mathbf{\Sigma}$) with the pointwise order, $\sqsubseteq$, **Poset**-enrich with respect to $\precsim$. Also, the category of partial maps induced by ($\mathbf{Cpo}_\wedge$,$\mathbf{\Sigma}_\wedge$) with the stable order, $\sqsubseteq_{st}$, **Poset**-enriches with respect to $\precsim_{st}$. Clearly, the category of partial maps induced by (**Poset**,**Close**) and (**Cpo**,**Close**) with the pointwise order do not **Poset**-enrich.

7. Cpo-Categories of Partial Maps

Convention. Whenever we say that a category of partial maps is **Cpo**-enriched we assume that so is the domain structure that induces it and that the order is $\precsim$.

We start the investigation on **Cpo**-categories of partial maps focusing on the properties enforced by the enrichment. In this respect, two important points are: the fact that the class of admissible subobjects of a **Cpo**-domain-structure inducing a **Cpo**-category of partial maps ought to be locally a cpo (i.e. the skeleton of every slice is a cpo), and a *trick* which reduces lubs of arbitrary partial maps to lubs of *graph included* partial maps (Subsection 7.1). Afterwards, with the purpose of obtaining lubs of admissible subobjects and lubs of partial maps in a canonical way, *uniform* domain structures are introduced and their properties studied (Subsection 7.2). This, together with the characterisation of **Poset**-categories of partial maps, will provide a characterisation of **Cpo**-categories of partial maps induced by uniform **Cpo**-domain-structures. Finally, as an application, a recipe for computing lubs of ω-chains of partial maps in these categories is given (Subsection 7.4).

*7.1. Properties of **Cpo**-Categories of Partial Maps*

Convention. We regard lubs (of subobjects and partial maps) as colimits. Thus we sometimes write $\bigsqcup\langle[m_k] \subseteq [m_{k+1}]\rangle = \langle[m_k] \subseteq [m]\rangle$ to mean that $\langle[m_k] \subseteq [m_{k+1}]\rangle$ is an ω-chain with lub $[m]$. (An analogous notation is used for partial maps.)

Proposition 7.1. For a **Poset**-domain-structure $(\mathcal{K}, \mathcal{D})$, if $p(\mathcal{K}, \mathcal{D})$ has hom-cpos with respect to $\sqsubsetapprox$ then $\mathcal{D}$ is locally a cpo.

Proof. If $\langle[m_k] \subseteq [m_{k+1}]\rangle$ is an ω-chain in $\mathcal{D}(A)$ then $\langle[m_k, m_k] \subseteq [m_{k+1}, m_{k+1}]\rangle$ is an ω-chain in $p\mathcal{K}(A, A)$. And, if $\bigsqcup[m_k, m_k] = [m, f]$, it follows that $\bigsqcup[m_k] = [m]$. □

The local completeness of the class of admissible subobjects has interesting consequences:

Proposition 7.2. In a domain structure $(\mathcal{K}, \mathcal{D})$, if $\mathcal{D}$ is locally a cpo then

1 If $\langle[m_k] \subseteq_{i_k} [m_{k+1}]\rangle$ is an ω-chain in $\mathcal{D}(A)$ with lub $\langle[m_k] \subseteq_{\mu_k} [m : D \rightarrowtail A]\rangle$ then $\langle[\mu_k] \subseteq_{i_k} [\mu_{k+1}]\rangle$ is an ω-chain in $\mathcal{D}(D)$ with lub $[\mathrm{id}_D]$.

2 For every $n : D \rightarrowtail A$ in $\mathcal{D}$, the mapping $[n \circ \_] : \mathcal{D}(D) \to \mathcal{D}(A)$ is continuous.

3 dom preserves lubs of ω-chains.

Proof.

(1) Clearly $\langle[\mu_k] \subseteq_{i_k} [\mu_{k+1}]\rangle$ is an ω-chain in $\mathcal{D}(D)$, and $\bigsqcup[\mu_k] = [\mathrm{id}_D]$ because for every $[n] \in \mathcal{D}(D)$,

$$\begin{aligned} \forall\, k \in |\,\omega\,|.\ \mu_k \subseteq n \quad &\Longleftrightarrow \quad \forall\, k \in |\,\omega\,|.\ m_k = m \circ \mu_k \subseteq m \circ n \\ &\Longleftrightarrow \quad m \subseteq m \circ n \\ &\Longleftrightarrow \quad [n] = [\mathrm{id}_D]. \end{aligned}$$

(2) First, observe that for every $[m] \in \mathcal{D}(D)$ and $[n'] \in \mathcal{D}(A)$,

$$n \circ m \subseteq n' \quad \Longleftrightarrow \quad m \subseteq n^{-1}(n'). \tag{9}$$

Now, let $\langle[m_k] \subseteq [m_{k+1}]\rangle$ be an ω-chain in $\mathcal{D}(D)$ with lub $[m]$. Then, for every $[n'] \in \mathcal{D}(A)$,

$$\begin{aligned} \forall\, k \in |\,\omega\,|.\ n \circ m_k \subseteq n' \quad &\Longleftrightarrow \quad \forall\, k \in |\,\omega\,|.\ m_k \subseteq n^{-1}(n') \quad &&\text{, by (9)} \\ &\Longleftrightarrow \quad m \subseteq n^{-1}(n') \\ &\Longleftrightarrow \quad n \circ m \subseteq n' &&\text{, by (9).} \end{aligned}$$

(3) Let $\langle[m_k, f_k] \sqsubsetapprox [m_{k+1}, f_{k+1}]\rangle$ be an ω-chain. If $\bigsqcup[m_k, f_k] = \langle[m_k, f_k] \sqsubsetapprox_{\mu'_k} [m', f]\rangle$ and $\bigsqcup[m_k] = \langle[m_k] \subseteq_{\mu_k} [m]\rangle$ then we are required to prove that $[m] = [m']$.

$[m']$ is an upper bound of the ω-chain $\langle[m_k] \subseteq [m_{k+1}]\rangle$ and then $[m] \subseteq_i [m']$. For every $k \in |\,\omega\,|$, since $[m_k] \subseteq_{\mu_k} [m] \subseteq_i [m']$ and $[m_k] \subseteq_{\mu'_k} [m']$, it follows that $\mu'_k = i \circ \mu_k$. Therefore $[m, f \circ i]$ is an upper bound of the ω-chain $\langle[m_k, f_k] \sqsubsetapprox [m_{k+1}, f_{k+1}]\rangle$, and then, $[m'] \subseteq [m]$. □

Convention. Let $\langle i_k : D_k \to D_{k+1}\rangle$ be an ω-chain, then

1 we write $\langle m_k \rangle : \langle i_k \rangle \stackrel{\cdot}{\to} A$ to indicate that $\langle m_k : D_k \to A \rangle$ is a cone for $\langle i_k \rangle$ (i.e. for all $k \in |\,\omega\,|$, $m_k = m_{k+1} \circ i_k$);
2 we write $\langle f_k \rangle : \langle i_k \rangle \xrightarrow[\leq]{\cdot} B$ to indicate that $\langle f_k : D_k \to B \rangle$ is a *lax* cone for $\langle i_k \rangle$ (i.e. for all $k \in |\,\omega\,|$, $f_k \sqsubseteq f_{k+1} \circ i_k$); and,
3 for cones $\langle \mu_k \rangle : \langle i_k \rangle \stackrel{\cdot}{\to} D$ and $\langle m_k \rangle : \langle i_k \rangle \stackrel{\cdot}{\to} A$, we write $m : \langle \mu_k \rangle \to \langle m_k \rangle$ to indicate that $m : D \to A$ is a morphism of cones (i.e. for all $k \in |\,\omega\,|$, $m \circ \mu_k = m_k$).

