Complete Cuboidal Sets in Axiomatic Domain Theory

(Extended Abstract)

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Abstract

We study the enrichment of models of axiomatic domain theory. To this end, we introduce a new and broader notion of domain, viz. that of complete cuboidal set, that complies with the axiomatic requirements. We show that the category of complete cuboidal sets provides a general notion of enrichment for a wide class of axiomatic domain-theoretic structures.

Introduction

The aim of *Axiomatic Domain Theory (ADT)* is to provide a conceptual understanding of why domains are adequate as mathematical models of computation. (For a discussion see [12, § Axiomatic Domain Theory].) The approach taken is to axiomatise the structure needed on a category so that its objects can be considered as domains, and its maps as continuous functions. The task of ADT is to explain the traditional approach, and also to provide new concepts and theorems. In particular, part of our agenda is to establish a *representation theory* for domains. Here, as a first step, we concentrate on the enrichment of models of ADT. The intention is that the enriched Yoneda-Grothendieck-Dedekind-Cayley embedding [27] will provide the desired representation (c.f. [15, 11]).

Axiomatic versions of various traditional results in domain theory can be found in e.g. [39, 16, 17, 38, 13, 9, 11, 32]. For instance, in [39], the crucial rôle of **Cpo**-enrichment in the solution of recursive domain equations was recognised and made the central concept of an abstract version of the limit/colimit coincidence theorem [36].

From an axiomatic viewpoint, however, it is desirable not to assume any order-theoretic structure but rather to derive it. The study of the enrichment of domain-theoretic structures in an axiomatic framework began in [11]. There, for a strong axiomatisation, every axiomatic domain-theoretic category was shown to enrich over **Cpo** (the category of small ω-cpos and continuous functions) with respect to an intensional notion of approximation, viz. the *path relation* (see e.g. [8]).

In this paper we study the issue of enrichment for the weakest possible axiomatisation (along the lines of [11] that is). To account for the notions of domain permitted by this axiomatic theory we are forced to abandon **Cpo**-enrichment. Indeed, already the axiomatic considerations of [11] suggested a category of domains that cannot be **Cpo**-enriched in a relevant sense. Thus, what is needed is to broaden the notion of domain!

We take a radical approach and consider domains with a notion of (higher-dimensional) path replacing the traditional notion of approximation formalised by order-theoretic structures. Further, these domains have an algebraic (rather than a universal) notion of passage to the limit (traditionally formalised by lubs). This results in a new notion of domain, viz. that of a *complete cuboidal set*, complying with the axiomatic requirements.

Cuboidal sets play a rôle similar to that played by posets in the traditional setting. They are the analogue of simplicial sets (see e.g. [28, 26]) but with the simplicial category enlarged to the *cuboidal category* $\square$ of *cuboids*, i.e. of finite products $\mathbb{O}_{n_1} \times \ldots \times \mathbb{O}_{n_i}$ of

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finite ordinals. These cuboids are the possible shapes of paths. A cuboidal set P has a set $P(C)$ of paths of every shape $C = \mathbb{O}_{n_1} \times \ldots \times \mathbb{O}_{n_i}$; indeed, it is a (rooted) presheaf over $\square$. The set of points of P is $P(\mathbb{O}_1)$. The set of (one-dimensional) paths of length n is $P(\mathbb{O}_{n+1})$; they can be thought of as (linear) computations conditional on the occurrence of n linearly ordered events $e_1 \leq \ldots \leq e_n$. Evidently, $\mathbb{O}_n$ is the partial order associated to this simple linear event structure [30], and can be considered as a sequential process of length n. At higher dimensions, $P(\mathbb{O}_{n_1} \times \ldots \times \mathbb{O}_{n_i})$ can be thought of as the set of computations conditional on the occurrence of $n_1 + \ldots + n_i$ events ordered by $e_{1,1} \leq \ldots \leq e_{1,n_1}; \ldots; e_{i,1} \leq \ldots \leq e_{i,n_i}$. This is the event structure which can be considered as i sequential processes, of respective lengths $n_1, \ldots, n_i$, running concurrently.

Complete cuboidal sets are cuboidal sets equipped with a *formal-lub* operator satisfying three algebraic laws, which are exactly those needed of the lub operator in order to prove the fixed-point theorem [35]. Computationally, the passage from cuboidal sets to complete cuboidal sets corresponds to allowing infinite processes. In fact, the formal-lub operator assigns paths of shape C to 'paths of shape $C \times \omega$', for every C. Here the set of paths of shape $C \times \omega$ is the colimit of the paths of shape $C \times \mathbb{O}_n$; such paths can be thought of as the higher-dimensional analogue of the increasing sequences of traditional domain theory.

Analogous to what happens in the traditional setting between cpos and posets, the category of complete cuboidal sets appears as the category of Eilenberg-Moore algebras for a certain *ideal-completion* monad on the category of cuboidal sets.

As a central result of the paper we provide an *enrichment theorem* stating that every axiomatic domain-theoretic lifting monad enriches over the category of complete cuboidal sets; as a corollary, so does every domain-theoretic model of recursive types in a wide class.

Organisation of the paper. Section 1 presents the basic structures needed in the rest of the paper and their relationships. In Section 2 we show how to construct models of ADT. To this end, we introduce an algebraic notion of passage to the limit, and describe the ideal-completion monad. This theory is applied in Section 3 to obtain the notion of complete cuboidal set. In Section 4, domain-theoretic lifting monads and domain-theoretic models of recursive types are shown to enrich with respect to the category of complete cuboidal sets. Section 5 indicates further directions for research.

1 Basic concepts

We review the basic structures for discussing models of ADT. Domain-theoretic commutative monads are the central concept of study; the special case of lifting is only needed for Section 4. (For the notion of monad see [28]; for the notion of commutative monad see [24, 29]; for the notion of lifting monad see Appendix A.)

Free algebras and fixed-point objects. An algebra for an endofunctor is said to be *free* [17] if it is initial and if its inverse is a final coalgebra.

A *fixed-point object* [7] for a monad $\mathbb{L} = (L, \eta, \mu)$ on a category with a terminal object 1 is an initial L-algebra $Lw \to w$ equipped with a global element $1 \to w$ invariant under the *successor* loop $\mathsf{succ} \stackrel{\text{def}}{=} w \xrightarrow{\eta_w} Lw \to w$.

Theorem 1.1 *For a commutative monad* $(L, \eta, \mu, \mathrm{t})$ *on a cartesian closed category, the following are equivalent:*

1. *The pair of arrows* $L\overline{\omega} \xrightarrow{\sigma} \overline{\omega} \xleftarrow{\infty} 1$ *is a fixed-point object.*

2. *The algebra* $\sigma : L\overline{\omega} \to \overline{\omega}$ *is free and* $\infty : 1 \to \overline{\omega}$ *is the unique mediating coalgebra morphism from* $\eta_1 : 1 \to L1$ *to* $\sigma^{-1} : \overline{\omega} \to L\overline{\omega}$.

Moreover, when either condition is satisfied, the diagram $1 \xrightarrow{\infty} \overline{\omega} \overset{\mathsf{succ}}{\underset{\mathrm{id}}{\rightrightarrows}} \overline{\omega}$ *is an equaliser.*

The fixed-point object for a monad provides a unique uniform fixed-point operator for endomorphisms on *objects with bottom* (viz. objects equipped with an Eilenberg-Moore algebra structure) —see [7, 17].

Proposition 1.2 *Let* $\mathbb{L}$ *be a commutative monad on a cartesian closed category* $\mathcal{C}$. *Assume that* L *has a free algebra. Then,*

1. *the terminal object* 1 *in* $\mathcal{C}$ *becomes a zero object in the category of Eilenberg-Moore algebras* $\mathcal{C}^{\mathbb{L}}$, *and*

2. *if* $\mathcal{C}$ *has an initial object* 0 *then* $L0 \cong 1$ *and so* 0 *becomes a zero object in the Kleisli category* $\mathcal{C}_{\mathbb{L}}$.

In the situation of the above proposition, one typically writes $\perp_{(A,a)}$ for the unique $\mathbb{L}$-algebra homomorphism from the zero $\mathbb{L}$-algebra $L1 \to 1$ to an $\mathbb{L}$-algebra $a : LA \to A$. Notice that the endofunctor L is *pointed* (i.e. it comes equipped with a global element). Indeed, the family $\{\perp_{(LA,\mu_A)} : 1 \to LA\}$ is a natural transformation $1 \dot{\to} L$.

