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An Enrichment Theorem for an Axiomatisation of Categories of Domains and Continuous Functions

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Domain-theoretic categories are axiomatised by means of categorical non-order-theoretic requirements on a cartesian closed category equipped with a commutative monad. An *enrichment theorem* showing that every axiomatic domain-theoretic category can be endowed with an *intensional* notion of approximation, the *path relation*, with respect to which the category **Cpo**-enriches is proved.
Our analysis suggests more liberal notions of domains. In particular, a category where the path order is *not* ω-complete, but in which the constructions of domain theory (as, for example, the existence of uniform fixed-point operators and the solution of domain equations) are available is presented.

Introduction

Domain theory (Sco70) is a mathematical theory of computation that provides models for spaces over which to define computable functions. The theory is based on the notion of *approximation* (or information order) between partial objects, which is formalised by means of order-theoretic structures. More precisely, domains are partially ordered sets equipped with a notion of *passage to the limit* (for appropriate subsets of partial objects) given by least upper bounds, and computable functions are taken to be continuous (i.e. to preserve the notions of approximation and passage to the limit). The simplest of these structures is **Cpo**, the category of small cpos (posets, possibly without bottom element, closed under lubs of ω-chains) and continuous functions (monotone functions that preserve lubs of ω-chains).

This paper studies the notions of approximation and passage to the limit in an axiomatic setting. Our axiomatisation is subject to the following criteria: the axioms should be *natural* (so that they are available in as many contexts as possible) and *non-order-theoretic* (so that they explain the order-theoretic structure). Our aim is

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1 to provide a justification of Scott's original consideration of ordered structures, and
2 to deepen our understanding of the notion of passage to the limit.

From a wider perspective, these investigations form part of a programme to formulate an *axiomatic theory of domains* —see (FJM^{+}96, § Axiomatic Domain Theory).

Concerning (1), in accordance with traditional views of domains, we set up a categorical axiomatic framework in which an *intensional* notion of approximation, the so-called *path relation*, can be defined and shown to be $\boldsymbol{\omega}$-complete.

Concerning (2), in order to broaden the concept of domain, we present a category in which the notion of approximation is *not* $\boldsymbol{\omega}$-complete, but that allows the constructions of domain theory.

In our exposition, we have decided to intertwine the introduction of assumptions and axioms with the development of the theory so that the contribution of each axiom is always clear. The *assumptions* are the means for introducing the categorical structure over which the *axioms* impose the extra requirements asked for in the models. Our axiomatisation can be classified as follows:

I. Axioms 1 and 2 introduce the *infinitary structure* allowing the definition of a uniform fixed-point operator for endomorphisms on pointed objects.
II. Axiom 3 forces the notion of *approximation* to be *transitive* and, together with Axioms 1 and 2, permits the definition of a notion of *passage to the limit* for $\boldsymbol{\omega}$-chains of approximations.
III. Axioms 4 and 5 force the passage-to-the-limit operator to be computed by *lubs*.

In Section 1 we introduce our basic assumption; viz. we assume a cartesian category of discourse equipped with a commutative monad $\mathbb{L} = (L, \eta, \mu, \mathrm{t}')$. This data captures the basic structure present in the leading example given by **Cpo** with the *lifting* monad on it. Other examples of categories with a lifting monad are introduced to aid the reader in understanding the axioms. (The status of these examples with respect to our axiomatisation is discussed in Section 9.) In Section 2 we axiomatise lifting. First, as in synthetic domain theory, we consider *dominances*; then, we introduce the notion of *lifting structure*, characterise it in terms of dominances, and show that they induce a commutative monad (henceforth called *lifting*). In Section 3 we consider the category of Eilenberg-Moore algebras for the monad $\mathbb{L}$ as a category of *pointed objects* and *strict maps*. In this view, every pointed object is equipped with a *bottom element* preserved by strict maps. In Section 4 we introduce the first two axioms and exemplify their use by defining a *uniform fixed-point operator*, fix, for endomorphisms on pointed objects. Axiom 1 postulates the existence of a *fixed-point object* consisting of an algebra $\sigma : L\overline{\omega} \to \overline{\omega}$, and Axiom 2 equips $\overline{\omega}$ with a global element ∞ *invariant under iteration*. The essential difference between our fixed-point object and the one previously considered in (CP92) is its *inductive* nature, in that we require it to arise as the colimit for the standard $\boldsymbol{\omega}$-chain generated by iterating L. An intensional notion of approximation, the *path relation* ($\sqsubseteq$), is introduced in Section 5 and characterised for the examples of Section 1. (In the context of synthetic domain theory the path relation has been considered briefly in (Pho90) (where it is called the *link relation*); it is also implicit in (Hyl91) and (Tay91), where it is required to coincide with the intrinsic (or specialisation) order.) The path relation interacts as expected with composition, L, pairing, and currying; but it need *not* be transitive. By introducing Axiom 3, transitivity is achieved and the path relation is shown to provide a **Preo**-enrichment for our category of discourse. Section 6 proposes axioms that provide a *passage-to-the-limit* operator $\bigvee$ for countably infinite sequences of approximations. With this structure we establish the first algebraic properties of the operator $\bigvee$. Surprisingly, we are far from having an $\boldsymbol{\omega}$-complete notion of approximation. Sections 7 and 8 introduce the last two axioms, Axioms 4 and 5, which make Σ ($\stackrel{\text{def}}{=} L1$) behave *externally* as the Sierpinsky space $\mathbf{2}$; as a consequence $\overline{\omega}$ can be regarded externally as the first infinite ordinal $\boldsymbol{\omega}$. We are then in a position to prove the *internal* completeness of the objects in our category of discourse and as a corollary, externalising via the Yoneda embedding, obtain the *enrichment theorem*: i.e. that every *domain-theoretic* category (viz. one that is a model of our axioms) **Cpo**-enriches with respect to the path relation. The leading example of a domain-theoretic category is —naturally— **Cpo** with the lifting monad.

The fact that categories of domains and stable functions fail to satisfy our last axiom suggests more liberal notions of domains. In particular, in Section 10, we present a category which contains the category $\mathbf{Cpo}_\wedge$ of cpos with continuous pullbacks and stable functions as a full subcategory, satisfies all but the last axiom, and does *not* **Cpo**-enrich with respect to the path order. Nevertheless, in this new category the constructions of domain theory (as, for example, the existence of uniform fixed-point operators and the solution of domain equations) are available.

In a companion paper we will exploit our axiomatisation and the enrichment theorem to get a *representation theorem* of domain-theoretic categories equipped with a lifting monad into categories of cpos and continuous functions (in a suitable intuitionistic set theory).

1. Basic assumptions

Our basic data is:

Assumption 1. *We assume a cartesian category of discourse $\mathcal{C}$ and a commutative monad $\mathbb{L} = (L, \eta, \mu, \mathsf{t}')$ on it.* □

To keep the paper reasonably self contained the above structure is defined; for details consult (Koc70; Koc71; Mog91).

Definition 1.1.

1 *Let $\mathcal{C}$ be a category. A* monad *$\mathbb{L} = (L, \eta, \mu)$ on $\mathcal{C}$ consists of a functor $L : \mathcal{C} \to \mathcal{C}$, and natural transformations $\eta_C : C \to LC$ (the* unit*) and $\mu_C : L^2C \to LC$ (the* multiplication*) such that*

— $\mu_C \circ \mu_{LC} = \mu_C \circ L\mu_C : L^3C \to LC$;

— $\mu_C \circ \eta_{LC} = id_{LC}$ *and* $\mu_C \circ L\eta_C = id_{LC}$.