We now analyse ω-chains of partial maps and their lubs. Let $\langle [m_k, f_k] \mathrel{\underset{\approx}{\sqsubset}}_{i_k} [m_{k+1}, f_{k+1}] \rangle$ be an ω-chain in $p\mathcal{K}(A,B)$. Then we have that $\langle i_k \rangle$ is an ω-chain in $\mathcal{D}$, $\langle m_k \rangle : \langle i_k \rangle \stackrel{\cdot}{\to} A$, and $\langle f_k \rangle : \langle i_k \rangle \xrightarrow[\leq]{\cdot} B$. In diagram form,

$$\begin{tikzcd}
 & D_{k+1} \arrow[ddl, bend right, "m_{k+1}"'] \arrow[ddr, bend left, "f_{k+1}"] & \\
 & D_k \arrow[u, tail, "i_k"] \arrow[dl, "m_k"] \arrow[dr, "f_k"'] \arrow[r, phantom, "\sqsubseteq"] & \\
A & & B
\end{tikzcd} \qquad (k \in |\,\omega\,|)$$

.

The **Cpo**-enrichment of $\mathcal{K}$ allows us to replace lax cones by cones.

Definition 7.1. Let $\langle i_k : D_k \to D_{k+1} \rangle$ be an ω-chain. Given a lax cone $\langle f_k \rangle : \langle i_k \rangle \xrightarrow[\leq]{\cdot} B$, for $k \in |\,\omega\,|$ we define $f_k^{\#} : D_k \to B$ as $\bigsqcup_{l \geq k} f_l \circ i_{k,l}$ where we write $i_{k,l}$ for the composite $i_{l-1} \circ \ldots \circ i_k$. □

Proposition 7.3. Let $\langle i_k \rangle$ be an ω-chain. Every lax cone $\langle f_k \rangle : \langle i_k \rangle \xrightarrow[\leq]{\cdot} B$ induces a cone $\langle f_k^{\#} \rangle : \langle i_k \rangle \stackrel{\cdot}{\to} B$.

Proof. For $k \in |\,\omega\,|$,

$$\begin{array}{rclcl} f_k^{\#} & = & \bigsqcup_{l \geq k+1} f_l \circ i_{k,l} & = & \bigsqcup_{l \geq k+1} (f_l \circ i_{k+1,l} \circ i_k) \\ & = & (\bigsqcup_{l \geq k+1} f_l \circ i_{k+1,l}) \circ i_k & = & f_{k+1}^{\#} \circ i_k . \end{array}$$ □

Intuitively, $f_k^{\#}$ improves f_k by taking into account the information of the f_l's, for $l \geq k$, restricted to D_k. But, technically, $f_k^{\#}$ may not be f restricted to D_k, as $f_k^{\#} \sqsubseteq f \circ \mu_k$ (see Proposition 7.4 (1)) but the equality *need not* hold (see Subsection 7.2, where this oddity is removed by imposing a *uniformity* condition on the domain structure).

We have thus seen that every ω-chain $\langle [m_k, f_k] \mathrel{\underset{\approx}{\sqsubset}}_{i_k} [m_{k+1}, f_{k+1}] \rangle$ induces the *graph included* ω-chain $\langle [m_k, f_k^{\#}] \subseteq_{i_k} [m_{k+1}, f_{k+1}^{\#}] \rangle$. As we show below, both chains have the same lub and hence the calculation of lubs of arbitrary chains of partial maps can be *reduced* to the calculation of lubs of chains of graph included partial maps.

Proposition 7.4.

1 $\bigsqcup [m_k, f_k]$ exists if and only if so does $\bigsqcup [m_k, f_k^{\#}]$ and they are equal.

2 For $[n,g] : X \rightharpoonup A$, $\bigsqcup([m_k, f_k] \circ [n,g])$ exists if and only if so does $\bigsqcup([m_k, f_k^{\#}] \circ [n,g])$ and they are equal.

Proof.

(1) Observe that for every $k \in |\,\omega\,|$,

$$[m_l, f_l] \sqsubseteq_{\mu_l} [m,f] \text{ for every } l \geq k \quad \Rightarrow \quad [m_k, f_k^{\#}] \sqsubseteq_{\mu_k} [m,f] \tag{10}$$

because assuming the antecedent, $f_l \sqsubseteq f \circ \mu_l$ and then, $f_l \circ i_{k,l} \sqsubseteq f \circ \mu_l \circ i_{k,l} = f \circ \mu_k$ for $l \geq k$, yielding $f_k^{\#} \sqsubseteq f \circ \mu_k$.

Then, by (10) and the fact that for every $k \in |\,\omega\,|$, $[m_k, f_k] \sqsubseteq_{\mathrm{id}} [m_k, f_k^{\#}]$ it follows that

$$[m_k, f_k] \sqsubseteq_{\mu_k} [m,f] \text{ for all } k \in |\,\omega\,| \quad \Leftrightarrow \quad [m_k, f_k^{\#}] \sqsubseteq_{\mu_k} [m,f] \text{ for all } k \in |\,\omega\,|$$

and we are done.

(2) By Proposition 7.4 (1), $\bigsqcup([m_k, f_k] \circ [n,g]) = \bigsqcup[n \circ g^{-1}(m_k), f_k \circ (m_k{}^*g)]$ exists if and only if so does $\bigsqcup[n \circ g^{-1}(m_k), \big(f_k \circ (m_k{}^*g)\big)^{\#}]$ and they are equal. Thus, to conclude the proof it is enough to show that $\big(f_k \circ (m_k{}^*g)\big)^{\#} = f_k^{\#} \circ (m_k{}^*g)$ because then $[n \circ g^{-1}(m_k), \big(f_k \circ (m_k{}^*g)\big)^{\#}] = [n \circ g^{-1}(m_k), f_k^{\#} \circ (m_k{}^*g)] = [n,g] \circ [m_k, f_k^{\#}]$.