Domain-theoretic commutative monads. An algebra $s : Lw \to w$ for an endofunctor L on a category with an initial object 0 is said to be *inductive* if the cone $\langle [s]_n \rangle : \langle L^n(0 \to L0) \rangle \dot{\to} w$ inductively defined by $[s]_0 \stackrel{\text{def}}{=} (0 \to w)$ and, for $n \geq 0$, by $[s]_{n+1} \stackrel{\text{def}}{=} s \circ L[s]_n$, is colimiting.

A *pre-domain-theoretic commutative monad* is a commutative monad $\mathbb{L}$ on a cartesian closed category with an initial object, equipped with an inductive initial L-algebra (henceforth denoted $\varsigma : L\omega \to \omega$).

Examples of pre-domain-theoretic commutative monads are the identity monad; the $(\_) + 1$ monad on **Set** (the category of small sets and functions); and the lifting monad $(\_)_\perp$ on **Preo** (the category of small preorders and monotone functions), on **Poset** (the category of small posets and monotone functions), and on $\mathbf{Poset}_\wedge$ (the category of small posets with pullbacks —i.e. binary bounded infs— and stable —i.e. pullback preserving— functions).

A *domain-theoretic commutative monad* $\mathbb{L}$ is a pre-domain-theoretic commutative monad in which the inductive initial L-algebra is free. In this context, the free L-algebra is denoted $\sigma : L\overline{\omega} \to \overline{\omega}$ and the global element $1 \to \overline{\omega}$ invariant under succ is denoted ∞.

Examples of domain-theoretic commutative monads are the lifting monad $(\_)_\perp$ on **Cpo** (the category of small ω-cpos and continuous functions) and on $\mathbf{Cpo}_\wedge$ (the category of small ω-cpos with continuous pullbacks and stable continuous functions). The identity monad is domain-theoretic only on trivial cartesian closed categories.

Models of linear type theory. A *symmetric monoidal functor* $(H, e, m) : \mathcal{V} \to \mathcal{V}'$ between symmetric monoidal categories (see [28] for this notion) consists of a functor $H : \mathcal{V} \to \mathcal{V}'$, a morphism $e : I' \to HI$, and a natural transformation $m_{A,B} : HA \otimes' HB \to H(A \otimes B)$ such that

$$\begin{aligned} & m_{A,B\otimes C} \circ (\mathrm{id}_{HA} \otimes' m_{B,C}) \circ a'_{HA,HB,HC} \\ & \qquad = H(a_{A,B,C}) \circ m_{A\otimes B,C} \circ (m_{A,B} \otimes' \mathrm{id}_{HC}), \\ & r'_{HA} = H(r_A) \circ m_{I,A} \circ (e \otimes \mathrm{id}_{HA}), \\ & m_{B,A} \circ c'_{HA,HB} = H(c_{A,B}) \circ m_{A,B}. \end{aligned}$$

A symmetric monoidal functor is said to be *strong* if e and $m_{A,B}$ are isomorphisms.

There is a standard process that to a symmetric monoidal functor $H : \mathcal{V} \to \mathcal{V}'$ associates a 2-functor $H_* : \mathcal{V}\text{-}\mathbf{CAT} \to \mathcal{V}'\text{-}\mathbf{CAT}$, where for a monoidal category $\mathcal{W}$ we write $\mathcal{W}\text{-}\mathbf{CAT}$ for the (possibly large) 2-category of $\mathcal{W}$-categories, $\mathcal{W}$-functors, and $\mathcal{W}$-natural transformations. For a $\mathcal{V}$-category $\mathcal{K}$, the $\mathcal{V}'$-category $H_*\mathcal{K}$ has the same objects as $\mathcal{K}$, hom-objects $H_*\mathcal{K}(A,B) \stackrel{\text{def}}{=} H\mathcal{K}(A,B)$, identities $I' \to HI \to H\mathcal{K}(A,A)$, and composition $H\mathcal{K}(B,C) \otimes' H\mathcal{K}(A,B) \to H(\mathcal{K}(B,C) \otimes \mathcal{K}(A,B)) \to H\mathcal{K}(A,C)$. If H is strong, the underlying ordinary categories $\mathcal{K}_0$ and $(H_*\mathcal{K})_0$ are isomorphic.

A *model of (intuitionistic) linear type theory* [3] is given by a strong symmetric monoidal functor $F : \mathcal{C} \to \mathcal{D}$ between a cartesian closed category $\mathcal{C}$ and a symmetric monoidal closed category $\mathcal{D}$ with finite products, such that the functor F has a right adjoint. It follows that the right adjoint is a symmetric monoidal functor and thus that the monad on $\mathcal{C}$ induced by the adjunction, henceforth denoted $\mathbb{L}$, is commutative.

The following *folklore* result (see [21] and the references therein) relates (accessible) commutative monads (on locally presentable cartesian closed categories) to models of linear type theory. (For the notions of locally presentable category and accessible monad see Appendix B.)

Theorem 1.3 *Let $\mathbb{L}$ be an accessible commutative monad on a locally presentable cartesian closed category $\mathcal{C}$. Then, $\mathcal{C} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathcal{C}^{\mathbb{L}}$ provides a model of linear type theory.*

In the above theorem, the symmetric monoidal structure on $\mathcal{C}^{\mathbb{L}}$ is given by the *tensor product* $\otimes$ of algebras; defined, for algebras X and Y, as the universal bilinear map $X \times Y \to X \otimes Y$. For details consult [25, 21].

Models of Axiomatic Domain Theory. Domain-theoretic commutative monads and models of linear type theory provide the enrichment structure for defining models of ADT.

A *domain-theoretic enrichment base* is a model of linear type theory such that its induced commutative monad is domain-theoretic.

A *domain-theoretic model of recursive types* with respect to a domain-theoretic enrichment base $F : \mathcal{C} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathcal{D} : U$ is a $\mathcal{D}$-category $\mathcal{M}$ such that the $\mathcal{C}$-category $U_*\mathcal{M}$ is $\mathcal{C}$-algebraically compact (see [17, 9, 32] for this notion). The rôle of algebraic compactness is to provide a universal approach for solving recursive type equations. What is domain-theoretic about these models is that fixed-points of endofunctors (viz. free algebras) are obtained by traditional domain-theoretic methods; viz. the basic lemma [39] and a version of the limit/colimit coincidence theorem [32].

Models of ADT are domain-theoretic models of recursive types admitting a rich type structure (e.g. suitable for interpreting products, higher types, and sums). Canonical examples of models of ADT are **pCpo** (the category of small ω-cpos and partial continuous functions [31]) and $\mathbf{Cppo}_\perp$ (the category of small ω-cpos with bottom element and strict continuous functions) with respect to the domain-theoretic enrichment base $(\_)_\perp : \mathbf{Cpo} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathbf{Cppo}_\perp$.

2 Constructing models of ADT

We explore the construction of models of ADT, first from domain-theoretic commutative monads, then from pre-domain-theoretic ones.

2.1 From domain-theoretic commutative monads

The models of ADT $\mathbf{Cppo}_\perp$ and $\mathbf{pCpo}$ are obtained as an application of the following result to the lifting monad on $\mathbf{Cpo}$.

Theorem 2.1 *Let $\mathbb{L}$ be an accessible commutative monad on a locally presentable cartesian closed category $\mathcal{C}$. Assume that L has an inductive free algebra. Then,*

1. *the category of Eilenberg-Moore algebras $\mathcal{C}^{\mathbb{L}}$ is a domain-theoretic model of recursive types with respect to the domain-theoretic enrichment base $\mathcal{C} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathcal{C}^{\mathbb{L}}$, and*
2. *when $\mathbb{L}$ is a lifting monad, the Kleisli category $\mathcal{C}_{\mathbb{L}}$ is also a domain-theoretic model of recursive types with respect to the domain-theoretic enrichment base $\mathcal{C} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathcal{C}^{\mathbb{L}}$.*

Another traditional domain-theoretic commutative monad satisfying the hypothesis of the theorem is the lifting monad on $\mathbf{Cpo}_\wedge$.