2 *Let $\mathcal{C}$ be a cartesian category. A* commutative monad *$\mathbb{L} = (L, \eta, \mu, \mathsf{t}')$ on $\mathcal{C}$ consists of a monad $\mathbb{L} = (L, \eta, \mu)$ on $\mathcal{C}$ and a natural transformation $\mathsf{t}'_{A,B} : A \times LB \to L(A \times B)$ (the* tensorial strength*) such that*

— *$L(r_C) \circ \mathsf{t}'_{1,C} = r_{LC} : 1 \times LC \to LC$ and $L(a_{A,B,C}) \circ \mathsf{t}'_{A\times B,C} = \mathsf{t}'_{A,B\times C} \circ (id_A \times \mathsf{t}_{B,C}) \circ a_{A,B,LC} : (A \times B) \times LC \to L\big((A \times B) \times C\big)$, where r_C and $a_{A,B,C}$ are respectively the canonical natural isomorphisms $1 \times C \cong C$ and $(A \times B) \times C \cong A \times (B \times C)$;*

— *$\mathsf{t}'_{A,B} \circ (id_A \times \eta_B) = \eta_{A\times B} : A \times B \to L(A \times B)$ and $\mathsf{t}'_{A,B} \circ (id_A \times \mu_B) = \mu_{A\times B} \circ (L\mathsf{t}_{A,B}) \circ \mathsf{t}_{A,LB} : A \times L^2 B \to L(A \times B)$;*

— *$\mu_{A\times B} \circ (L\mathsf{t}''_{A,B}) \circ \mathsf{t}'_{LA,B} = \mu_{A\times B} \circ (L\mathsf{t}'_{A,B}) \circ \mathsf{t}''_{A,LB} : L(A) \times L(B) \to L(A \times B)$, where $\mathsf{t}''_{A,B} \stackrel{\text{def}}{=} L(c_{B,A}) \circ \mathsf{t}'_{B,A} \circ c_{LA,B} : L(A) \times B \to L(A \times B)$ with $c_{A,B}$ the canonical natural isomorphism $A \times B \cong B \times A$.* □

Examples of the categorical structure of Assumption 1 are *lifting* monads (see Example 1.2 and Section 2) and *powerdomain* monads (see (HP79; AJ96)).

Example 1.2. (Lifting monads)

1 *Let* **Set** *be the category of small sets and functions.*
The lifting monad *on* **Set** *can be described as follows: it has underlying functor $(\_)+1$ associating to a set A the set $A + 1 = \{*\} \cup \{[a] \mid a \in A\}$ with unit the inclusion $A \hookrightarrow A + 1 : a \mapsto [a]$, multiplication $(A + 1) + 1 \to A + 1$ sending $* \mapsto *$, $[*] \mapsto *$, $[[a]] \mapsto [a]$, and tensorial strength $A \times (B + 1) \to (A \times B) + 1$ sending $\langle a, * \rangle \mapsto *$ and $\langle a, [b] \rangle \mapsto [\langle a, b \rangle]$.*

2 *Let (***Preo***)* **Poset** *be the category of small (preorders) posets and monotone functions.*
The lifting monad *on (***Preo***)* **Poset** *can be described as follows: it has underlying functor $(\_)_\perp$ associating to a (preorder) poset P the (preorder) poset $P_\perp = \{\perp\} \cup \{[x] \mid x \in P\}$ where $\perp \leq_{P_\perp} [x]$ and $[x] \leq_{P_\perp} [y] \Leftrightarrow x \leq_P y$ with unit the inclusion $P \hookrightarrow P_\perp : x \mapsto [x]$, multiplication $(P_\perp)_\perp \to P_\perp$ sending $\perp \mapsto \perp$, $[\perp] \mapsto \perp$, $[[x]] \mapsto [x]$, and tensorial strength $P \times Q_\perp \to (P \times Q)_\perp$ sending $\langle x, \perp \rangle \mapsto \perp$ and $\langle x, [y] \rangle \mapsto [\langle x, y \rangle]$.*

3 *Let* **Cpo** *be the category of small cpos and continuous functions.*
The lifting monad on **Poset** *restricts to a* lifting monad *on* **Cpo***.*

4 Convention. *To emphasise the distinction between* ordinary *and* order-enriched *categories the latter are subscripted with the order providing the enrichment. (Thus, for example, for a category $\mathcal{K}$, $\mathcal{K}_=$ denotes the order-enriched category induced by $\mathcal{K}$ by discretely ordering each hom-set.) The ordinary* underlying *category of an order-enriched category $\mathcal{W}_\leq$ is denoted $\mathcal{W}$.*

Let $\mathbf{Cpo}_\sqsubseteq$ be the **Cpo***-category of small cpos and continuous functions ordered pointwise (i.e. $f \sqsubseteq g : P \to Q \Leftrightarrow \forall\, x \in P.\, fx \leq_Q gx$).*
For $\mathcal{W}_\leq$ a small **Cpo***-category, let $[\mathcal{W}_\leq{}^{\mathrm{op}}, \mathbf{Cpo}_\sqsubseteq]$ be the functor category of* **Cpo***-functors and natural transformations.*
The pointwise lifting monad *on $[\mathcal{W}_\leq{}^{\mathrm{op}}, \mathbf{Cpo}_\sqsubseteq]$ has underlying functor $[\mathcal{W}_\leq{}^{\mathrm{op}}, (\_)_\perp] : F \mapsto (\_)_\perp \circ F$, with unit, multiplication, and tensorial strengths given pointwise.*

5 *Let (***Poset***$\wedge$) $\mathbf{Cpo}_\wedge$ be the category of small (posets) cpos with (monotone) continuous pullbacks and stable (i.e. (monotone) continuous pullback preserving) functions.*
The lifting monad on **Poset** *restricts to a* lifting monad *on ($\mathbf{Poset}_\wedge$) $\mathbf{Cpo}_\wedge$.* □

The reader should keep in mind that **Cpo** equipped with the lifting monad is considered the leading example of a domain-theoretic category; and hence it will guide our axiomatisation.

2. Lifting

In this section we review a, by now standard, axiomatisation of *lifting* monads in terms of *dominances*. (For the connection between this axiomatisation and *partiality* consult (Fio96).) Lifting monads are a fundamental example of Assumption 1, and hence important for understanding it. However, since the constructions of this section are only needed for treating Examples 5.2 (5), 5.8, and 5.9, on a first reading, it may be skipped.

Convention. Subobjects obtained as pullbacks of a monomorphism m are called *m-subobjects*.

In the vein of synthetic domain theory (Ros86; Hyl91; Tay91), we consider a *generic observable property*, $\top : 1 \hookrightarrow \Sigma$, whose role is to internalise the notion of *termination* and hence determine the *observable properties* ($\top$*-subobjects*) of all objects.

Definition 2.1. (Dominance (Ros86)**)** *Let $\mathcal{K}$ be a category with a terminal object 1. An object Σ equipped with a subobject $\top : 1 \hookrightarrow \Sigma$ such that pullbacks of $\top$ always exist is called a* $\top$-suboject classifier *if every $\top$-subobject $D \hookrightarrow A$ appears in a pullback*

$$\begin{tikzcd} D \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner", very near start] & 1 \arrow[d, hook, "\top"] \\ A \arrow[r, dotted] & \Sigma \end{tikzcd}$$

for a unique characteristic map $A \to \Sigma$.
A $\top$-subobject classifier is called a dominance *if*

— *$\top^* : \mathcal{K}/\Sigma \to \mathcal{K}$ has right adjoint, and*
— *$\top$-subobjects are closed under composition.*

Remark. *In the context of SDT, dominances are only defined by the second requirement, as the first requirement generally holds in the universes of sets considered (e.g. locally cartesian closed categories with finite limits, as toposes and quasitoposes).* □

Remark.