Let $[n \circ g^{-1}(m_k), f_k \circ (m_k{}^*g)] \sqsubseteq_{j_k} [n \circ g^{-1}(m_{k+1}), f_{k+1} \circ (m_{k+1}{}^*g)]$ as indicated in the following diagram

$$\begin{tikzcd}
 & & \bullet \arrow[d, tail, "j_k"'] \arrow[rr, "m_k{}^*g"] & & \bullet \arrow[d, tail, "i_k"] \arrow[ddl, tail, bend right, "m_k"'] \arrow[ddr, bend left, "f_k"] \arrow[dr, phantom, "\sqsupseteq"] & \\
 & \bullet \arrow[dl, tail, "n"'] \arrow[drr, "g"] & \bullet \arrow[l] \arrow[rr, "m_{k+1}{}^*g"'] & & \bullet \arrow[dl, tail, "m_{k+1}"] \arrow[dr, "f_{k+1}"] & \\
\bullet & & & \bullet & & \bullet
\end{tikzcd} \tag{11}$$

From (11), by induction, it follows that $(m_l{}^*g) \circ j_{k,l} = i_{k,l} \circ (m_k{}^*g)$ for every $l \geq k$ and hence $\big(f_k \circ (m_k{}^*g)\big)^{\#} = \bigsqcup_{l \geq k} f_l \circ (m_l{}^*g) \circ j_{k,l} = \bigsqcup_{l \geq k} f_l \circ i_{k,l} \circ (m_k{}^*g) = f_k^{\#} \circ (m_k{}^*g)$. $\square$

As mentioned at the beginning of Section 6, checking for continuity of composition on the left is more difficult than checking for continuity of composition on the right. This is reflected in the fact that the technique used to prove Proposition 7.4 does not extend to prove that $\bigsqcup([n,g] \circ [m_k, f_k]) = \bigsqcup([n,g] \circ [m_k, f_k^{\#}])$ for every $[n,g] : B \rightharpoonup Y$. For instance, a straight reduction of this equality to Proposition 7.4 (1) —as done for proving Proposition 7.4 (2)— is not possible because, for a start, there is no reason for which $\mathrm{dom}([n,g] \circ [m_k, f_k]) = [m_k \circ f_k{}^{-1}(n)]$ equals $\mathrm{dom}([n,g] \circ [m_k, f_k^{\#}]) = [m_k \circ f_k^{\#^{-1}}(n)]$.

7.2. Uniform Domain Structures

There are **Cpo**-categories of partial maps in which lubs do not behave as expected, the worst case being that the lub of an ω-chain of partial identities need not be a partial identity (i.e. for every $A \in |\mathcal{K}|$, the partial identity mapping $[\_, \_] : \mathcal{D}(A) \to p\mathcal{K}(A, A)$ need not be continuous). For an example, consider the **Cpo**-category $\mathcal{K}$ pictured as

$$\begin{tikzcd} 0 \arrow[r] \arrow[drr, "0"'] & 1 \arrow[r] \arrow[dr, "1"] & \cdots \arrow[r] & k \arrow[r] \arrow[dl, "k"] & \cdots \\ & & \omega \arrow[loop below, "\perp"] & & \end{tikzcd} \qquad (k \in |\omega|)$$

where $\perp \sqsubset \omega$, and hence forced to be idempotent. Then $\mathbf{Monos}_{\mathcal{K}} = \mathcal{K} \setminus \{\perp\}$ and $p(\mathcal{K}, \mathbf{Monos})$ **Cpo**-enriches. The only infinite chain is $\langle [k, k] \subseteq [k+1, k+1] \rangle$ in $p\mathcal{K}(\omega, \omega)$ and, contradicting all expectations, $\bigsqcup [k, k] = [\omega, \perp] \sqsubset [\omega, \omega]$.

The problem is that for an ω-chain $\langle [m_k] \subseteq_{i_k : D_k \rightarrowtail D_{k+1}} [m_{k+1}] \rangle$ in $\mathcal{D}(A)$ with lub $\langle [m_k] \subseteq_{\mu_k : D_k \rightarrowtail D} [m] \rangle$ the universal property of lubs is not enough to capture the intuition that $D = \bigcup D_k$. Pursuing this intuition, given an $|\omega|$-indexed family $\langle f_k : D_k \to B \rangle$ such that f_{k+1} extends f_k (i.e. $f_k = f_{k+1} \restriction D_k$) there ought to be a *unique uniform extension* $f : D \to B$ of the f_k's (i.e. $f_k = f \restriction D_k$).

In categorical terms, we are asking for the following property to hold

$$\text{If } \bigsqcup \langle [m_k] \subseteq_{i_k} [m_{k+1}] \rangle = \langle [m_k] \subseteq_{\mu_k : D_k \rightarrowtail D} [m] \rangle \text{ then } \langle \mu_k \rangle : \langle i_k \rangle \stackrel{.}{\to} D \text{ is colimiting in } \mathcal{K}. \tag{12}$$

A colimit as in (12) can be used to obtain lubs of ω-chains of admissible subobjects:

Lemma 7.1. Let $\langle [m_k] \subseteq_{i_k} [m_{k+1}] \rangle$ be an ω-chain in $\mathcal{D}(A)$, and let $\langle \mu_k \rangle$ be a colimit in $\mathcal{K}$ for $\langle i_k \rangle$. If the unique mediating morphism $m : \langle \mu_k \rangle \to \langle m_k \rangle$ is admissible, then $\bigsqcup [m_k] = [m]$.

Proof. $\langle \mu_k \rangle : \langle m_k, i_k \rangle \stackrel{.}{\to} m$ is colimiting in $\mathcal{D}/A$ because the forgetful functor $U_A : \mathcal{D}/A \to \mathcal{K}$ reflects colimits. □

When (12) holds for all ω-chains of admissible subobjects, the domain structure is said to be uniform.

Definition 7.2. A domain structure $(\mathcal{K}, \mathcal{D})$ is said to be *uniform* if the forgetful functor $U : \Sigma_{A \in |\mathcal{D}|} \mathcal{D}/A \to \mathcal{K}$ preserves colimits of ω-chains. □

The following should help in understanding uniformity:

Lemma 7.2. The domain structure $(\mathcal{K}, \mathcal{D})$ is uniform and $\mathcal{D}$ is locally a cpo if and only if $p(\mathcal{K}_{\mathbf{Poset}}, \mathcal{D})$ has hom-cpos.

Proof.
$(\Rightarrow)$ $\bigsqcup\langle[m_k, F_k] \subseteq [m_{k+1}, F_{k+1}]\rangle = [m, f]$ where $\bigsqcup[m_k] = \langle[m_k] \subseteq_{\mu_k} [m]\rangle$ and $f : \langle\mu_k\rangle \to \langle F_k\rangle$.
$(\Leftarrow)$ By Proposition 7.1, $\mathcal{D}$ is locally a cpo. Let $\langle[m_k] \subseteq [m_{k+1}]\rangle$ be an ω-chain in $\mathcal{D}(A)$ with lub $\langle[m_k] \subseteq_{\mu_k} [m]\rangle$, and let $\langle F_k\rangle : \langle\mu_k\rangle \dot{\to} B$ in $\mathcal{K}$. Then the ω-chain $\langle[m_k, F_k] \subseteq [m_{k+1}, F_{k+1}]\rangle$ in $p\mathcal{K}(A, B)$ has lub $\langle[m_k, F_k] \subseteq_{\mu_k} [m, f]\rangle$ for some f (because if $\bigsqcup[m_k, F_k] = [n, g]$ then $[m] = \bigsqcup[m_k] = [n]$). Clearly $f : \langle\mu_k\rangle \to \langle F_k\rangle$. To see that it is the unique such, let $g : \langle\mu_k\rangle \to \langle F_k\rangle$ then for every $k \in |\,\omega\,|$, $[m_k, F_k] \subseteq [m, g]$ and therefore $[m, f] \subseteq [m, g]$. Thus, $f = g$. □

It follows from the lemma that $(\mathbf{Set}, \mathbf{Monos})$, $(\mathbf{Poset}, \mathbf{Open})$, $(\mathbf{Cpo}, \mathbf{\Sigma})$, and $(\mathbf{Cpo}_\wedge, \mathbf{\Sigma}_\wedge)$ are uniform. A non-example was given at the beginning of the subsection.