2.2 From pre-domain-theoretic commutative monads

Formally ω-complete objects. Domains are spaces equipped with a notion of approximation (the information order) and a notion of passage to the limit (the lub operator). We now consider objects with an *algebraic* (rather than a universal) notion of passage to the limit. To this purpose we introduce a notion of *formal-lub operator* [12, § Axiomatic Domain Theory].

Consider a pre-domain-theoretic commutative monad $\mathbb{L} = (L, \eta, \mu, \mathrm{t})$ on $\mathcal{C}$. Think of the lifting monad on $\mathbf{Poset}$. Recall that we write $\varsigma : L\omega \to \omega$ for the inductive initial L-algebra, and write succ for the *successor* loop $\omega \xrightarrow{\eta_\omega} L\omega \xrightarrow{\varsigma} \omega$. A *formal-lub operator* (for ω-chains under the invariance succ) on an object D in $\mathcal{C}$ is given by a map $\bigvee : D^\omega \to D$ in $\mathcal{C}$ satisfying the following three algebraic laws:

(Constant)
$$\begin{tikzcd} D \arrow[r, "D^!"] \arrow[dr, "\mathrm{id}_D"'] & D^\omega \arrow[d, "\bigvee"] \\ & D \end{tikzcd} \quad \text{(i.e. } \textstyle\bigvee_n\langle x\rangle = x\text{)},$$

(Diagonal)
$$\begin{tikzcd} D^{\omega\times\omega} \cong (D^\omega)^\omega \arrow[r, "\bigvee^\omega"] \arrow[d, "D^\Delta"'] & D^\omega \arrow[d, "\bigvee"] \\ D^\omega \arrow[r, "\bigvee"'] & D \end{tikzcd}$$

$$\text{(i.e. } \textstyle\bigvee_n\langle\bigvee_m\langle x_{m,n}\rangle\rangle = \bigvee_n\langle x_{n,n}\rangle\text{)},$$

(Shift)
$$\begin{tikzcd} D^\omega \arrow[r, "D^{\mathsf{succ}}"] \arrow[dr, "\bigvee"'] & D^\omega \arrow[d, "\bigvee"] \\ & D \end{tikzcd}$$

$$\text{(i.e. } \textstyle\bigvee_n\langle x_n\rangle = \bigvee_n\langle x_{\mathsf{succ}(n)}\rangle\text{)}.$$

A *formally ω-complete object* is an object equipped with a formal-lub operator. For formally ω-complete objects $(P, \bigvee_P)$ and $(Q, \bigvee_Q)$, a map $f : P \to Q$ in $\mathcal{C}$ is said to be *ω-continuous* if it satisfies the following law:

(Continuity)
$$\begin{tikzcd} P^\omega \arrow[r, "f^\omega"] \arrow[d, "\bigvee_P"'] & Q^\omega \arrow[d, "\bigvee_Q"] \\ P \arrow[r, "f"'] & Q \end{tikzcd}$$

$$\text{(i.e. } f(\textstyle\bigvee_P\langle x_n\rangle) = \bigvee_Q\langle f(x_n)\rangle\text{)}.$$

We write $\bigvee(\mathcal{C})$ for the category of formally ω-complete objects and ω-continuous maps. For our running example, $\bigvee(\mathbf{Poset})$ is $\mathbf{Cpo}$, the category of ω-complete posets and ω-continous functions. If we perform the construction on $\mathbf{Preo}$ (the category of preorders and monotone functions) we obtain the same result; $\bigvee(\mathbf{Preo})$ is again $\mathbf{Cpo}$. The real surprise comes when we consider the construction on structures with not only a one-dimensional notion of approximation (as the above examples) but also with *higher-dimensional* notions of approximation. For instance, consider $\mathbf{Poset}_\wedge$; the pullback "squares" in the posets provide a (particular) "two-dimensional" notion of path. Then, the category $\bigvee(\mathbf{Poset}_\wedge)$ strictly contains the category $\mathbf{Cpo}_\wedge$ as a full subcategory and cannot be enriched over $\mathbf{Cpo}$ in a relevant sense; see [11]. However, in

$\bigvee(\mathbf{Poset}_\wedge)$ the constructions of domain theory are possible (e.g. $\bigvee(\mathbf{Poset}_\wedge)$ has finite sums, is cartesian closed, and admits a lifting monad which satisfies the hypothesis of Theorem 2.1). In the next section we will generalise this to cuboidal sets, our structures with (general) finite higher-order notions of path.

A more succint presentation of the $\bigvee$-construction is as follows. Consider the *exponentiation* commutative monad $\mathbb{W}$ on $\mathcal{C}$ with underlying endofunctor $(\_)^\omega$, unit $D^! : D \to D^\omega$, multiplication $(D^\omega)^\omega \cong D^{\omega\times\omega} \xrightarrow{D^\Delta} D^\omega$, and tensorial strength $A \times D^\omega \to (A \times D)^\omega$ the exponential transpose of the composite $(A \times D^\omega) \times \omega \cong A \times (D^\omega \times \omega) \xrightarrow{\mathsf{id}_A \times \mathsf{eval}} A \times D$ (where eval denotes the evaluation map). Then $\bigvee(\mathcal{C})$ is the full subcategory of the category of Eilenberg-Moore algebras $\mathcal{C}^{\mathbb{W}}$ consisting of the algebras satisfying (Shift).

As in traditional domain theory the formal-lub operator allows the definition of a fixed-point operator for endomorphisms on *objects with bottom*. Define a *formally ω-complete pointed object* to be a triple $(D, d : LD \to D, \bigvee_D : D^\omega \to D)$ where (D, d) is an $\mathbb{L}$-algebra and $(D, \bigvee_D)$ is a formally ω-complete object. For an endomorphism f on D, let $\mathsf{it}_{(D,d)}(f) : \omega \to D$ be the unique L-algebra morphism from the initial L-algebra (ω, ς) to the L-algebra $(D, f \circ d)$, and set $\mathsf{fix}_{(D,d,\bigvee_D)} \stackrel{\text{def}}{=} \bigvee_D \circ \ulcorner \mathsf{it}(f) \urcorner$. The algebraic laws of the formal-lub operator are enough to derive formal versions of the *fixed-point* and *uniformity* properties of the fixed-point operator.

Proposition 2.2 *Let $(A, a, \bigvee_A)$ and $(B, b, \bigvee_B)$ be formally ω-complete pointed objects, let f be an endomophism on A and g an endomorphism on B, and let h be an $\mathbb{L}$-algebra morphism from (A, a) to (B, b). Then,*

1. $f \circ \mathsf{fix}(f) = \mathsf{fix}(f)$, *and*
2. *if* $g \circ h = h \circ f$ *then* $h \circ \mathsf{fix}(f) = \mathsf{fix}(g)$.

The $\bigvee$-construction. As we have seen above, sometimes the $\bigvee$-construction takes us from a pre-domain-theoretic commutative monad (e.g. the lifting monad on **Poset**) to a domain-theoretic one (e.g. the lifting monad on **Cpo**). What happens if we perform the $\bigvee$-construction on a domain-theoretic commutative monad?

Theorem 2.3 *For a domain-theoretic commutative monad on a category $\mathcal{C}$, the forgetful functor $\bigvee(\mathcal{C}) \to \mathcal{C}$ is an isomorphism with inverse the functor sending an object D in $\mathcal{C}$ to the formally $\overline{\omega}$-complete object $(D, D^\infty : D^{\overline{\omega}} \to D)$ in $\bigvee(\mathcal{C})$.*

Consider a domain-theoretic commutative monad $\mathbb{L}$ on $\mathcal{C}$. By the above theorem, every object D in $\mathcal{C}$ admits a unique formal-lub operator (viz. $D^\infty : D^{\overline{\omega}} \to D$) and so, for an $\mathbb{L}$-algebra (D, d), we may simply write $\mathsf{fix}_{(D,d)}$ for $\mathsf{fix}_{(D,d,D^\infty)}$. With this convention, for every endomorphism f on D, we have that $\mathsf{fix}_{(D,d)}(f) = \mathsf{it}_{(D,d)}(f) \circ \infty$ (c.f. [7]).

Since the exponentiation functor $(\_)^\omega$ preserves binary products, under a mild assumption the $\bigvee$-construction preserves cartesian closure.

Proposition 2.4 *Consider a pre-domain-theoretic commutative monad on a category $\mathcal{C}$.*

The forgetful functor $\bigvee(\mathcal{C}) \to \mathcal{C}$ creates products, and hence $\bigvee(\mathcal{C})$ is cartesian. If in addition $\mathcal{C}$ has equalisers then $\bigvee(\mathcal{C})$ is cartesian closed.