1 Pullbacks of $\top$-subobjects always exist.
2 In a cartesian category, if $m : D \hookrightarrow A$ is a $\top$-subobject then so is $m \times B : D \times B \rightarrowtail A \times B$ because if m has characteristic map $f : A \to \Sigma$ then $m \times B$ has characteristic map $f \circ \pi_1 : A \times B \to \Sigma$. Hence, if m and n are $\top$-subobjects then so is $m \times n$.

Examples of dominances:

Example 2.2. (Dominances)

1 *In* **Set**, *let* $1 = \{0\}$ *and* $2 = \{0, 1\}$. *Then,* $1 \hookrightarrow 2 : 0 \mapsto 1$ *classifies all subsets.*
2 *In (***Preo***)* **Poset**, *let* $\mathbf{1} = \{0\}$ *and* $\mathbf{2} = \{0 \leq 1\}$. *Then,* $\mathbf{1} \hookrightarrow \mathbf{2} : 0 \mapsto 1$ *classifies upper-closed subsets.*

3 In $\mathbf{Cpo}$, $\mathbf{1} \hookrightarrow \mathbf{2} : 0 \mapsto 1$ *classifies Scott-open subsets.*

4 For $\mathcal{W}_{\leq}$ *a small* $\mathbf{Cpo}$*-category, let* $\mathbf{1} : \mathcal{W}_{\leq}{}^{\mathrm{op}} \to \mathbf{Cpo}_{\sqsubseteq} : w \mapsto \mathbf{1}$.

(a) Let $\mathbf{2} : \mathcal{W}_{\leq}{}^{\mathrm{op}} \to \mathbf{Cpo}_{\sqsubseteq} : w \mapsto \mathbf{2}$.
Then, $\top : \mathbf{1} \hookrightarrow \mathbf{2}$ *defined by* $\top_W(0) = 1$ *classifies the pointwise-Scott-open cartesian natural (i.e. natural transformations in which the squares required to commute by naturality are pullbacks) monos.*

(b) Let $\mathfrak{S} : \mathcal{W}_{\leq}{}^{\mathrm{op}} \to \mathbf{Cpo}_{\sqsubseteq}$ *be the* $\mathbf{Cpo}$*-functor defined*

— *on objects: as* $\mathfrak{S}(W) = (\textit{Scott-sieves}(W), \subseteq)$ *where a sieve* R *on* W *(i.e. a subset of* $\coprod_{V \in |\mathcal{W}|} \mathcal{W}(V, W)$ *satisfying the closure property: for every* $f : V \to W$ *in* R *and* $g : V' \to V$ *in* $\mathcal{W}$, $f \circ g$ *is in* R*) is said to be* Scott *if for every* $V \in |\mathcal{W}|$, $R \cap \mathcal{W}(V, W)$ *is Scott-open in* $\mathcal{W}_{\leq}(V, W)$*; and*

— *on morphisms: as* $\mathfrak{S}(f : W' \to W)(R) = \{h : V \to W' \mid f \circ h \in R\}$.

Then, $\top : \mathbf{1} \hookrightarrow \mathfrak{S}$ *defined by* $\top_W(0) = \coprod_{V \in |\mathcal{W}|} \mathcal{W}_{\leq}(V, W)$ *classifies the pointwise-Scott-open natural monos.*

5 In ($\mathbf{Poset}_{\wedge}$*)* $\mathbf{Cpo}_{\wedge}$, $\mathbf{1} \hookrightarrow \mathbf{2} : 0 \mapsto 1$ *classifies (upper-closed) Scott-open subsets closed under pullbacks.*

6 In a topos, $\mathsf{true} : 1 \to \Omega$ *classifies subobjects.* □

The above dominances satisfy the extra requirement of classifying the unique map $0 \to 1$; thus allowing the notion of *non-termination* to be internalised as a map $1 \to \Sigma$. Now we introduce an alternative description of these dominances in terms of so-called *lifting structures* (see Theorem 2.4).

Definition 2.3. (Lifting structure) *Let* $\mathcal{K}$ *be a category with an initial object 0 and a terminal object 1.*
A lifting structure $((\_)_\perp, [\_])$ *on* $\mathcal{K}$ *consists of a pointed endofunctor* $[\_] : Id \stackrel{\cdot}{\to} (\_)_\perp : \mathcal{K} \to \mathcal{K}$ *such that*

— *pullbacks of* $[\_]_1$ *always exist,*

— $[\_]$ *is cartesian (that is, for every* $f : A \to B$,

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle [\_]_A}\downarrow & \lrcorner & \downarrow{\scriptstyle [\_]_B} \\ A_\perp & \xrightarrow[f_\perp]{} & B_\perp \end{array}$$

is a pullback square),

— *for every* $B \in |\mathcal{K}|$, $[\_]_B$ *classifies* $[\_]_1$*-partial maps with target* B *(that is, every span* $A \hookleftarrow D \to B$ *with* $D \hookrightarrow A$ *an* $[\_]_1$*-subobject appears in a pullback*

$$\begin{array}{ccc} D & \longrightarrow & B \\ \downarrow & \lrcorner & \downarrow{\scriptstyle [\_]_B} \\ A & \longrightarrow & B_\perp \end{array}$$

for a unique characteristic map $A \to B_\perp$*), and*

— *the composite* $1 \xhookrightarrow{[\_]_1} 1_\perp \xhookrightarrow{[\_]_{1_\perp}} (1_\perp)_\perp$ *and the unique map* $0 \to 1$ *are* $[\_]_1$*-subobjects.* □

Theorem 2.4. *Lifting structures and dominances that classify* $0 \to 1$ *are in bijective correspondence.*

Proof. See (Fio96, Chapter 3). □

Lifting structures are easily seen to provide the structure of a commutative monad:

Definition 2.5. (Lifting) *Every lifting structure* $((\_)_\perp, [\_])$ *on* $\mathcal{K}$ *induces a* lifting monad $((\_)_\perp, [\_], \mu)$ *on* $\mathcal{K}$ *with* multiplication *defined as:*

$$
\begin{tikzcd}
A \arrow[r, "A"] \arrow[d, hook, "{[\_]_A}"'] \arrow[dr, phantom, "\lrcorner"] & A \arrow[dd, hook, "{[\_]_A}"] \\
A_\perp \arrow[d, hook, "{[\_]_{A_\perp}}"'] & \\
(A_\perp)_\perp \arrow[r, "\mu_A"'] & A_\perp
\end{tikzcd}
\quad .
$$

Moreover, for cartesian $\mathcal{K}$*, the lifting monad is commutative with* tensorial strength *defined as:*

$$
\begin{tikzcd}
A \times B \arrow[r, "A \times B"] \arrow[d, hook, "{A \times [\_]_B}"'] \arrow[dr, phantom, "\lrcorner"] & A \times B \arrow[d, hook, "{[\_]_{A \times B}}"] \\
A \times B_\perp \arrow[r, "\mathsf{t}'_{A,B}"'] & (A \times B)_\perp
\end{tikzcd}
\quad .
$$