The next proposition exhibits the appropriateness of uniformity by showing that, for uniform domain structures, the partial identity mapping is continuous. Various other consequences are also presented.

Proposition 7.5. Let $(\mathcal{K}, \mathcal{D})$ be a uniform **Poset**-domain-structure. If $p\mathcal{K}$ has hom-cpos w.r.t. $\precsim$, then for every $A \in |\,\mathcal{K}\,|$, the *partial identity* mapping $[\_, \_] : \mathcal{D}(A) \to p\mathcal{K}(A, A)$ is continuous.

Proof. Monotonicity is clear. For preservation of lubs, let $\langle[m_k] \subseteq [m_{k+1}]\rangle$ be an ω-chain in $\mathcal{D}(A)$.
Assume that $\bigsqcup[m_k, m_k] = \langle[m_k, m_k] \precsim_{\mu_k} [m, f]\rangle$. Then, by Proposition 7.1, $\bigsqcup[m_k] = [m]$ and therefore we just need to check that $f = m$.
Since, $[m, m]$ is an upper bound of the ω-chain $\langle[m_k, m_k] \subseteq [m_{k+1}, m_{k+1}]\rangle$ it follows that $[m, f] \precsim [m, m]$ and therefore $f \sqsubseteq m$. Thus, $f \circ \mu_k \sqsubseteq m \circ \mu_k$ and since $m \circ \mu_k = m_k \sqsubseteq f \circ \mu_k$, invoking uniformity the proof is finished. □

Proposition 7.6. Let $(\mathcal{K}, \mathcal{D})$ be a uniform **Cpo**-domain-structure. If $p\mathcal{K}$ **Cpo**-enriches, then

1 $\mathcal{D}(\_)$ is a **Cpo**-functor $\mathcal{K}^{op} \to \mathbf{Cpo}$.
2 If $\langle[m_k] \subseteq [m_{k+1}]\rangle$ is an ω-chain in $\mathcal{D}(A)$ with lub $[m]$, then for every $f, g : A \to B$,

$$f \circ m_k \sqsubseteq g \circ m_k \text{ for every } k \in |\,\omega\,| \text{ if and only if } f \circ m \sqsubseteq g \circ m.$$

3 $\subseteq$ is a subcpo of $\precsim$.

Proof.
(1) By Proposition 7.1, $\mathcal{D}$ is locally a cpo. And $\mathcal{D}(\_)(=)$ is a composition of continuous maps, namely, $\mathrm{dom}([=, =] \circ \_)$.

(2) ($\Rightarrow$) For every $k \in |\,\omega\,|$, $f \circ m_k \sqsubseteq g \circ m_k$ implies $f \circ [m_k, m_k] \precsim g \circ [m_k, m_k]$. Then $f \circ [m, m] \;=\; \bigsqcup f \circ [m_k, m_k] \;\precsim\; \bigsqcup g \circ [m_k, m_k] \;=\; g \circ [m, m]$ and therefore $f \circ m \sqsubseteq g \circ m$.

($\Leftarrow$) For every $k \in |\,\omega\,|$, let $[m_k] \subseteq_{\mu_k} [m]$. Then,

$$f \circ m_k \;=\; f \circ m \circ \mu_k \;\sqsubseteq\; g \circ m \circ \mu_k \;=\; g \circ m_k.$$

(3) For every $A, B \in |\,p\mathcal{K}\,|$, by Propositions 7.1 and 7.2, $(p\mathcal{K}(A, B), \subseteq)$ is a cpo. As $\subseteq$ is a subposet of $\precsim$, we only need to show that for every ω-chain $\langle [m_k, F_k] \subseteq [m_{k+1}, F_{k+1}] \rangle$ with lub $\langle [m_k, F_k] \subseteq_{\mu_k} [m, f] \rangle$ in $(p\mathcal{K}(A, B), \subseteq)$

$$[m_k, F_k] \precsim_{\nu_k} [n, g] \text{ for every } k \in |\,\omega\,| \quad \Rightarrow \quad [m, f] \precsim [n, g].$$

Assuming the antecedent, for every $k \in |\,\omega\,|$, $[m_k] \subseteq_{\nu_k} [n]$ and, because $\bigsqcup [m_k] = [m]$, we have that $[m] \subseteq_i [n]$. Finally, to show that $f \sqsubseteq g \circ i$ observe that $[m_k] \subseteq_{\nu_k} [n]$ and $[m_k] \subseteq_{\mu_k} [m] \subseteq_i [n]$ imply $\nu_k = i \circ \mu_k$. Then, $f \circ \mu_k = F_k \sqsubseteq g \circ \nu_k = g \circ i \circ \mu_k$ and since, by Proposition 7.2 (1), $\bigsqcup [\mu_k] = [\mathrm{id}]$, using Proposition 7.6 (2) we are done. $\square$

*7.3. Characterisation of **Cpo**-Categories of Partial Maps induced by Uniform Domain Structures*

Theorem 7.1. Let $(\mathcal{K}, \mathcal{D})$ be a uniform **Cpo**-domain-structure. Then,

$p(\mathcal{K}, \mathcal{D})$ **Cpo**-enriches with respect to $\precsim$

if and only if

1 $p(\mathcal{K}, \mathcal{D})$ **Poset**-enriches with respect to $\precsim$ (see Theorem 6.1),

2 $\mathcal{D}(\_)$ is a **Cpo**-functor $\mathcal{K}^{op} \to$ **Cpo**, and

3 the colimits of ω-chains preserved by the forgetful functor $U : \Sigma_{A \in |\,\mathcal{D}\,|} \mathcal{D}/A \to \mathcal{K}$ are **Poset**-colimits. $\square$

That the three conditions above are necessary follows from Propositions 7.6 (1), 7.6 (2), 7.2 (1) and 2.1. We show that they are sufficient in a series of steps.

Remark. The idea behind the proof is simple though one has to go through a series of long calculations.