Given a pre-domain-theoretic commutative monad $\mathbb{L}$ on $\mathcal{C}$, we will be interested in *extending* $\mathbb{L}$ to $\bigvee(\mathcal{C})$. This is not always possible; e.g. for the $(\_) + 1$ monad on **Set**, the category $\bigvee(\mathbf{Set})$ is equivalent to the *arrow category* $\boxed{0 \to 1}$. However, one can always extend $\mathbb{L}$ to $\mathcal{C}^{\mathbb{W}}$; this is equivalent to asking for a *distributive law* of $\mathbb{L}$ over $\mathbb{W}$ (see [2] for this notion).

Proposition 2.5 *The transformation $L(\_^\omega) \to (L\_)^\omega$ with components $L(D^\omega) \to (LD)^\omega$ defined as the exponential transpose of $L(D^\omega) \times \omega \xrightarrow{\mathsf{t}'} L(D^\omega \times \omega) \xrightarrow{L(\mathsf{eval})} LD$ yields a* distributive law *of $\mathbb{L}$ over $\mathbb{W}$.*

The ideal-completion monad. Consider a pre-domain-theoretic commutative monad $\mathbb{L}$ on a category $\mathcal{C}$. Every object in $\mathcal{C}$ admits a *free* formal ω-completion in $\bigvee(\mathcal{C})$ if the forgetful functor $\bigvee(\mathcal{C}) \to \mathcal{C}$ is monadic. That is, if it has a left adjoint delivering free constructions such that the category of Eilenberg-Moore algebras for the monad on $\mathcal{C}$ induced by the adjunction $\mathcal{C} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \bigvee(\mathcal{C})$ is equivalent to $\bigvee(\mathcal{C})$. In this case, the monad $\mathbb{I}$ representing $\bigvee(\mathcal{C})$ as $\mathcal{C}^{\mathbb{I}}$ is called the *ideal-completion* monad (as it generalises the case of posets).

The question arises as to when the ideal-completion monad $\mathbb{I}$ exists and, if so, whether there exists a distributive law of $\mathbb{L}$ over $\mathbb{I}$ to extend $\mathbb{L}$ to $\bigvee(\mathcal{C})$. Details of a general theory addressing these issues will appear elsewhere.

3 A standard domain-theoretic enrichment base

Cuboidal sets. We write $\mathbb{O}_\kappa$ for the ordinal associated to the cardinal κ.

The *cuboidal category* $\square$ is defined as the full subcategory of $\mathbf{Poset}_\wedge$ determined by the objects (henceforth called *cuboids*) $\mathbb{O}_{n_1} \times \ldots \times \mathbb{O}_{n_i}$ where $i \geq 1$, and $0 \leq n_j < \aleph_0$ for all $1 \leq j \leq i$.

For a small category $\mathcal{C}$ with an initial object 0, we write $\widetilde{\mathcal{C}}$ for the full subcategory of the presheaf topos $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ consisting of the presheaves P such that $P(0) \cong 1$. When the initial object in $\mathcal{C}$ is strict, $\widetilde{\mathcal{C}}$ is the sheaf topos obtained by declaring that the empty cover covers the initial object.

We call $\widetilde{\square}$ the category of *cuboidal sets*. There are two other presentations of $\widetilde{\square}$. Let $\mathbf{F}$ be the full subcategory of $\mathbf{Poset}_\wedge$ containing the initial object and closed under finite products and lifting, and let $\mathbf{FDL}$ be the full subcategory of $\mathbf{Poset}_\wedge$ consisting of the finite distributive lattices. Since $\mathbf{FDL}$ is the Cauchy completion [27] of both $\square$ and $\mathbf{F}$, the full inclusions $\square \hookrightarrow \mathbf{F} \hookrightarrow \mathbf{FDL}$ induce the following equivalences $\widetilde{\square} \simeq \widetilde{\mathbf{F}} \simeq \widetilde{\mathbf{FDL}}$. The category $\mathbf{F}$ plays a crucial rôle in the enrichment theorem of Section 4.

The *Yoneda embedding* $\square \hookrightarrow \widetilde{\square}$ associating to a cuboid C the *representable* cuboidal set $\widetilde{\square}(\_, C)$ provides a full and faithful representation of the cuboidal category into the category of cuboidal sets. As a notational convention, we *identify* the cuboidal category $\square$ with its image under the Yoneda embedding in the category of cuboidal sets $\widetilde{\square}$.

The internal structure of a cuboidal set may be grasped by looking at its paths. A *path* of shape a cuboid C in a cuboidal set P is a map $C \to P$ in $\widetilde{\square}$ (or equivalently, by Yoneda, an element of $P(C)$). To understand paths in some concrete cases it is instructive to look at the following embeddings. Let $\mathcal{C}$ be either $\mathbf{Preo}$, $\mathbf{Poset}$, or $\mathbf{Poset}_\wedge$ and let J denote the inclusion functor $\square \to \mathcal{C}$. Then, the functor $N : \mathcal{C} \to \widetilde{\square}$ associating to an object D in $\mathcal{C}$ the cuboidal set $N(D) \stackrel{\text{def}}{=} \mathcal{C}(J\_, D)$, called the *nerve* of D (c.f. [26]), is full and faithful.

A subset of a poset with pullbacks is said to be a *stable open* if it is upper-closed and closed under pullbacks. The set of stable opens of D in $\mathbf{Poset}_\wedge$ is denoted $\mathcal{O}_\wedge(D)$. In $\mathbf{Poset}_\wedge$ and $\square$, stable opens are closed under identities, composition, and pullbacks along arbitrary maps (i.e. they provide a class of *admissible* subobjects [33].)

The stable opens of a cuboid are easy to visualise; they are either empty or determined by a *vertex* of the cuboid. The notion of stable open can be extended from the cuboidal category to the category of cuboidal sets. The quickest way to this is by declaring a subobject $O \rightarrowtail Q$ of a cuboidal set Q to be *locally stable open* if it appears in a pullback

$$\begin{tikzcd} O \arrow[r] \arrow[d, tail] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{O}_1 \arrow[d, hook] \\ Q \arrow[r] & \mathbb{O}_2 \end{tikzcd} \tag{1}$$

where $\mathbb{O}_1 \hookrightarrow \mathbb{O}_2$ is the unique mono with stable-open image. Equivalently, a subobject $O \rightarrowtail Q$ in $\widetilde{\square}$ is locally stable open if and only if, for every cuboid C in $\square$ and every path $C \to Q$ in $\widetilde{\square}$, we have a pullback square

$$\begin{tikzcd} V \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner", very near start] & O \arrow[d, tail] \\ C \arrow[r] & Q \end{tikzcd}$$

for some stable open V of C and a (necessarily unique) map $V \to O$ in $\widetilde{\square}$.

The stable open $\mathbb{O}_1 \hookrightarrow \mathbb{O}_2$ not only determines the locally stable opens, but also *classifies* them. Indeed, every locally stable open $O \hookrightarrow Q$ in $\widetilde{\square}$ appears in a pullback (1) for a unique *characteristic map* $Q \to \mathbb{O}_2$. Moreover, as the initial object in $\square$ is strict, the category $\widetilde{\square}$ is a topos and in it the *Sierpinski space* $\mathbb{O}_2$ is a *dominance* (see [34] for a general treatment). It follows that the topos of cuboidal sets admits a *lifting* monad $\mathbb{L}$ which we now make explicit.

To understand the following definitions think that a path $C \to LP$ of shape C in LP is described by a pair consisting of its *degree of definedness* represented by a stable open V of the cuboid C and a *total* path $V \to P$ of shape V in P.

- The underlying functor L is given by $(LP)(C) \stackrel{\text{def}}{=} \{(V, x) \mid V \in \mathcal{O}_\wedge(C), x : V \to P\}$ with action on morphisms $(LP)(C' \to C)$ sending $(V, V \to P)$, with V a stable open of C, to $(V', V' \to V \to P)$ where the square

$$\begin{tikzcd} V' \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner", very near start] & V \arrow[d, hook] \\ C' \arrow[r] & C \end{tikzcd}$$

  is a pullback.

- The unit $\eta_P : P \to LP$ sends a path $C \to P$ to $(C, C \to P)$.