□

Example 2.6. *The lifting monads induced by the dominances of Example 2.2 are:*

— *for Example 2.2 (1), that of Example 1.2 (1);*
— *for Example 2.2 (2), that of Example 1.2 (2);*
— *for Example 2.2 (3), that of Example 1.2 (3);*
— *for Example 2.2 (4a), that of Example 1.2 (4);*
— *for Example 2.2 (5), that of Example 1.2 (5);*
— *for Example 2.2 (6), the* partial map classifier monad *(see (Joh77)).* □

Proposition 2.7. *The category of Eilenberg-Moore algebras for a lifting monad on a category with a* strict *initial object has a zero object.* □

3. Pointed objects and strict maps

We write $\mathcal{C}^{\mathbb{L}}$ for the category of Eilenberg-Moore algebras for the monad $\mathbb{L}$. In the leading example, **Cpo** with the lifting monad, we have that $\mathbf{Cpo}^{\mathbb{L}}$ is the category of small cpos with a least element and strict continuous functions. Thus the following:

Convention. An algebra for the monad $\mathbb{L}$ is called a *pointed* object. The algebra structure on a pointed object X is denoted by ρ_X but generally left implicit; moreover ρ_{LA} is taken to be μ_A unless specified otherwise. Algebra homomorphisms are called *strict* and are denoted with $\circ\!\!\rightarrow$. When we write $X \circ\!\!\rightarrow Y$, X and Y are assumed to be pointed.

Remark. If $\mathcal{C}^{\mathbb{L}}$ has a zero object 1 then $\mathcal{C}$ has a terminal object 1 (as the forgetful functor $\mathcal{C}^{\mathbb{L}} \to \mathcal{C}$ has a left adjoint and therefore preserves limits); moreover if $\mathcal{C}$ has an initial object 0 then $L0 \cong 1$, and η_0, μ_0, $\mathsf{t}'_{A,0}$, $\mathsf{t}'_{0,B}$ have target 1 and hence are uniquely determined by their source.

Assumption 2. *We assume $\mathcal{C}^{\mathbb{L}}$ has a zero object 1.* □

Definition 3.1. (Bottom) *For a pointed object X we define $\perp_X$ to be the unique strict map $1 \circ\!\!\rightarrow X$.* □

Remark. For every $f : X \circ\!\!\rightarrow Y$, $f \circ \perp_X = \perp_Y$ and hence $\perp_X = \rho_X \circ \perp_{LX}$.

4. Inductive fixed-point object

The axioms of this section introduce the *infinitary structure* allowing the definition of a uniform fixed-point operator, fix, for endomorphisms on pointed objects.

Convention. We set: $\Sigma = L1$, $\Sigma_n = L^n 1$, $\perp = (1 \circ\!\!\rightarrow \Sigma)$, $\top = \eta_1 : 1 \to \Sigma$.

To motivate the first axiom it is convenient to regard Σ_n as a *canonical chain* of $n+1$ elements. In fact, as we show below, for $n \geq 0$, every Σ_n comes equipped with global elements $\underline{k}_{\Sigma_n} : 1 \to \Sigma_n$ $(0 \leq k \leq n)$ such that, for $0 \leq i \leq j \leq n$, we have $\underline{i}_{\Sigma_n} \sqsubseteq \underline{j}_{\Sigma_n}$ for a suitable notion of approximation $\sqsubseteq$ (defined in Section 5).

Definition 4.1. ($\underline{i}_{\Sigma_n}$) *For $0 \leq i \leq n$, we inductively define $\underline{i}_{\Sigma_n} : 1 \to \Sigma_n$ as follows:*
$\underline{0}_{\Sigma_n} \stackrel{\text{def}}{=} (1 \circ\!\!\rightarrow \Sigma_n)$ *and* $\underline{i+1}_{\Sigma_{n+1}} \stackrel{\text{def}}{=} (1 \xrightarrow{\top} \Sigma \xrightarrow{L(\underline{i}_{\Sigma_n})} \Sigma_{n+1}) = (1 \xrightarrow{\underline{i}_{\Sigma_n}} \Sigma_n \xrightarrow{\eta_{\Sigma_n}} \Sigma_{n+1})$. □

In view of the above intuition, Axiom 1 postulates the existence of a *countably infinite chain*, $\overline{\omega}$, obtained by *pasting* the Σ_n's so that they become the finite prefixes of $\overline{\omega}$. Formally:

Axiom 1. ($\overline{\omega}$) *The functor $L : \mathcal{C} \to \mathcal{C}$ has an algebra $\sigma : L\overline{\omega} \to \overline{\omega}$ such that the cone $\langle \mathsf{s}_n \rangle : \langle L^n \perp : \Sigma_n \to \Sigma_{n+1} \rangle \dot{\to} \overline{\omega}$, inductively defined as: $\mathsf{s}_0 = \sigma \circ \perp_{L\overline{\omega}}$ and $\mathsf{s}_{n+1} = \sigma \circ L(\mathsf{s}_n)$, is colimiting. (That is, the following*

$$
\begin{tikzcd}
1 \arrow[r, "\perp"] \arrow[d, "\sigma \circ \perp_{L\overline{\omega}}"'] \arrow[dr, "\perp_{L\overline{\omega}}"] & \Sigma \arrow[r, "L\perp"] \arrow[d, "L(\sigma \circ \perp_{L\overline{\omega}})"] & \Sigma_2 \arrow[r, "L^2\perp"] \arrow[d, "L^2(\sigma \circ \perp_{L\overline{\omega}})"] & \cdots \\
\overline{\omega} & L\overline{\omega} \arrow[l, "\sigma"] & L^2\overline{\omega} \arrow[l, "L\sigma"] & \cdots \arrow[l, "L^2\sigma"]
\end{tikzcd}
$$

is a colimit diagram.) □

In the leading example, **Cpo** with the lifting monad, we take $\overline{\omega}$ to be $\boldsymbol{\omega}+1$ (the successor ordinal of $\boldsymbol{\omega}$) and σ to be the successor map sending $\perp \mapsto 0$, $n \mapsto n+1$ $(n \geq 0)$, and $\omega \mapsto \omega$.

Also in the axiomatic framework the *successor* operation can be defined:

Definition 4.2.

1 $(\underline{n}_{\overline{\omega}})$ *For every* $n \geq 0$, *we define* $\underline{n}_{\overline{\omega}} \stackrel{\text{def}}{=} \mathsf{s}_n \circ \underline{n}_{\Sigma_n} : 1 \to \overline{\omega}$.

2 (succ) *We define* $\mathsf{succ} : \overline{\omega} \to \overline{\omega}$ *as the composite* $\sigma \circ \eta_{\overline{\omega}}$. □

Proposition 4.3. *For* $n \geq 0$, $\mathsf{succ} \circ \underline{n}_{\overline{\omega}} = \underline{n+1}_{\overline{\omega}}$. □

Axiom 2 equips $\overline{\omega}$ with a *point at infinity*, ∞, characterised by its *invariance under iteration*:

Axiom 2. (∞) *$\overline{\omega}$ has a global element ∞ which is a fixed-point of* succ. □

Convention. For $n \geq 0$, we inductively define $\eta_A^{(n)} : A \to L^n A$ as $\eta_A^{(0)} = A$ and $\eta_A^{(n+1)} = \eta_{L^n A} \circ \eta_A^{(n)} = \eta_{LA}^{(n)} \circ \eta_A$.