(i) For showing that every hom-poset is a cpo, by Proposition 7.4 (1), we reduce the general case to finding lubs of chains of graph inclusions. Intuitively, the lub of a chain of graph inclusions is the partial map with domain of definition the union of the domains of definition of the partial maps in the chain, and with total component the unique uniform extension of the total components of the partial maps in the chain. (Recall that such extensions exist by the uniformity of the domain structure.)

(ii) For showing that composition on the right is continuous, by Proposition 7.4 (2), the general case is reduced to the graph inclusion case which, in turn, is proved by calculating the relevant lub with the above recipe.

(iii) For showing that composition on the left is continuous the problem is broken down in more subcases. We show that composition on the left with total maps is continuous, and that composition on the left with partial inclusions is continuous for total maps. This allows us to reduce the general case to the graph inclusion case. Then, after proving that composition on the left is continuos for chains of graph inclusions the general case is proved.

Until the end of this subsection, let $(\mathcal{K}, \mathcal{D})$ be a uniform domain structure.

Hom-posets are cpos

Lemma 7.3. (The graph inclusion case) Let $\mathcal{D}$ be locally a cpo, and let $\mathcal{K}$ be **Poset**-enriched. Assume that U maps colimits of ω-chains to **Poset**-colimits. For an ω-chain $\langle [m_k, F_k] \subseteq [m_{k+1}, F_{k+1}] \rangle$,

$$\bigsqcup [m_k, F_k] = \langle [m_k, F_k] \sqsubseteq_{\mu_k} [m, f] \rangle \quad \Leftrightarrow \quad \bigsqcup [m_k] = \langle [m_k] \subseteq_{\mu_k} [m] \rangle \text{ and } f : \langle \mu_k \rangle \to \langle F_k \rangle.$$

Proof.
($\Rightarrow$) $\bigsqcup [m_k] = \langle [m_k] \subseteq_{\mu_k} [m] \rangle$ because dom preserves lubs of ω-chains —see Proposition 7.2 (3). Since U preserves colimits of ω-chains, it follows that $\langle \mu_k \rangle$ is a colimiting cone. Let $g : \langle \mu_k \rangle \to \langle F_k \rangle$, then $[m, g]$ is an upper bound of the ω-chain $\langle [m_k, F_k] \subseteq [m_{k+1}, F_{k+1}] \rangle$ and therefore $f \sqsubseteq g$. Thus, $f \circ \mu_k \sqsubseteq g \circ \mu_k = F_k$, and since $F_k \sqsubseteq f \circ \mu_k$, it follows that $f : \langle \mu_k \rangle \to \langle F_k \rangle$.
($\Leftarrow$) By construction, $[m, f]$ is an upper-bound of $\langle [m_k, F_k] \subseteq [m_{k+1}, F_{k+1}] \rangle$. It is also the least, for assuming that $[m_k, F_k] \sqsubseteq_{\nu_k} [n, g]$, it follows that $[m_k] \subseteq_{\nu_k} [n]$, and therefore $[m] \subseteq_i [n]$. Thus, $\nu_k = i \circ \mu_k$ and then for every $k \in |\, \omega \,|$, $f \circ \mu_k \sqsubseteq g \circ i \circ \mu_k$. Finally, since $\langle \mu_k \rangle$ is a **Poset**-colimit, we conclude that $f \sqsubseteq g \circ i$. □

Proposition 7.7. (Hom-posets are cpos) Let $\mathcal{D}$ be locally a cpo, and let $\mathcal{K}$ be **Cpo**-enriched. If U maps colimits of ω-chains to **Poset**-colimits, then every hom-poset in $p\mathcal{K}$ is a cpo.

Proof. Follows from Lemma 7.3 and Proposition 7.4 (1). □

Composition on the right is continuous

Proposition 7.8. Let $\mathcal{K}$ be **Cpo**-enriched. Assume that U maps colimits of ω-chains to **Poset**-colimits, and that $\mathcal{D}(\_)$ is a functor $\mathcal{K}^{op} \to$ **Cpo**. Then, 1 **(Composition on the right is continuous for ω-chains of graph inclusions)** for every ω-chain $\langle [m_k, F_k] \subseteq [m_{k+1}, F_{k+1}] \rangle$ in $p\mathcal{K}(B, C)$, and every $[n, g] : A \rightharpoonup B$,

$$(\bigsqcup [m_k, F_k]) \circ [n, g] \quad = \quad \bigsqcup ([m_k, F_k] \circ [n, g]).$$

2 **(Composition on the right is continuous)** for every $A \in |\, p\mathcal{K} \,|$, $p\mathcal{K}(\_, A)$ is a functor $p\mathcal{K}^{op} \to \mathbf{Cpo}$.

Proof.
(1) Let $\bigsqcup [m_k, F_k] = \langle [m_k, F_k] \sqsubseteq_{\mu_k} [m, f] \rangle$. By Lemma 7.3, $\bigsqcup [m_k] = \langle [m_k] \subseteq_{\mu_k} [m] \rangle$ and $f : \langle \mu_k \rangle \to \langle F_k \rangle$.
By Corollary 6.1, $\langle [m_k, F_k] \circ [n, g] \subseteq [m_{k+1}, F_{k+1}] \circ [n, g] \rangle$ is an ω-chain. To prove

$$\bigsqcup [n \circ g^{-1}(m_k), F_k \circ ({m_k}^* g)] \;=\; [n \circ g^{-1}(m), f \circ (m^* g)]$$

we will, again, use Lemma 7.3.
First, $\bigsqcup [n \circ g^{-1}(m_k)] \;=\; [n \circ g^{-1}(m)]$ by Proposition 7.2 (2) and the continuity of $\mathcal{D}(g)$.
Second, assume that $n \circ g^{-1}(m_k) \subseteq_{\nu_k} n \circ g^{-1}(m)$. Since $f : \langle \mu_k \rangle \to \langle F_k \rangle$, it follows that

$$\begin{tikzcd}
\cdot \arrow[r, "{m_k}^* g"] \arrow[d, tail, "\nu_k"'] \arrow[dr, phantom, "\lrcorner", very near start] & \cdot \arrow[d, tail, "\mu_k"] \arrow[r, "F_k"] & \cdot \\
\cdot \arrow[r, "m^* g"'] & \cdot \arrow[ur, "f"'] &
\end{tikzcd}$$

and therefore $f \circ (m^* g) : \langle \nu_k \rangle \to \langle F_k \circ ({m_k}^* g) \rangle$.
(2) For every ω-chain $\langle [m_k, f_k] \sqsubseteq [m_{k+1}, f_{k+1}] \rangle$ in $p\mathcal{K}(C, A)$ and $[n, g] : B \rightharpoonup C$, by Lemma 6.1, $\langle [m_k, f_k] \circ [n, g] \sqsubseteq [m_{k+1}, f_{k+1}] \circ [n, g] \rangle$ is an ω-chain in $p\mathcal{K}(B, A)$. And, we have

$$\begin{aligned}
(\bigsqcup [m_k, f_k]) \circ [n, g] \quad &= \quad (\bigsqcup [m_k, f_k^{\#}]) \circ [n, g] && \text{, by Proposition 7.4 (1)} \\
&= \quad \bigsqcup ([m_k, f_k^{\#}] \circ [n, g]) && \text{, by Proposition 7.8 (1)} \\
&= \quad \bigsqcup ([m_k, f_k] \circ [n, g]) && \text{, by Proposition 7.4 (2).}
\end{aligned}$$

□

7.3.1. Composition on the left is continuous

Proposition 7.9. Let $\mathcal{D}$ be locally a cpo, let $\mathcal{K}$ be **Cpo**-enriched, and let $p\mathcal{K}$ be **Poset**-enriched. Assume that U maps colimits of ω-chains to **Poset**-colimits. Then,

1 **(Composition on the left with total maps is continuous)** for every $A \in |\, p\mathcal{K} \,|$, $p\mathcal{K}(A, \_)$ is a functor $\mathcal{K} \to \mathbf{Cpo}$.