- The multiplication $\mu_P : L^2P \to LP$ sends $(V, (V', V' \to P))$, with V a stable open of C and V' a stable open of V, to $(V', V' \to P)$.

- The tensorial strength $\mathrm{t}_{Q,P} : Q \times LP \to L(Q \times P)$ sends $\langle C \to Q, (V, V \to P)\rangle$, with V a stable open of C, to $(V, \langle V \hookrightarrow C \to Q, V \to P\rangle)$.

Theorem 3.1 *([14]) The lifting functor on the category of cuboidal sets preserves non-empty connected colimits.*

The category of cuboidal sets is locally presentable (as it is a sheaf topos) and, by the above theorem, the lifting monad is accessible. It also follows from the above theorem, e.g. using the basic lemma [39], that the lifting functor has an inductive initial algebra. Since connected colimits in the category of cuboidal sets are given pointwise, the inductive initial L-algebra $\omega = \mathsf{colim}\, \langle \mathbb{O}_n \overset{l_n}{\rightarrowtail} \mathbb{O}_{n+1} \rangle_n$, where l_n denotes the inclusion of $\mathbb{O}_n$ into $\mathbb{O}_{n+1}$, has as its set $\omega(C)$ of paths of shape C the set of sequences of the form

$$C \hookleftarrow V_0 \hookleftarrow \ldots \hookleftarrow V_n \hookleftarrow \emptyset \hookleftarrow \ldots \hookleftarrow \emptyset \hookleftarrow \ldots$$

with each V_i a stable open of the cuboid C.

Corollary 3.2 *1. The adjunction $\widetilde{\Box} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \widetilde{\Box}^{\mathbb{L}}$ is a model of linear type theory.*

2. The lifting monad $\mathbb{L}$ on $\widetilde{\Box}$ is pre-domain-theoretic.

Complete cuboidal sets. A cuboidal set is given a notion of passage to the limit by equipping it with a formal-lub operator. This results in the category $\bigvee(\widetilde{\Box})$ of formally ω-complete cuboidal sets (henceforth simply called *complete cuboidal sets*).

In this subsection, we study the domain-theoretic structure of $\bigvee(\widetilde{\Box})$. We start by observing that $\bigvee(\widetilde{\Box})$ embeds the categories **Cpo** and $\bigvee(\mathbf{Poset}_\wedge)$; hence also $\mathbf{Cpo}_\wedge$. Let $\mathcal{C}$ be either **Poset** or $\mathbf{Poset}_\wedge$. For every object D in $\mathcal{C}$, there is a canonical isomorphism $(ND)^\omega \overset{\cdot}{\cong} N(D^{\mathbb{O}_{\aleph_0}})$ in $\widetilde{\Box}$ (c.f. (4) below), where N is the nerve functor $\mathcal{C} \to \widetilde{\Box}$ defined in the previous subsection. Then, the functor $\bigvee(N) : \bigvee(\mathcal{C}) \to \bigvee(\widetilde{\Box})$ associating to an object $(D, \bigvee)$ in $\bigvee(\mathcal{C})$ the complete cuboidal set $(N(D), (ND)^\omega \cong N(D^{\mathbb{O}_{\aleph_0}}) \xrightarrow{N(\bigvee)} ND)$ is full and faithful.

Theorem 3.3 *1. The category $\bigvee(\widetilde{\Box})$ is locally presentable and cartesian closed.*

2. The lifting monad $\mathbb{L}$ on $\widetilde{\Box}$ extends to an accessible commutative monad $\check{\mathbb{L}}$ on $\bigvee(\widetilde{\Box})$.

The above result follows from a general theory alluded to in Section 2, § The ideal-completion monad. Here we only give an idea of the proof.

One shows that $\widetilde{\Box}$ admits an *accessible* ideal-completion monad $\mathbb{I}$; hence $\bigvee(\widetilde{\Box}) \simeq \widetilde{\Box}^{\mathbb{I}}$ is locally presentable (see Appendix B). The cartesian closure of $\bigvee(\widetilde{\Box})$ follows from Proposition 2.4.

In fact, one can construct the ideal-completion monad $\mathbb{I}$ as a quotient of the exponentiation monad $\mathbb{W}$, and show that the distributive law $\mathbb{L}\mathbb{W} \overset{\cdot}{\to} \mathbb{W}\mathbb{L}$ *collapses* to an isomorphism $\mathbb{L}\mathbb{I} \overset{\cdot}{\cong} \mathbb{I}\mathbb{L}$ yielding another distributive law. Then, the functor L on $\widetilde{\Box}$ extends to $\widetilde{\Box}^{\mathbb{I}}$; if $\bigvee_P : I(P) \to P$ is an $\mathbb{I}$-algebra then so is the composite $ILP \cong LIP \xrightarrow{L(\bigvee_P)} LP$. We conjecture that the commutative monad $\check{\mathbb{L}}$ on $\bigvee(\widetilde{\Box})$ is a lifting monad.

For illustrative purposes, for a complete cuboidal set $\bigvee_P : P^\omega \to P$, we give an explicit description of the formal-lub operator $\bigvee_{LP} : (LP)^\omega \to LP$ in two steps. First, to every generalised chain $C \to (LP)^\omega$ of shape C in LP we associate a path $C \to L(P^\omega)$ again of shape C in $L(P^\omega)$ by (intuitively) removing bottom elements from the chain whenever possible. Second, we define the action of the formal-lub operator $\bigvee_{LP}$ on generalised chains $C \to (LP)^\omega$ as the composite $C \to L(P^\omega) \xrightarrow{L(\bigvee_P)} LP$. We describe the passage from $C \to (LP)^\omega$ to $C \to L(P^\omega)$. We have the following bijective correspondence:

$$\begin{array}{c}
C \to (LP)^\omega \\
\hline\hline
C \times \omega \to LP \\
\hline\hline
\begin{tikzcd}
C \times \mathbb{O}_n \arrow[d, tail, "\mathrm{id}_C \times l_n"'] & U_n \arrow[l, hook'] \arrow[r] \arrow[d, tail] \arrow[dl, phantom, "\lrcorner"] & P \\
C \times \mathbb{O}_{n+1} & U_{n+1} \arrow[l, hook'] \arrow[ur] &
\end{tikzcd}
\qquad (n \geq 0)
\end{array}$$

with the crucial property that there exists a *least* j such that, for all $n < j$, $U_n = \emptyset$ and, for all n, $U_{j+n} \cong V \times \mathbb{O}_{1+n}$ for a unique $V \in \mathcal{O}_\wedge(C)$. Then we define $C \to L(P^\omega)$ as the characteristic map of the partial map $C \hookleftarrow V \to P^\omega$ where $V \to P^\omega$ is the exponential transpose of the unique mediating map $V \times \omega \to P$ from the colimiting cone $\langle V \times \mathbb{O}_{n+1} \to V \times \omega \rangle_n$ to the cone $\langle V \times \mathbb{O}_{n+1} \cong U_{j+n} \to P \rangle_n$.

Theorem 3.4 *The commutative monad $\check{\mathbb{L}}$ on $\bigvee(\widetilde{\Box})$ is domain-theoretic.*

Notice that since $I\mathbb{O}_0 \cong \mathbb{O}_0$ and $ILP \cong LIP$, for finite n, we have that $\mathbb{O}_n$ is isomorphic to its formal ideal completion $I\mathbb{O}_n$. On the other hand, the formal ideal completion $\overline{\omega} \overset{\text{def}}{=} I(\omega)$ of ω with structure map $\sigma \overset{\text{def}}{=} (L(I\omega) \cong I(L\omega) \underset{\cong}{\xrightarrow{I\varsigma}} I\omega)$ and global element $\infty \overset{\text{def}}{=} (1 \xrightarrow{\ulcorner \iota_\omega \urcorner} (I\omega)^\omega \xrightarrow{\bigvee} I\omega)$, where we write ι for the unit of $\mathbb{I}$, provides an inductive $\check{L}$-fixed-point object.