Proposition 4.4. (The invariance of ∞) *For* $n \geq 0$, $\sigma \circ \ldots \circ (L^n \sigma) \circ \eta_{\overline{\omega}}^{(n+1)} \circ \infty = \infty$. □

The rest of the section is devoted to exemplifying the use of the axioms by showing how one can define a uniform fixed-point operator fix. To this purpose, given an endomorphism f on a pointed object X we first *internalise* the sequence $\langle \perp_X, f \circ \perp_X, \ldots, f^n \circ \perp_X, \ldots \rangle$ of iterations of f at $\perp_X$ and then *extract* its value at infinity. Precisely:

Definition 4.5.

1 (it) *For every endomorphism f on a pointed object X, we define* $\mathsf{it}(f) : \overline{\omega} \to X$ *as the unique mediating morphism from the colimiting cone* $\langle \mathsf{s}_n \rangle$ *to the cone* $\langle \mathsf{it}_n(f) \rangle$ *inductively defined as:* $\mathsf{it}_0(f) = \perp_X$ *and* $\mathsf{it}_{n+1}(f) = \rho_X \circ L(f \circ \mathsf{it}_n(f))$.

2 (fix) *For every endomorphism f on a pointed object X, we define* $\mathsf{fix}(f) : 1 \to X$ *as the composite* $\mathsf{it}(f) \circ \infty$. □

Proposition 4.6. *For every endomorphism f on a pointed object X,*

1 $f \circ \mathsf{it}(f) = \mathsf{it}(f) \circ \mathsf{succ}$, *and*

2 **(fix is a fixed-point operator)** $f \circ \mathsf{fix}(f) = \mathsf{fix}(f)$.

Proof.

(1) Because for $n \geq 0$,

$$\begin{array}{rclcl} \mathsf{it}(f) \circ \sigma \circ \eta_{\overline{\omega}} \circ \mathsf{s}_n & = & \mathsf{it}(f) \circ \sigma \circ L(\mathsf{s}_n) \circ \eta_{\Sigma_n} & = & \mathsf{it}(f) \circ \mathsf{s}_{n+1} \circ \eta_{\Sigma_n} \\ & = & \mathsf{it}_{n+1}(f) \circ \eta_{\Sigma_n} & = & \rho_X \circ L(f \circ \mathsf{it}_n(f)) \circ \eta_{\Sigma_n} \\ & = & \rho_X \circ \eta_X \circ f \circ \mathsf{it}_n(f) & = & f \circ \mathsf{it}(f) \circ \mathsf{s}_n. \end{array}$$

(2) We calculate: $f \circ \mathsf{fix}(f) = f \circ \mathsf{it}(f) \circ \infty = \mathsf{it}(f) \circ \mathsf{succ} \circ \infty = \mathsf{it}(f) \circ \infty = \mathsf{fix}(f)$. □

Proposition 4.7. (Uniformity) *The fixed-point operator* fix *is uniform. That is, if* $(X \xrightarrow{f} X \circ\!\!\xrightarrow{h} Y) = (X \circ\!\!\xrightarrow{h} Y \xrightarrow{g} Y)$ *then* $h \circ \mathsf{fix}(f) = \mathsf{fix}(g)$.

Proof. Because $h \circ \mathsf{it}(f) = \mathsf{it}(g)$. □

For our subsequent development σ need not be an isomorphism; when it is (as happens in the leading example and in *domain-theoretic* categories —see Section 9) we have:

Proposition 4.8. *Assume σ is an isomorphism. Then,*

1 $(\overline{\omega}, \sigma \circ \mu_{\overline{\omega}} \circ L\sigma^{-1})$ *is a pointed object and σ is strict;*
2 *if ∞ is the unique fixed-point of* succ *then the uniformity property characterises* fix*;*
3 $\sigma : L\overline{\omega} \to \overline{\omega}$ *is an initial L-algebra and, for every endomorphism f on a pointed object X,* $\mathsf{it}(f)$ *is characterised as the unique L-algebra homomorphism $(\overline{\omega}, \sigma) \to (X, \rho_X \circ Lf)$.*

Proof.
(2) Let Y be a uniform fixed-point operator.
Since $\mathsf{it}(f) \circ \sigma = \rho_X \circ L(f \circ \mathsf{it}(f))$, it follows that $\mathsf{it}(f)$ is a strict morphism $\overline{\omega} \circ\!\!\rightarrow X$ because

$$\begin{tikzcd}
L\overline{\omega} \arrow[rr, "L(\mathsf{it}(f))"] \arrow[d, "L\sigma^{-1}"'] & & LX \arrow[ddd, "\rho_X"] \\
L^2\overline{\omega} \arrow[r, "L^2(f \circ \mathsf{it}(f))"] \arrow[d, "\mu_{\overline{\omega}}"'] & L^2X \arrow[ur, "L\rho_X"] \arrow[d, "\mu_X"'] & \\
L\overline{\omega} \arrow[r, "L(f \circ \mathsf{it}(f))"'] \arrow[d, "\sigma"'] & LX \arrow[dr, "\rho_X"] & \\
\overline{\omega} \arrow[rr, "\mathsf{it}(f)"'] & & X
\end{tikzcd} .$$

Then, as $\mathsf{it}(f) \circ \mathsf{succ} = f \circ \mathsf{it}(f)$ we have that $\mathsf{Y}(f) = \mathsf{it}(f) \circ \mathsf{Y}(\mathsf{succ}) = \mathsf{it}(f) \circ \infty = \mathsf{fix}(f)$.
(3) Given $\alpha : LA \to A$ define the cone $\langle \mathsf{a}_n \rangle : \langle \Sigma_n \rangle \dot{\to} A$ inductively as follows: $\mathsf{a}_0 = \alpha \circ \perp_{LA}$ and $\mathsf{a}_{n+1} = \alpha \circ L(\mathsf{a}_n)$. Then, for $h : \overline{\omega} \to A$ it follows by induction that, for $n \geq 0$, $\alpha \circ (Lh) \circ \sigma^{-1} \circ \mathsf{s}_n = \mathsf{a}_n$ and hence $h : (\overline{\omega}, \sigma) \to (A, \alpha)$ if and only if $h : \langle \mathsf{s}_n \rangle \to \langle \mathsf{a}_n \rangle$.
The second part follows because $\mathsf{it}(f) = \rho_X \circ L(f \circ \mathsf{it}(f)) \circ \sigma^{-1}$. □

5. Path order

The notion of approximation, viz. the *path relation*, is introduced; for further analysis consult (Fio95). After studying examples (both for concrete models and in the axiomatic setting) the properties of the path relation and its interaction with relevant categorical structure are considered. In particular, we show that the path relation need *not* be transitive. By introducing Axiom 3, transitivity is achieved and the path relation is shown to provide a **Preo**-enrichment for our category of discourse.