2 **(Composition on the left with partial inclusions is continuous for total maps)** For $A \in |\,\mathcal{K}\,|$ and $m \in \mathcal{D}(D, B)$, if the mapping $\mathcal{D}(\_)[m] : \mathcal{K}(A, B) \to \mathcal{D}(A)$ is continuous, then so is the mapping $m^R \circ \_ : \mathcal{K}(A, B) \to p\mathcal{K}(A, D)$.

3 **(Reduction to the graph inclusion case)** If $\mathcal{D}(\_)$ is a **Cpo**-functor $\mathcal{K}^{op} \to$ **Cpo**, then for every ω-chain $\langle [m_k, f_k] \sqsubseteq_{i_k} [m_{k+1}, f_{k+1}] \rangle$ in $p\mathcal{K}(A, B)$ and $[n, g] : B \rightharpoonup C$,

$$\bigsqcup([n, g] \circ [m_k, f_k]) \;=\; \bigsqcup([n, g] \circ [m_k, f_k^{\#}]).$$

Proof.

(1) Let $\langle [m_k, f_k] \sqsubseteq_{i_k} [m_{k+1}, f_{k+1}] \rangle$ be an ω-chain in $p\mathcal{K}(A, B)$.
By Proposition 7.4 (1) and Lemma 7.3, we have that $\bigsqcup [m_k, f_k] = [m, f]$ where $\bigsqcup [m_k] = \langle [m_k] \subseteq_{\mu_k} [m] \rangle$ and $f : \langle \mu_k \rangle \to \langle f_k^{\#} \rangle$.
Then, $g \circ f : \langle \mu_k \rangle \to \langle g \circ f_k^{\#} \rangle$ and since $g \circ f_k^{\#} = (g \circ f_k)^{\#}$, by Proposition 7.4 (1) and Lemma 7.3, $\bigsqcup [m_k, g \circ f_k] = [m, g \circ f]$.
(2) $m^R \circ \_$ is monotone because $p\mathcal{K}$ is **Poset**-enriched.
For the preservation of lubs, if $\langle f_k \rangle$ is an ω-chain in $\mathcal{K}(A, B)$, writing f for $\bigsqcup f_k$, we must prove $\bigsqcup [{f_k}^{-1}(m), m^* f_k] \;=\; [f^{-1}(m), m^* f]$.
The ω-chain $\langle [{f_k}^{-1}(m), m^* f_k] \sqsubseteq [{f_{k+1}}^{-1}(m), m^* f_{k+1}] \rangle$ in $p\mathcal{K}(A, D)$ has $[f^{-1}(m), m^* f]$ as an upper bound because the composition of partial maps is monotone. To see that it is the least assume $[{f_k}^{-1}(m), m^* f_k] \sqsubseteq_{\mu_k} [f^{-1}(m), m^* f]$ and $[{f_k}^{-1}(m), m^* f_k] \sqsubseteq_{\nu_k} [n, g]$.
First, $[{f_k}^{-1}(m)] \subseteq_{\nu_k} [n]$ and therefore $[f^{-1}(m)] \subseteq_i [n]$, by the continuity of $\mathcal{D}(\_)[m]$.
Second, we will show that $m^* f \sqsubseteq g \circ i$.
The inequalities $[{f_k}^{-1}(m)] \subseteq_{\mu_k} [f^{-1}(m)] \subseteq_i [n]$ and $[{f_k}^{-1}(m)] \subseteq_{\nu_k} [n]$, imply $\nu_k = i \circ \mu_k$.
So we have

$$\begin{tikzcd}
 & {} \arrow[dddl, bend right, "n"'] \arrow[dddr, bend left, "g"] & \\
 & {} \arrow[u, "i"'] \arrow[r, phantom, "\sqsubseteq"] & {} \\
 & {} \arrow[u, "\mu_k"'] \arrow[dl] \arrow[dr] & \\
{} \arrow[dr, "f_k"'] & & {} \arrow[dl, "m"] \\
 & {} &
\end{tikzcd}$$

.

Thus, for every $k \in |\,\omega\,|$, $f_k \circ n \circ i \circ \mu_k \sqsubseteq m \circ g \circ i \circ \mu_k$ and since $\langle \mu_k \rangle$ is a **Poset**-colimit, we conclude that $m \circ g \circ i$ is an upper bound of the ω-chain $\langle f_k \circ n \circ i \rangle$. On the other hand, $\bigsqcup (f_k \circ n \circ i) = f \circ n \circ i = f \circ f^{-1}(m) = m \circ (m^* f)$.
Then, $m \circ (m^* f) \sqsubseteq m \circ g \circ i$, and since m is full we are done.

(3) We proceed via two calculations:
First,

$$\begin{aligned}
\textstyle\bigsqcup_{k\geq 0}(n^R \circ [m_k, f_k^\#]) &= \textstyle\bigsqcup_{k\geq 0}(n^R \circ (\bigsqcup_{l\geq k} f_l \circ i_{k,l}) \circ {m_k}^R) \\
&= \textstyle\bigsqcup_{k\geq 0}\left((\bigsqcup_{l\geq k} n^R \circ f_l \circ i_{k,l}) \circ {m_k}^R\right) && \text{, by Proposition 7.9 (2)} \\
&= \textstyle\bigsqcup_{k\geq 0}\bigsqcup_{l\geq k}(n^R \circ f_l \circ i_{k,l} \circ {m_k}^R) && \text{, by Proposition 7.8 (2)} \\
&= \textstyle\bigsqcup_{k\geq 0}(n^R \circ f_k \circ {m_k}^R) \\
&= \textstyle\bigsqcup_{k\geq 0}(n^R \circ [m_k, f_k]).
\end{aligned}$$

Second,

$$\begin{aligned}
\textstyle\bigsqcup([n, g] \circ [m_k, f_k]) &= \textstyle\bigsqcup(g \circ n^R \circ [m_k, f_k]) \\
&= g \circ \textstyle\bigsqcup(n^R \circ [m_k, f_k]) && \text{, by Proposition 7.9 (1)} \\
&= g \circ \textstyle\bigsqcup(n^R \circ [m_k, f_k^\#]) && \text{, by the previous calculation} \\
&= \textstyle\bigsqcup(g \circ n^R \circ [m_k, f_k^\#]) && \text{, by Proposition 7.9 (1)} \\
&= \textstyle\bigsqcup([n, g] \circ [m_k, f_k^\#]).
\end{aligned}$$