We give an explict description of $\overline{\omega}$ and ∞. We will present $\overline{\omega}$ as a quotient of ω^ω. The following bijective correspondences describe the paths of ω^ω:

$$C \to \omega^\omega$$

$$\overline{\overline{}}$$

$$C \times \omega \to \omega$$

$$\overline{\overline{}}$$

$$\begin{tikzcd} C \times \mathbb{O}_n \arrow[r] & \omega \\ C \times \mathbb{O}_{n+1} \arrow[u, leftarrowtail, "\mathrm{id}_c \times l_n"'] \arrow[ur] & \end{tikzcd} \qquad (n \geq 0)$$

$$\overline{\overline{}}$$

$$\begin{tikzcd} V_{n,j+1} \arrow[r, hook] \arrow[d, tail] \arrow[dr, phantom, "\lrcorner", very near start] & V_{n,j} \arrow[r, hook] \arrow[d, tail] \arrow[dr, phantom, "\lrcorner", very near start] & C \times \mathbb{O}_n \arrow[d, tail, "\mathrm{id}_C \times l_n"] \\ V_{n+1,j+1} \arrow[r, hook] & V_{n+1,j} \arrow[r, hook] & C \times \mathbb{O}_{n+1} \end{tikzcd}$$

$$(n, j \geq 0)$$

$$\overline{\overline{}}$$

$$C \times \mathbb{O}_{\aleph_0} \hookleftarrow V_0 \hookleftarrow \ldots \hookleftarrow V_n \hookleftarrow \ldots$$

$$\overline{\overline{}}$$

$$v_0 \leq \ldots \leq v_n \leq \ldots \text{ in } C \times \mathbb{O}_{\aleph_0}$$

Then the quotient of ω^ω yielding $\overline{\omega}$ is done by *ideal-completing locally*: for every cuboid C, the set $\overline{\omega}(C)$, of paths of shape C in $\overline{\omega}$, may be identified with $C \times \mathbb{O}_{\aleph_0+1}$. And $\infty_C \in C \times \mathbb{O}_{\aleph_0+1}$ is the pair consisting of the least element of C and the greatest element of $\mathbb{O}_{\aleph_0+1}$.

As $\overline{\omega}$ admits an $\check{\mathbb{L}}$-algebra structure (viz. the composite $\check{L}\overline{\omega} \xrightarrow{L\sigma^{-1}} \check{L}^2\overline{\omega} \xrightarrow{\mu_{\overline{\omega}}} \check{L}\overline{\omega} \xrightarrow{\sigma} \overline{\omega}$), by the formal fixed-point property (Proposition 2.2 (1)), endomorphisms on it have fixed-points. It follows that $\mathsf{fix}(\mathsf{succ}) = \infty$.

Corollary 3.5 *The adjunction* $\bigvee(\widetilde{\Box}) \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \bigvee(\widetilde{\Box})^{\check{\mathbb{L}}}$ *is a domain-theoretic enrichment base with respect to which the category of Eilenberg-Moore algebras* $\bigvee(\widetilde{\Box})^{\check{\mathbb{L}}}$ *is a domain-theoretic model of recursive types.*

4 An enrichment theorem for domain-theoretic lifting monads

We show that every domain-theoretic *lifting* monad enriches over the category of complete cuboidal sets.

More precisely, say that a $\mathcal{V}$-category $\mathcal{K}$ provides an enrichment of the ordinary category $\mathcal{C}$ if the underlying ordinary category $\mathcal{K}_0$ (with hom-sets $\mathcal{K}_0(A, B) \stackrel{\text{def}}{=} \mathcal{V}(I, \mathcal{K}(A, B))$) and $\mathcal{C}$ are isomorphic. Then, for a domain-theoretic lifting monad on $\mathcal{C}$ we will construct a $\bigvee(\widetilde{\Box})$-category providing an enrichment of the ordinary category $\mathcal{C}$ in such a way that the domain-theoretic structure also enriches. It will follow as a corollary that every domain-theoretic model of recursive types in a wide class also enriches over $\bigvee(\widetilde{\Box})$.

For a symmetric monoidal category $\mathcal{W}$, write $\gamma_{\mathcal{W}}$ (omitting the subscript if it is clear from the context) for the *global-sections* functor $\mathcal{W}(I, \_) : \mathcal{W} \to \mathbf{Set}$. For a symmetric monoidal functor $H : \mathcal{V} \to \mathcal{V}'$ such that $(\mathcal{V} \xrightarrow{H} \mathcal{V}' \xrightarrow{\gamma_{\mathcal{V}'}} \mathbf{Set}) \stackrel{.}{\cong} (\mathcal{V} \xrightarrow{\gamma_{\mathcal{V}}} \mathbf{Set})$, the $\mathcal{V}'$-category $H_*\mathcal{K}$ (defined in Section 1, § Models of linear type theory) provides an enrichment of $\mathcal{K}_0$.

Thus, for a domain-theoretic lifting monad on $\mathcal{C}$ we aim at producing a cartesian functor $\mathcal{C} \to \bigvee(\widetilde{\Box})$ such that $(\mathcal{C} \to \bigvee(\widetilde{\Box}) \xrightarrow{\gamma} \mathbf{Set}) \stackrel{.}{\cong} (\mathcal{C} \xrightarrow{\gamma} \mathbf{Set})$. Then, since $\mathcal{C}$ enriches over itself (as it is cartesian closed) the above process will provide the aforementioned enrichment result.

We start our considerations with respect to the pre-domain-theoretic case.

Enrichment for pre-domain-theoretic lifting monads. Let $\mathbb{L} = (L, \eta, \mu)$ be a pre-domain-theoretic lifting monad on $\mathcal{C}$.

We define a functor $S : \Box \to \mathcal{C}$. For $n \geq 0$, we write Σ_n for $L^n 0$ and define the action of S on objects by the mapping sending $\mathbb{O}_{n_1} \times \ldots \times \mathbb{O}_{n_i}$ to $\Sigma_{n_1} \times \ldots \times \Sigma_{n_i}$. To define the action of S on morphisms we proceed in two steps. First, for $n \geq 0$, we define $S_n : \Box(C, \mathbb{O}_n) \to \mathcal{C}(S(C), \Sigma_n)$. Second, we let $S : \Box(C, \mathbb{O}_{n_1} \times \ldots \times \mathbb{O}_{n_i}) \to \mathcal{C}(S(C), \Sigma_{n_1} \times \ldots \times \Sigma_{n_i})$ be the mapping $f \mapsto \langle S_{n_1}(f_1), \ldots, S_{n_i}(f_i)\rangle$ where $f_j \stackrel{\text{def}}{=} (C \xrightarrow{f} \mathbb{O}_{n_1} \times \ldots \times \mathbb{O}_{n_i} \xrightarrow{\pi_j} \mathbb{O}_{n_j})$. We give an inductive definition of S_n. For $n = 0$, the action $S_0 : \Box(C, \mathbb{O}_0) \to \mathcal{C}(S(C), 0)$ is uniquely determined by its target. For $n = m + 1$ with $m \geq 0$, let $C = \mathbb{O}_{n_1} \times \ldots \times \mathbb{O}_{n_i}$ and consider $f : C \to \mathbb{O}_{m+1}$ for which the following diagram

$$\begin{tikzcd} \mathbb{O}_{n'_1} \times \ldots \times \mathbb{O}_{n'_i} \arrow[r, "f'"] \arrow[d, hook] & \mathbb{O}_m \arrow[d, hook] \\ \mathbb{O}_{n_1} \times \ldots \times \mathbb{O}_{n_i} \arrow[r, "f"'] & \mathbb{O}_{m+1} \end{tikzcd}$$

is a pullback in $\Box$. Then, we let $S_{m+1}(f)$ be the unique characteristic map making the diagram

$$\begin{tikzcd} \Sigma_{n'_1} \times \ldots \times \Sigma_{n'_i} \arrow[r, "S_m(f')"] \arrow[d, hook, "u_1 \times \ldots \times u_i"'] & \Sigma_m \arrow[d, hook, "\eta_{\Sigma_m}"] \\ \Sigma_{n_1} \times \ldots \times \Sigma_{n_i} \arrow[r, "S_{m+1}(f)"'] & \Sigma_{m+1} \end{tikzcd} ,$$

where $u_j \stackrel{\text{def}}{=} (\Sigma_{n'_j} \xhookrightarrow{\eta} \cdots \xhookrightarrow{\eta} \Sigma_{n_j})$, a pullback in $\mathcal{C}$. By construction, S is indeed a functor; moreover, it preserves the initial object, finite products, and pullbacks of stable opens along arbitrary maps.