We have equipped Σ with two global elements $\perp$ and $\top$ intuitively corresponding to non-termination and termination. Thus, $\perp$ approximates $\top$. Informally, let us visualise this fact as the existence of a *directed path* $\perp \rightsquigarrow \top$ from $\perp$ to $\top$ in Σ, and say that such a path is a *proof* that $\perp$ approximates $\top$. With this viewpoint, it is natural to say, for two global elements x and y of an object C, that x approximates y if the *canonical* path $\perp \rightsquigarrow \top$ can be transformed, via a map $\Sigma \to C$, into a path $x \rightsquigarrow y$ from x to y in C. In symbols, $x \sqsubseteq y : 1 \to C$ if there exists (a *proof*) $\alpha : \Sigma \to C$ such that $\alpha \circ \bot = x$ and $\alpha \circ \top = y$. A formalisation of these intuitions leads to the following:

Definition 5.1. *For $f, g : A \to B$ and $\alpha : \Sigma \times A \to B$, we set $\alpha : f \rightsquigarrow g : A \to B$ if and only if*

$$
\begin{tikzcd}
1 \times A \cong A \arrow[d, "\top \times A"'] \arrow[dr, "g"] & \\
\Sigma \times A \arrow[r, "\alpha"] & B \\
1 \times A \cong A \arrow[u, "\bot \times A"] \arrow[ur, "f"'] &
\end{tikzcd}
.
$$

Moreover, we define the path relation $f \sqsubseteq g : A \to B$ *if and only if there exists* $\alpha : \Sigma \times A \to B$ *such that* $\alpha : f \rightsquigarrow g$. □

We characterise the path relation for some of our examples:

Example 5.2.

1 *For the lifting monad on* **Set** *(see Example 1.2 (1)), the path relation is* indiscrete *(i.e. $f \sqsubseteq g$ for all f and g).*
2 *For the lifting monad on (***Preo***)* **Poset** *and* **Cpo** *(see Examples 1.2 (2)–(3)), the path relation is the* pointwise (pre)order *(i.e. $f \sqsubseteq g : P \to Q \Leftrightarrow \forall x \in P.\, fx \leq_Q gx$).*
3 *Let $\mathcal{W}_\leq$ be a small* **Cpo***-category. For the pointwise lifting monad on $[\mathcal{W}_\leq{}^{\mathrm{op}}, \mathbf{Cpo}_\sqsubseteq]$ (see Example 1.2 (4)), the path relation is the* pointwise pointwise order*; that is,*

$$f \sqsubseteq g : P \to Q \Leftrightarrow \forall W \in |\,\mathcal{W}_\leq\,|.\, \forall x \in PW.\, f_W x \leq_{QW} g_W x.$$

4 *For the lifting monad on* $\mathbf{Poset}_\wedge$ *and* $\mathbf{Cpo}_\wedge$ *(see Example 1.2 (5)), the path relation is the* stable order*; that is, $f \sqsubseteq g : P \to Q \Leftrightarrow \forall x, y \in P.\, x \leq_P y \Rightarrow fx = gx \sqcap_Q fy$.*
5 *For the monad induced by the dominance $1 \hookrightarrow \mathfrak{S}$ on $[\mathbf{2}_=, \mathbf{Cpo}_\sqsubseteq]$ (see Example 2.2 (4b)), the path relation is the pointwise pointwise order.* □

Example 5.3.

1 *For $0 \leq i \leq j \leq n$, $\underline{i}_{\Sigma_n} \sqsubseteq \underline{j}_{\Sigma_n} : 1 \to \Sigma_n$.*
2 *For $0 \leq i \leq j$, $\underline{i}_{\overline{\omega}} \sqsubseteq \underline{j}_{\overline{\omega}} : 1 \to \overline{\omega}$.*
3 **Definition 5.4. ($\boldsymbol{\infty_n}$)** *For $n \geq 0$, we define $\infty_n : \Sigma \to \overline{\omega}$ as the composite*

$$\sigma \circ \ldots \circ (L^n \sigma) \circ (L^{n+1} \infty) \circ \eta_\Sigma^{(n)}.$$ □

For $n \geq 0$, $\infty_n : \underline{n} \rightsquigarrow \infty : 1 \to \overline{\omega}$. □

Example 5.5. *Given an endomorphism f on a pointed object X, the sequence $\langle \bot_X, f \circ \bot_X, \ldots, f^n \circ \bot_X, \ldots \rangle$ of iterations of f at $\bot_X$ becomes a* chain *with respect to the path relation as $f^n \circ \rho_X \circ L(f^i \circ \bot_X) : f^n \circ \bot_X \rightsquigarrow f^{n+i} \circ \bot_X : 1 \to X$.* □

We now incorporate exponentials into our discussion:

Assumption 3. *We assume $\mathcal{C}$ cartesian closed.* □

Convention. In a cartesian closed category $\mathcal{K}$ we write $\mathsf{e} : A^B \times B \to A$ for the *evaluation* map and $\lambda : \mathcal{K}(C \times A, B) \cong \mathcal{K}(C, B^A)$ for the *currying* operator.

Proposition 5.6. (The calculus of proofs (Fio95)**)**

1 $id_f : f \rightsquigarrow f : A \to B$ *where* $id_f = f \circ \pi_2$.
2 *If* $\alpha : f \rightsquigarrow f' : A \to B$ *and* $\beta : g \rightsquigarrow g' : B \to C$ *then* $\beta \star \alpha : g \circ f \rightsquigarrow g' \circ f' : A \to C$ *where* $\beta \star \alpha = \beta \circ \langle \pi_1, \alpha \rangle$.
3 *For every pointed object* X *and every* $x : 1 \to X$, $\rho_X \circ (Lx) : \bot_X \rightsquigarrow x$.
4 *If* $\alpha : f \rightsquigarrow g : A \to B$ *then* $(L\alpha) \circ \mathsf{t}' : Lf \rightsquigarrow Lg : LA \to LB$.
5 $\alpha : f \rightsquigarrow f' : C \to A$ *and* $\beta : g \rightsquigarrow g' : C \to B$ *if and only if* $\langle \alpha, \beta \rangle : \langle f, g \rangle \rightsquigarrow \langle f', g' \rangle : C \to A \times B$.
6 $\alpha : f \rightsquigarrow g : A \times B \to C$ *if and only if* $\lambda(\alpha \circ a) : \lambda(f) \rightsquigarrow \lambda(g) : A \to C^B$ *where* a *is the canonical isomorphism* $(\Sigma \times A) \times B \cong \Sigma \times (A \times B)$. □

Corollary 5.7.

1 *For every* $f : A \to X$ *with* X *pointed,* $(A \to 1 \xrightarrow{\bot_X} X) \rightsquigarrow f : A \to X$.
2 $\alpha : f \rightsquigarrow g : A \to B$ *if and only if* $\lambda(\alpha) : \ulcorner f \urcorner \rightsquigarrow \ulcorner g \urcorner : 1 \to B^A$.

Proof. (1) By Proposition 5.6 (3), because X^A is pointed (with $\rho_{X^A} = \lambda(\rho_X \circ (L\mathsf{e}) \circ \mathsf{t}''))$ and $\bot_{X^A} = \ulcorner A \to 1 \xrightarrow{\bot_X} X \urcorner$ (using Definition 3.1). □

Contrary to what Example 5.2 might suggest, the path relation need *not* be transitive:

Example 5.8. *In* $[\swarrow\searrow, \mathbf{Cpo}]$, *the category of spans over* **Cpo**, *with the dominance* $1 \hookrightarrow \mathfrak{S}$ *(see Section 2, in particular Example 2.2 (4b)) the path relation is* not *transitive. To see this observe that*

$$1 \quad = \quad \begin{tikzcd}[column sep=small] & (*,*) \arrow[dl, dashed] \arrow[dr, dashed] & \\ * & & * \end{tikzcd}$$