□

Proposition 7.10. (Composition on the left is continuous for ω-chains of graph inclusions) Let $\mathcal{K}$ be **Cpo**-enriched. Assume that U maps colimits of ω-chains to **Poset**-colimits, and that $\mathcal{D}(\_)$ is a functor $\mathcal{K}^{op} \to$ **Cpo**. Then, for every ω-chain $\langle [m_k, F_k] \subseteq_{i_k} [m_{k+1}, F_{k+1}]\rangle$ in $p\mathcal{K}(A, B)$ and $[n, g] : B \rightharpoonup C$,

$$\bigsqcup([n, g] \circ [m_k, F_k]) \quad = \quad [n, g] \circ \bigsqcup [m_k, F_k].$$

Proof. Let $\bigsqcup [m_k, F_k] = \langle [m_k, F_k] \sqsubseteq_{\mu_k} [m, f]\rangle$.
By Corollary 6.1, $\langle [n, g] \circ [m_k, F_k] \subseteq [n, g] \circ [m_{k+1}, F_{k+1}]\rangle$ is an ω-chain. To prove

$$\bigsqcup [m_k \circ {F_k}^{-1}(n), g \circ (n^* F_k)] \quad = \quad [m \circ f^{-1}(n), g \circ (n^* f)]$$

we will use Lemma 7.3.
If $[m_k \circ {F_k}^{-1}(n)] \subseteq_{j_k} [m_{k+1} \circ {F_{k+1}}^{-1}(n)]$ then to show $\bigsqcup [m_k \circ {F_k}^{-1}(n)] = [m \circ f^{-1}(n)]$, by Lemma 7.1, it is enough to find a colimiting cone for $\langle j_k\rangle$, say $\langle \nu_k\rangle$, such that $m \circ f^{-1}(n) : \langle \nu_k\rangle \to \langle m_k \circ {F_k}^{-1}(n)\rangle$.
We have

$$
\begin{tikzcd}
{} \arrow[rr, tail, "j_k"] \arrow[d, tail, "F_k{}^{-1}(n)"'] \arrow[drrrr, "n^*F_k"] & & {} \arrow[d, tail, "F_{k+1}{}^{-1}(n)"'] \arrow[drr, "n^*F_{k+1}"] & & \\
{} \arrow[rr, tail, "i_k"] \arrow[dr, "m_k"'] \arrow[drrrr, "F_k"'] & & {} \arrow[dl, "m_{k+1}"] \arrow[drr, "F_{k+1}"] & & {} \arrow[d, tail, "n"] \\
& {} & & & {}
\end{tikzcd}
$$

and so

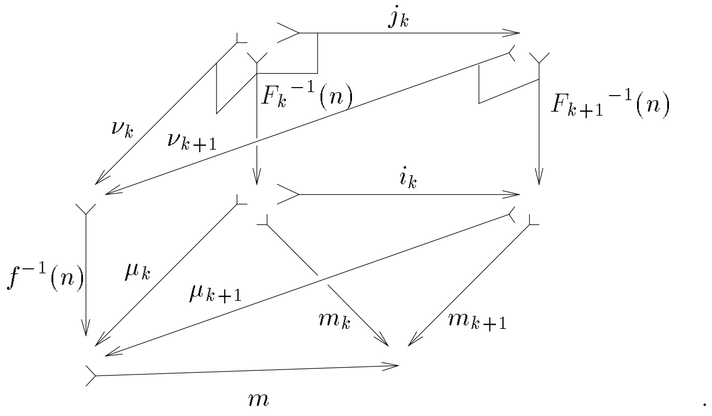
$$
\begin{tikzcd}
{} \arrow[r, tail, "j_k"] \arrow[d, tail, "F_k{}^{-1}(n)"'] & {} \arrow[d, tail, "F_{k+1}{}^{-1}(n)"] \\
{} \arrow[r, tail, "i_k"'] & {}
\end{tikzcd}
\quad .
$$

Moreover, by Lemma 7.3, $f : \langle \mu_k \rangle \to \langle F_k \rangle$, and therefore for every $k \in |\, \omega \,|$ we have

$$
\begin{tikzcd}
{} \arrow[r, tail, "\nu_k"] \arrow[d, tail, "F_k{}^{-1}(n)"'] \arrow[rr, bend left, "n^*F_k"] & {} \arrow[r, "n^*f"] \arrow[d, tail, "f^{-1}(n)"] & {} \arrow[d, tail, "n"] \\
{} \arrow[r, tail, "\mu_k"'] \arrow[rr, bend right, "F_k"'] & {} \arrow[r, "f"'] & {}
\end{tikzcd}
\tag{13}
$$

for some unique ν_k.

The complete picture is as follows



Thus, $\langle \nu_k \rangle$ is a cone for $\langle j_k \rangle$ and $m \circ f^{-1}(n) : \langle \nu_k \rangle \to \langle m_k \circ F_k{}^{-1}(n) \rangle$.
Finally, recall that by Lemma 7.3, $\bigsqcup [m_k] = \langle [m_k] \subseteq_{\mu_k} [m] \rangle$ and so, by Proposition 7.2 (1), $\bigsqcup [\mu_k] = [\mathrm{id}]$. Since, by construction, ν_k is obtained by pulling back μ_k along $f^{-1}(n)$, and $\mathcal{D}\big(f^{-1}(n)\big)$ is continuous, it follows that $\bigsqcup [\nu_k] = [\mathrm{id}]$. Hence, since U preserves colimits of ω-chains, $\langle \nu_k \rangle$ is a colimit for $\langle j_k \rangle$.
To complete the proof, by Lemma 7.3, we need only check whether $g \circ (n^* f) : \langle \nu_k \rangle \to \langle g \circ (n^* F_k) \rangle$. This follows because $(n^* f) \circ \nu_k = n^* F_k$ —see diagram (13). □

Proposition 7.11. (Composition on the left is continuous) Let $\mathcal{K}$ be **Cpo**-enriched and let $p\mathcal{K}$ be **Poset**-enriched. Assume that U maps colimits of ω-chains to **Poset**-colimits and that $\mathcal{D}(\_)$ is a **Cpo**-functor $\mathcal{K}^{op} \to$ **Cpo**. Then, for every $A \in |\, p\mathcal{K}\,|$, $p\mathcal{K}(A, \_)$ is a functor $p\mathcal{K} \to$ **Cpo**.