We remark on a more conceptual definition of the functor S. The category $\mathbf{F}$ is the *free* cartesian category with an initial object and a lifting monad $\mathbb{L}$ such that $L0 \cong 1$; and, the functor $S : \square \to \mathcal{C}$ is the composite $\square \hookrightarrow \mathbf{F} \to \mathcal{C}$ where $\mathbf{F} \to \mathcal{C}$ is the unique structure preserving functor given by freeness.

Define the *nerve* functor $N : \mathcal{C} \to \widetilde{\square}$ with action given by the mapping $f \mapsto \mathcal{C}(S\_, f)$. Thus for a cuboid C in $\square$ and an object D in $\mathcal{C}$, we have the following bijective correspondence

$$\frac{C \to N(D)}{\overline{S(C) \to D}}$$

stating that the paths of shape C in the cuboidal set $N(D)$ are the paths of shape $S(C)$ in the object D. For instance, the C-parameterised paths $C \times \mathbb{O}_n \to N(D)$ in $\widetilde{\square}$ are in bijective correspondance with the $S(C)$-parameterised paths $S(C) \times \Sigma_n \to D$ in $\mathcal{C}$. It follows that,

$$\gamma_{\widetilde{\square}} \circ N \quad \dot{\cong} \quad \gamma_{\mathcal{C}} \tag{2}$$

$$(ND)^{\mathbb{O}_n} \quad \cong \quad N(D^{\Sigma_n}) \tag{3}$$

The nerve functor has the crucial property of preserving limits. We thus have the following two consequences.

1. Applying the N_* functor to $\mathcal{C}$ regarded as a $\mathcal{C}$-category we obtain the $\widetilde{\square}$-category $N_*\mathcal{C}$ which, by (2), provides an enrichment of $\mathcal{C}$.

2. The isomorphism (3) extends to the limit; that is,

$$(ND)^{\omega} \quad \cong \quad N(D^{\omega_{\mathcal{C}}}) \tag{4}$$

   as $(ND)^{\omega} \cong (ND)^{\mathsf{colim}\,\mathbb{O}_n} \cong \mathsf{lim}\,(ND)^{\mathbb{O}_n} \cong \mathsf{lim}\,N(D^{\Sigma_n}) \cong N(\mathsf{lim}\,D^{\Sigma_n}) \cong N(D^{\mathsf{colim}\,\Sigma_n}) \cong N(D^{\omega_{\mathcal{C}}})$.

In fact, from (4) we obtain a natural isomorphism $\nu : (N\_)^{\omega} \dot{\cong} N(\_^{\omega_{\mathcal{C}}})$ making the pair (N, ν) into a morphism $\mathbb{W}_{\mathcal{C}} \to \mathbb{W}$ of monads (see [40] for this notion). We thus get a functor $\mathcal{C}^{\mathbb{W}_{\mathcal{C}}} \to \widetilde{\square}^{\mathbb{W}}$ sending a $\mathbb{W}_{\mathcal{C}}$-algebra $D^{\omega_{\mathcal{C}}} \xrightarrow{\bigvee} D$ to the $\mathbb{W}$-algebra $(ND)^{\omega} \xrightarrow[\cong]{\nu} N(D^{\omega_{\mathcal{C}}}) \xrightarrow{N(\bigvee)} ND$. Moreover, the diagram

$$\begin{tikzcd}
(N_)^{\omega} \arrow[r, "\nu", "\cong"'] \arrow[d, "(N_)^{\mathsf{succ}}"'] & N(_^{\omega_{\mathcal{C}}}) \arrow[d, "N(_^{\mathsf{succ}_{\mathcal{C}}})"] \\
(N_)^{\omega} \arrow[r, "\nu", "\cong"'] & N(_^{\omega_{\mathcal{C}}})
\end{tikzcd}$$

commutes, and so the functor $\mathcal{C}^{\mathbb{W}_{\mathcal{C}}} \to \widetilde{\square}^{\mathbb{W}}$ cuts down to a functor $\bigvee(\mathcal{C}) \to \bigvee(\widetilde{\square})$.

Enrichment for domain-theoretic lifting monads. Applying the above discussion to a domain-theoretic lifting monad $\mathbb{L}$ on $\mathcal{C}$, we get a limit preserving functor $\bigvee(N) : \mathcal{C} \cong \bigvee(\mathcal{C}) \to \bigvee(\widetilde{\square})$, sending an object D to the complete cuboidal set $(N(D), (ND)^{\omega} \xrightarrow[\cong]{\nu} N(D^{\overline{\omega}}) \xrightarrow{N(D^{\infty})} ND)$, such that $(\mathcal{C} \xrightarrow{\bigvee(N)} \bigvee(\widetilde{\square}) \xrightarrow{\gamma} \mathbf{Set}) \dot{\cong} (\mathcal{C} \xrightarrow{\gamma} \mathbf{Set})$.

We have thus obtained the first part of the following result.

Theorem 4.1 (Enrichment theorem) *Let $\mathbb{L}$ be a domain-theoretic* lifting *monad on $\mathcal{C}$.*

1. *The $\bigvee(\widetilde{\square})$-category $\bigvee(N)_*\mathcal{C}$ provides an enrichment of $\mathcal{C}$.*

2. *In $\bigvee(N)_*\mathcal{C}$, the cartesian closed structure of $\mathcal{C}$, the commutative monad $\mathbb{L}$ on $\mathcal{C}$, and the colimit defining $\overline{\omega}$ also enrich over $\bigvee(\widetilde{\square})$.*

Enrichment for domain-theoretic models of recursive types. Every domain-theoretic model of recursive types with respect to a domain-theoretic enrichment base inducing a lifting monad enriches over the category of complete cuboidal sets.

Corollary 4.2 *Let $\mathcal{M}$ be a domain-theoretic model of recursive types with respect to a domain-theoretic enrichment base $F : \mathcal{C} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathcal{D} : U$ inducing a* lifting *monad.*

The $\bigvee(\widetilde{\square})$-category $\bigvee(N)_*U_*\mathcal{M}$ provides an enrichment of $\mathcal{M}_0$.

We conjecture that (possibly under some mild assumptions) domain-theoretic models of recursive types enrich over $\bigvee(\widetilde{\square})^{\mathbb{L}}$.

5 Concluding remarks

We have introduced a new notion of domain, viz. that of *complete cuboidal set*, that complies with the requirements of ADT. To this end, we have provided an algebraic notion of passage to the limit with respect to which a definition of ideal completion was given. Further, we have shown that the category of complete cuboidal sets provides a general notion of enrichment, traditionally attributed to the category of cpos, for a wide class of domain-theoretic structures (viz. domain-theoretic lifting monads and certain domain-theoretic models of recursive types).

Many further directions of research are possible; we mention a few here.

There are two orthogonal directions for extending the enrichment theorem: by incorporating binary sums into the domain-theoretic lifting monads, or by considering arbitrary commutative monads (rather than lifting ones).

As in [10], the enrichment theorem seems likely to provide the basis for developing a representation theory. This possibility is under investigation.

Connections with synthetic domain theory are envisaged. In particular, a synthetic characterisation of the category of complete cuboidal sets seems to be available. This might help in settling our first conjecture (see Section 3, § Complete cuboidal sets), which is important for our representation programme.

At a more speculative level, we wonder what the relationship is between this work and presheaf or higher-dimensional models of concurrency [23, 42, 6, 41, 19, 18, 5, 20].

References

[1] J. Adámek and J. Rosický. *Locally Presentable and Accessible Categories.* Cambridge University Press, 1994.

[2] M. Barr and C. Wells. *Toposes, Triples and Theories.* Springer-Verlag, 1985.

[3] N. Benton. A mixed linear and non-linear logic: Proofs, terms and models. In *CSL'94 Conf.*, volume 933 of *Lecture Notes in Computer Science*, 1995.

[4] F. Borceux. *Handbook of Categorical Algebra.* Cambridge University Press, 1994.

[5] G.L. Cattani and V. Sassone. Higher dimensional transition systems. In *11*$^{\text{th}}$ *LICS Conf.*, pages 55–62. IEEE, Computer Society Press, 1996.

[6] G.L. Cattani and G. Winskel. Presheaf models for concurrency. Presented at *CSL'96 Conf.*, 1996.

[7] R.L. Crole and A.M. Pitts. New foundations for fixpoint computations: FIX-hyperdoctrines and the FIX-logic. *Information and Computation*, 98:171–210, 1992.