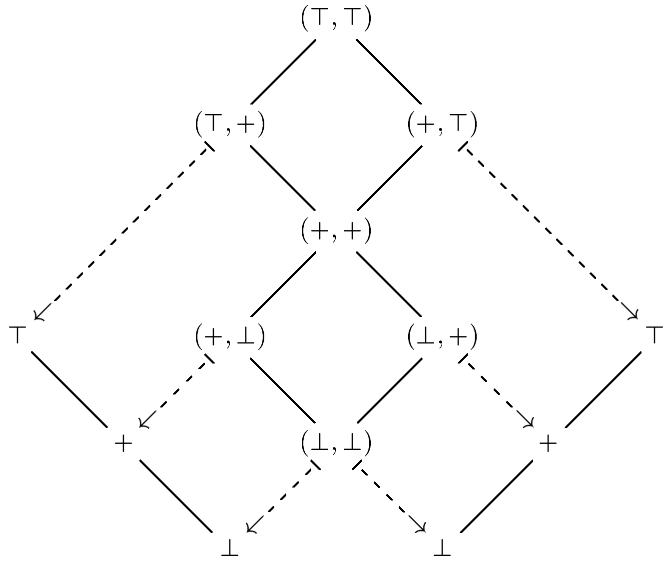
and

$$\mathfrak{S} \quad = \quad \begin{tikzcd}[column sep=small] & & & (\top,\top) \arrow[dl, no head] \arrow[dr, no head] & & & \\ & & (\top,\bot) \arrow[dl, dashed] \arrow[dr, no head] & & (\bot,\top) \arrow[dl, no head] \arrow[dr, dashed] & & \\ & \top \arrow[dr, no head] & & (\bot,\bot) \arrow[dl, dashed] \arrow[dr, dashed] & & \top \arrow[dl, no head] & \\ & & \bot & & \bot & & \end{tikzcd}$$

with the images of $\bot : 1 \to \mathfrak{S}$ *and* $\top : 1 \hookrightarrow \mathfrak{S}$ *respectively corresponding to*

$$\begin{tikzcd}[column sep=small] & (\bot,\bot) \arrow[dl, dashed] \arrow[dr, dashed] & \\ \bot & & \bot \end{tikzcd} \quad \text{and} \quad \begin{tikzcd}[column sep=small] & (\top,\top) \arrow[dl, dashed] \arrow[dr, dashed] & \\ \top & & \top \end{tikzcd} .$$

Thus, the pushout of $\mathfrak{S} \xleftarrow{\top} 1 \xrightarrow{\perp} \mathfrak{S}$ *can be described as*



and in its hom of global elements we have that

$$\begin{tikzcd}[column sep=small] & (\perp,\perp) \arrow[dl, dashed] \arrow[dr, dashed] & & & & (+,+) \arrow[dl, dashed] \arrow[dr, dashed] & & & & (\top,\top) \arrow[dl, dashed] \arrow[dr, dashed] & \\ \perp & & \perp & \rightsquigarrow & + & & + & \rightsquigarrow & \top & & \top \end{tikzcd}$$

but

$$\begin{tikzcd}[column sep=small] & (\perp,\perp) \arrow[dl, dashed] \arrow[dr, dashed] & & & & (\top,\top) \arrow[dl, dashed] \arrow[dr, dashed] & \\ \perp & & \perp & \not\rightsquigarrow & \top & & \top \end{tikzcd} \ .$$

□

The following axiom forces the *transitivity* of the path relation. Intuitively, thinking of Σ_n ($n \geq 0$) as a *chain* of $n+1$ elements, the axiom and the fact that pushouts compose imply that Σ_{i+k+j} ($i, j, k \geq 0$) results from gluing the upper $k+1$ elements of Σ_{i+k} with the lower $k+1$ elements of Σ_{k+j}.

Axiom 3. *For* $n \geq 0$, *the commuting square*

$$\begin{tikzcd} \Sigma_n \arrow[r, "L^n\perp", tail] \arrow[d, hook, "\eta_{\Sigma_n}"'] & \Sigma_{n+1} \arrow[d, hook, "\eta_{\Sigma_{n+1}}"] \\ \Sigma_{n+1} \arrow[r, "L^{n+1}\perp"', tail] & \Sigma_{n+2} \end{tikzcd}$$

is a pushout. □

Remark. Since pushouts compose, for $i, j, k \geq 0$, the commuting square

$$\begin{tikzcd}
\Sigma_k \arrow[r, "L^k\perp"] \arrow[d, hook, "\eta^{(i)}_{\Sigma_k}"'] & \Sigma_{k+j} \arrow[d, hook, "\eta^{(i)}_{\Sigma_{k+j}}"] \\
\Sigma_{k+i} \arrow[r, "L^{k+i}\perp"'] & \Sigma_{k+i+j}
\end{tikzcd}$$

is a pushout.

Example 5.9. *As the path relation in* $[\swarrow\searrow, \mathbf{Cpo}]$ *with the dominance* $1 \hookrightarrow \mathfrak{S}$ *is not transitive, Axiom 3 cannot hold there. Indeed, we have*

$$\mathfrak{S}_2 \quad = \quad \begin{tikzcd}[column sep=small]
 & & & (\top,\top) \arrow[dl, dash] \arrow[dr, dash] & & & \\
 & & (\top,+) \arrow[dl, dash] \arrow[dr, dash] & & (+,\top) \arrow[dl, dash] \arrow[dr, dash] & & \\
 & (\top,\perp) \arrow[dl, dashed] \arrow[dr, dash] & & (+,+) \arrow[dl, dash] \arrow[dr, dash] & & (\perp,\top) \arrow[dl, dash] \arrow[dr, dashed] & \\
\top \arrow[dr, dash] & & (+,\perp) \arrow[dl, dashed] \arrow[dr, dash] & & (\perp,+) \arrow[dl, dash] \arrow[dr, dashed] & & \top \arrow[dl, dash] \\
 & + \arrow[dr, dash] & & (\perp,\perp) \arrow[dl, dashed] \arrow[dr, dashed] & & + \arrow[dl, dash] & \\
 & & \perp & & \perp & &
\end{tikzcd} .$$

□

In the rest of the section we will only be concerned with the instance of Axiom 3 for $n = 0$; the case for general n will play a crucial rôle in the following section when internalising finite chains. Note that, as $\_ \times A$ preserves colimits (by Assumption 3), it follows that the commuting square

$$\begin{tikzcd}
1 \times A \arrow[r, "\perp \times A"] \arrow[d, "\top \times A"'] & \Sigma \times A \arrow[d, "\eta_\Sigma \times A"] \\
\Sigma \times A \arrow[r, "L\perp \times A"'] & \Sigma_2 \times A
\end{tikzcd} \tag{1}$$

is a pushout.

Convention. For a pushout

$$\begin{tikzcd} \cdot \arrow[r] \arrow[d] & B \arrow[d] \\ A \arrow[r] & C \end{tikzcd}$$

and a cone $A \xrightarrow{f} D \xleftarrow{g} B$ for it, we write $[f, g]$ for the unique mediating morphism $C \to D$ given by the universal property.

Remark. For a cone $\Sigma \times A \xrightarrow{\alpha} B \xleftarrow{\beta} \Sigma \times A$ for (1), we have $\lambda([\alpha, \beta]) = [\lambda(\alpha), \lambda(\beta)]$.