Proof. Because for every ω-chain $\langle [m_k, f_k] \sqsubsetapprox [m_{k+1}, f_{k+1}] \rangle$ in $p\mathcal{K}(A, B)$, and every $[n, g] : B \rightharpoonup C$, we have

$$\begin{aligned}
[n,g] \circ \bigsqcup [m_k, f_k] &= [n,g] \circ \bigsqcup [m_k, f_k^{\#}] && \text{, by Proposition 7.4 (1)} \\
&= \bigsqcup \big([n,g] \circ [m_k, f_k^{\#}]\big) && \text{, by Proposition 7.10} \\
&= \bigsqcup \big([n,g] \circ [m_k, f_k]\big) && \text{, by Proposition 7.9 (3).}
\end{aligned}$$

□

7.4. *Computing Lubs of ω-Chains of Partial Maps*

A by-product of the characterisation provided by Theorem 7.1 is a recipe for computing lubs of ω-chains of partial maps in **Cpo**-categories of partial maps:

Proposition 7.12. Let $(\mathcal{K}, \mathcal{D})$ be a uniform **Cpo**-domain-structure such that $p\mathcal{K}$ **Cpo**-enriches. For every ω-chain $\langle [m_k, f_k] \sqsubsetapprox_{i_k} [m_{k+1}, f_{k+1}] \rangle$,

$$\bigsqcup [m_k, f_k] = \langle [m_k, f_k] \sqsubsetapprox_{\mu_k : D_k \rightharpoonup D} [m, f] \rangle$$

if and only if

$\langle \mu_k \rangle : \langle D_k, i_k \rangle \dot{\rightarrow} D$ is colimiting in $\mathcal{K}$, $m : \langle \mu_k \rangle \rightarrow \langle m_k \rangle$ and $f : \langle \mu_k \rangle \rightarrow \langle f_k^{\#} \rangle$.

Proof. Follows from Proposition 7.4 (1) and Lemma 7.3. $\square$

8. Enrichment with respect to Specialisation

In this section the characterisation theorems of order-enriched categories of partial maps (Theorems 6.1 and 7.1) are applied when the order is specialisation. As will be seen, **Poset**-enrichment is immediate, whilst **Cpo**-enrichment requires that the operation of pulling back admissible monos along arbitrary maps be continuous.

Proposition 8.1. Let $(\mathcal{K}, \mathcal{D})$ be a domain structure. Then, $\mathcal{K}$ and $p\mathcal{K}$ **Preo**-enrich with respect $\sqsubseteq^s$ and $\precsim^s$ respectively.
Thus, if every object in $\mathcal{K}$ is T_0 (Definition 4.3) then $\mathcal{K}$ and $p\mathcal{K}$ **Poset**-enrich with respect $\sqsubseteq^s$ and $\precsim^s$ respectively.

Proof. It is enough to see that $p\mathcal{K}$ **Preo**-enriches with respect to $\precsim^s$.
This follows because $\precsim^s$ is a preorder and, for $t : X \rightharpoonup A$, $u \precsim^s v : A \rightharpoonup B$ and $w : B \rightharpoonup Y$, we have that

$$\begin{array}{rcl} u \precsim^s v & \Leftrightarrow & \mathcal{D}(u) \subseteq \mathcal{D}(v) \\ & \Rightarrow & \mathcal{D}(w \circ u \circ t) = \mathcal{D}(t) \circ \mathcal{D}(u) \circ \mathcal{D}(w) \subseteq \mathcal{D}(t) \circ \mathcal{D}(v) \circ \mathcal{D}(w) \\ & \Leftrightarrow & w \circ u \circ t \precsim^s w \circ v \circ t. \end{array}$$ $\square$

Proposition 8.2. Let $(\mathcal{K}, \mathcal{D})$ be a uniform domain structure. If $\mathcal{D}$ is locally a cpo, $\mathcal{K}$ has hom-cpos with respect to $\sqsubseteq^s$, and for every $A, B \in |\mathcal{K}|$, the mapping $\mathcal{D}(\_)(=) : \mathcal{K}(A, B) \times \mathcal{D}(B) \rightarrow \mathcal{D}(A)$ is continuous, then $\mathcal{K}$ and $p\mathcal{K}$ **Cpo**-enrich with respect to $\sqsubseteq^s$ and $\precsim^s$ respectively.

Proof. By Proposition 8.1, $\mathcal{K}$ **Poset**-enriches. By hypothesis every hom in $\mathcal{K}$ is a cpo. Moreover, using the fact that $\mathcal{D}(\_) : \mathcal{K}(A, B) \rightarrow \mathbf{Cpo}(\mathcal{D}(B), \mathcal{D}(A))$ is injective, we have that

— since for every $m : D \rightarrowtail B$ in $\mathcal{D}$, the mapping $\mathcal{D}(\_)[m] : \mathcal{K}(A, B) \rightarrow \mathcal{D}(A)$ is continuous, then so is composition on the left in $\mathcal{K}$; and,
— since, in addition, for every $f : A \rightarrow B$, the mapping $\mathcal{D}(f) : \mathcal{D}(B) \rightarrow \mathcal{D}(A)$ is continuous, it follows that so is composition on the right in $\mathcal{K}$.

Therefore, $\mathcal{K}$ **Cpo**-enriches.
By Proposition 8.1, $p\mathcal{K}$ **Poset**-enriches.
Finally, for every ω-chain $\langle [m_k] \subseteq [m_{k+1}] \rangle$ for which $[m_k] \subseteq_{\mu_k : D_k \rightarrow D} \bigsqcup [m_k]$ we show that for $f, f' : D \rightarrow B$,

$$f \circ \mu_k \sqsubseteq^s f' \circ \mu_k \text{ for every } k \in |\omega| \quad \Rightarrow \quad f \sqsubseteq^s f'. \tag{14}$$

This will establish item (3) of Theorem 7.1 and, by invoking this theorem, conclude the proof of the proposition.

For $[m] \in \mathcal{D}(B)$, let $\mathcal{D}(f)[m] = [n]$ and let $\mathcal{D}(f')[m] = [n']$. Assuming the antecedent of (14), it follows that $\mathcal{D}(\mu_k)[n] = \mathcal{D}(f \circ \mu_k)[m] \subseteq \mathcal{D}(f' \circ \mu_k)[m] = \mathcal{D}(\mu_k)[n']$ and therefore $[n \circ n^{-1}(\mu_k)] = [\mu_k \circ {\mu_k}^{-1}(n)] \subseteq [\mu_k \circ {\mu_k}^{-1}(n')] = [n' \circ {n'}^{-1}(\mu_k)]$. Then,

$$
\begin{aligned}
\mathcal{D}(f)[m] &= \bigsqcup [n \circ n^{-1}(\mu_k)] && \text{, because } [n \circ \_] \text{ and } \mathcal{D}(n) \text{ are continuous,} \\
& && \bigsqcup [\mu_k] = [\mathrm{id}] \text{ and } n^{-1}[\mathrm{id}] = [\mathrm{id}] \\
&\subseteq \bigsqcup [n' \circ {n'}^{-1}(\mu_k)] \\
&= \mathcal{D}(f')[m].
\end{aligned}
$$

□

Acknowledgements

This work was done under the supervision of Gordon Plotkin and Barry Jay. I would like to thank them both for their general guidance and specific ideas. I would also like to thank John Power and Eugenio Moggi for intense discussions. Finally, thanks to the referees for comments and suggestions.

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