[8] M.P. Fiore. Lifting as a KZ-doctrine. In D. Pitt, D. Rydeheard, and P. Johnstone, editors, *CTCS'95 Conf.*, volume 953 of *Lecture Notes in Computer Science*, pages 146–158. Springer-Verlag, 1995.

[9] M.P. Fiore. *Axiomatic Domain Theory in Categories of Partial Maps.* Cambridge University Press Distinguished Dissertations in Computer Science, 1996.

[10] M.P. Fiore. Enrichment and representation theorems for categories of domains and continuous functions. Manuscript (available from http://www.dcs.ed.ac.uk/home/mf/), 1996.

[11] M.P. Fiore. An enrichment theorem for an axiomatisation of categories of domains and continuous functions. To appear in *Mathematical Structures in Computer Science* —Proceedings of the Workshop LDPL'95, 1997.

[12] M.P. Fiore, A. Jung, E. Moggi, P. O'Hearn, J. Riecke, G. Rosolini, and I. Stark. Domains and denotational semantics: History, accomplishments and open problems. In number 59 of *Bulletin of the European Association for Theoretical Computer Science*, pages 227–256, 1996.

[13] M.P. Fiore and G.D. Plotkin. An axiomatisation of computationally adequate domain-theoretic models of FPC. In *9*$^{\text{th}}$ *LICS Conf.*, pages 92–102. IEEE, Computer Society Press, 1994.

[14] M.P. Fiore and G.D. Plotkin. An extension of models of axiomatic domain theory to models of synthetic domain theory. Presented at *CSL'96 Conf.*, 1996.

[15] P. Freyd. *Abelian Categories: An Introduction to the Theory of Functors.* Harper & Row, 1964.

[16] P.J. Freyd. Recursive types reduced to inductive types. In *5*$^{\text{th}}$ *LICS Conf.*, pages 498–507. IEEE, Computer Society Press, 1990.

[17] P.J. Freyd. Algebraically complete categories. In A. Carboni, M.C. Pedicchio, and G. Rosolini, editors, *Category Theory*, volume 1488 of *Lecture Notes in Mathematics*, pages 131–156. Springer-Verlag, 1991.

[18] E. Goubault. Domains of higher dimensional automata. In *CONCUR'93 Conf.*, volume 715 of *Lecture Notes in Computer Science*, pages 293–307, 1993.

[19] E. Goubault and T.P. Jensen. Homology of higher dimensional automata. In *CONCUR'92 Conf.*, volume 630 of *Lecture Notes in Computer Science*, pages 254–268, 1992.

[20] M. Herlihy and S. Rajsbaum. Algebraic topology and distributed computing: A primer. In J. van Leeuwen, editor, *Computer Science Today: Recent Trends and Developments*, volume 1000 of *Lecture Notes in Computer Science*, pages 203–217, 1995.

[21] B. Jacobs. Semantics of weakening and contraction. *Annals of Pure and Applied Logic*, 69:73–106, 1994.

[22] P.T. Johnstone. *Topos Theory.* Academic Press, 1977.

[23] A. Joyal, M. Nielsen, and G. Winskel. Bisimulation from open maps. *Information and Computation*, 127(2):164–185, 1996.

[24] A. Kock. Monads on symmetric monoidal closed categories. *Arch. Math. (Basel)*, 21:1–10, 1970.

[25] A. Kock. Bilinearity and cartesian closed monads. *Math. Scand.*, 29:161–174, 1971.

[26] S. Mac Lane and I. Moerdijk. *Sheaves in Geometry and Logic: A First Introduction to Topos Theory.* Springer-Verlag, 1992.

[27] F.W. Lawvere. Qualitative distinctions between some toposes of generalized graphs. *Contemporary Mathematics*, 92:261–299, 1989.

[28] S. Mac Lane. *Categories for the Working Mathematician.* Springer-Verlag, 1971.

[29] E. Moggi. Notions of computation and monads. *Information and Computation*, 93(1):55–92, 1991.

[30] M. Nielsen, G. Plotkin, and G. Winskel. Petri nets, event structures and domains I. *Theoretical Computer Science*, 13(1):85–108, 1981.

[31] G.D. Plotkin. Denotational semantics with partial functions. Lecture at C.S.L.I. Summer School, 1985.

[32] G.D. Plotkin. Algebraic completeness and compactness in an enriched setting. Invited lecture given at the Workshop on Logic, Domains, and Programming Languages. Darmstadt, 1995.

[33] E. Robinson and G. Rosolini. Categories of partial maps. *Information and Computation*, 79:95–130, 1988.

[34] G. Rosolini. *Continuity and Effectiveness in Topoi.* PhD thesis, University of Oxford, 1986.

[35] D.S. Scott. A type-theoretical alternative to CUCH, ISWIM, OWHY. Manuscript (available as [37]), 1969.

[36] D.S. Scott. Continuous lattices. In F.W. Lawvere, editor, *Toposes, Algebraic Geometry and Logic*, volume 274 of *Lecture Notes in Mathematics*, pages 97–136. Springer-Verlag, 1972.

[37] D.S. Scott. A type-theoretical alternative to CUCH, ISWIM, OWHY. *Theoretical Computer Science*, 121:411–440, 1993. (Reprint of a manuscript).

[38] A.K. Simpson. Recursive types in Kleisli categories. Manuscript (available from http://www.dcs.ed.ac.uk/home/als/), 1992.

[39] M.B. Smyth and G.D. Plotkin. The category-theoretic solution of recursive domain equations. *SIAM Journal of Computing*, 11(4):761–783, 1982.

[40] R. Street. The formal theory of monads. *Journal of Pure and Applied Algebra*, 2:149–168, 1972.

[41] R. van Glabbeek. Bisimulations for higher dimensional automata. Communication to the concurrency mailing list (available from http://theory.stanford.edu/people/rvg/hda), 1991.

[42] G. Winskel. Presheaf semantics for value passing. In *CONCUR'96*, volume 1119 of *Lecture Notes in Computer Science*, pages 98–114, 1996.

A Lifting monads

A monad $\mathbb{L} = (L, \eta, \mu)$ on a category with terminal object is said to be a *lifting* monad if

- pullbacks of η_1 along arbitrary maps exist;
- the unit η is cartesian (i.e. the squares required to commute by naturality are pullbacks) and a *partial map classifier* (i.e. in the situation

$$\begin{tikzcd} 1 \arrow[d, hook, "\eta_1"'] & D \arrow[l] \arrow[r] \arrow[d, hook] \arrow[dl, phantom, "\lrcorner"] & B \\ L1 & A \arrow[l] \end{tikzcd}$$

there exists a unique *characteristic map* $A \to LB$ such that the diagram

$$\begin{tikzcd} D \arrow[r] \arrow[d, hook] & B \arrow[d, hook, "\eta_B"] \\ A \arrow[r] & LB \end{tikzcd}$$

is a pullback).

Examples of lifting monads are the identity monad; the $(\_)+1$ monad on **Set**; the traditional lifting monad $(\_)_\perp$ on **Preo**, **Poset**, $\mathbf{Poset}_\wedge$, **Cpo**, and $\mathbf{Cpo}_\wedge$; and the *partial map classifier monad* [22] on a topos.

Recall that lifting monads on cartesian categories are commutative (see e.g. [9]).

B Locally presentable categories

We recall some basic definitions and facts of the theory of *locally presentable categories* used throughout the paper. For a thorough treatment consult [1, 4].

Let λ be a *regular cardinal* (i.e. an infinite cardinal which is not a sum of a smaller number of smaller cardinals). An object C of a category $\mathcal{C}$ is *λ-presentable* when its covariant hom-functor $\mathcal{C}(C, \_)$ preserves λ-directed colimits. A category $\mathcal{C}$ is said to be *locally λ-presentable* if it is cocomplete, and has a small set $\mathsf{P}_\lambda\mathcal{C}$ of λ-presentable objects such that every object in the category is a λ-directed colimit of objects from $\mathsf{P}_\lambda\mathcal{C}$. A functor between locally λ-presentable categories is called *λ-accessible* if it preserves λ-directed colimits.

A category is said to be *locally presentable* if it is locally λ-presentable for some regular cardinal λ. Every locally presentable category is complete.

A monad is called *accessible* if its underlying functor is λ-accessible for some regular cardinal λ. The category of Eilenberg-Moore algebras for an accessible monad on a locally presentable category is locally presentable.