Proposition 5.10. (Transitivity) *If $\alpha : f \rightsquigarrow g : A \to B$ and $\beta : g \rightsquigarrow h : A \to B$ then $[\alpha, \beta] \circ (L\top \times A) : f \rightsquigarrow h : A \to B$.*

Proof. As $[\alpha, \beta] \circ (L\top \times A) \circ (\top \times A) = [\alpha, \beta] \circ (\eta_\Sigma \times A) \circ (\top \times A) = \beta \circ (\top \times A) = h \circ \pi_2$ and $[\alpha, \beta] \circ (L\top \times A) \circ (\bot \times A) = [\alpha, \beta] \circ (L\bot \times A) \circ (\bot \times A) = \alpha \circ (\bot \times A) = f \circ \pi_2$. □

As a corollary we have:

Theorem 5.11. *The category $\mathcal{C}$ equipped with $\sqsubseteq$ (henceforth denoted $\mathcal{C}_\sqsubseteq$) is a* **Preo**-*cartesian-closed-category and L determines a* **Preo**-*endofunctor on it.*

Proof. By Propositions 5.6 (1) and 5.10, the path relation gives a preorder on each hom-set. And, by Proposition 5.6 (2), the composition of morphisms is a monotone operation with respect to the path preorder. Finally, by Proposition 5.6 (5–6), the cartesian closed structure **Preo**-enriches. L determines a **Preo**-endofunctor by Proposition 5.6 (4). □

It follows that $\mathbb{L}$ is a **Preo**-monad and hence:

Corollary 5.12. *$\mathcal{C}^{\mathbb{L}}$ and $\mathcal{C}_{\mathbb{L}}$* **Preo**-*enrich with respect to $\sqsubseteq$.* □

6. Canonical upper bounds of ω-chains

First, using Axiom 3, to every $\boldsymbol{\omega}$-chain $\overline{f} = \langle f_n \rightsquigarrow f_{n+1} \rangle$ in $\mathcal{C}_\sqsubseteq(A, B)$ we associate maps $\Sigma_n \times A \to B$ $(n \geq 0)$ internalising the finite chains $\langle f_0 \rightsquigarrow \ldots \rightsquigarrow f_n \rangle$ (see Definition 6.1); then, using Axiom 1, we internalise the whole $\boldsymbol{\omega}$-chain $\overline{f}$ (see Definition 6.2); and finally, using Axiom 2, we extract its value at infinity. This process defines a passage-to-the-limit operator $\bigvee$ yielding canonical upper bounds for $\boldsymbol{\omega}$-chains of approximations. Various *algebraic* properties of $\bigvee$ are established.

Definition 6.1. *For an $\boldsymbol{\omega}$-chain $\overline{f} = \langle \alpha_n : f_n \rightsquigarrow f_{n+1} \rangle$ in $\mathcal{C}_\sqsubseteq(A, B)$ we inductively define, for $n \geq 0$, $[\overline{f}]_n : \Sigma_n \times A \to B$ satisfying $[\overline{f}]_n \circ (\eta_1^{(n)} \times A) = \alpha_n \circ (\bot \times A)$ as follows:*

— $[\overline{f}]_0 = (1 \times A \cong A \xrightarrow{f_0} B)$ *and* $[\overline{f}]_0 \circ (\eta_1^{(0)} \times A) = (1 \times A \cong A \xrightarrow{f_0} B) = \alpha_0 \circ (\bot \times A)$.
— $[\overline{f}]_{n+1} = \big[[\overline{f}]_n, \alpha_n\big] : (\Sigma_{n+1} \times A) \to B$ *(that is,*

$$
\begin{tikzcd}
1 \times A \arrow[r, "\perp \times A"] \arrow[d, "\eta_1^{(n)} \times A"'] & \Sigma \times A \arrow[d, "\eta_\Sigma^{(n)} \times A"'] \arrow[ddr, bend left, "\alpha_n"] & \\
\Sigma_n \times A \arrow[r, "L^n \perp \times A"] \arrow[drr, bend right, "{[\overline{f}]_n}"'] & \Sigma_{n+1} \times A \arrow[dr, "{[\overline{f}]_{n+1}}"'] & \\
& & B
\end{tikzcd}
\quad)
$$

and $[\overline{f}]_{n+1} \circ (\eta_1^{(n+1)} \times A) \; = \; [\overline{f}]_{n+1} \circ (\eta_\Sigma^{(n)} \times A) \circ (\top \times A) \; = \; \alpha_n \circ (\top \times A) \; =$ $(1 \times A \cong A \xrightarrow{f_{n+1}} B) \; = \; \alpha_{n+1} \circ (\perp \times A)$. □

We have constructed $\langle [\overline{f}]_n \rangle$ so that it is a cone for $\langle L^n(\perp) \times A : \Sigma_n \times A \to \Sigma_{n+1} \times A \rangle$. This together with the requirement that $\overline{\omega}$ be constructed as a colimit allows us to internalise $\overline{f}$:

Definition 6.2. *For every $\boldsymbol{\omega}$-chain $\overline{f}$ in $\mathcal{C}_{\sqsubseteq}(A, B)$ we define, $[\overline{f}] : \overline{\omega} \times A \to B$ as the unique mediating morphism from $\langle \mathsf{s}_n \times A \rangle$ to $\langle [\overline{f}]_n \rangle$.* □

Example 6.3. *Let $f : A \to B$. Writing $\langle f \rangle$ for the constantly f $\boldsymbol{\omega}$-chain $\langle f \overset{\mathrm{id}_f}{\rightsquigarrow} f \overset{\mathrm{id}_f}{\rightsquigarrow} \ldots \rangle$ in $\mathcal{C}_{\sqsubseteq}(A, B)$, for $n \geq 0$, $[\langle f \rangle]_n = (\Sigma_n \times A \xrightarrow{\pi_2} A \xrightarrow{f} B)$ and hence $[\langle f \rangle] = (\overline{\omega} \times A \xrightarrow{\pi_2} A \xrightarrow{f} B)$.* □

Example 6.4. *Let $\overline{c}$ be the $\boldsymbol{\omega}$-chain $\langle \mathsf{s}_{n+1} \circ \eta_\Sigma^{(n)} : \underline{n} \rightsquigarrow \underline{n+1} \rangle$ in $\mathcal{C}_{\sqsubseteq}(1, \overline{\omega})$. Then, for every $n \geq 0$, $[\overline{c}]_n = \mathsf{s}_n$ $\big($because, by induction, the case $n = 0$ reduces to the equality $[\overline{c}]_0 = \mathsf{s}_0 \circ \eta_1^{(0)} = \mathsf{s}_0$; and for the case $n = m + 1$ ($m \geq 0$) we have that*

$$
\begin{aligned}
[\overline{c}]_{m+1} \;&=\; \big[[\overline{c}]_m, \mathsf{s}_{m+1} \circ \eta_\Sigma^{(m)}\big] \\
&=\; [\mathsf{s}_m, \mathsf{s}_{m+1} \circ \eta_\Sigma^{(m)}] && \text{, by induction} \\
&=\; [\mathsf{s}_{m+1} \circ L^m \perp, \mathsf{s}_{m+1} \circ \eta_\Sigma^{(m)}] \\
&=\; \mathsf{s}_{m+1} \circ [L^m \perp, \eta_\Sigma^{(m)}] \\
&=\; \mathsf{s}_{m+1} && \text{, by Axiom 3}\big)
\end{aligned}
$$

and hence $[\overline{c}] = id_{\overline{\omega}}$. □

Example 6.5. *For every endomorphism f on a pointed object X,*

$$
\mathsf{it}(f) \;=\; [\langle f^n \circ \rho_X \circ L(f \circ \perp_X) : f^n \circ \perp_X \rightsquigarrow f^{n+1} \circ \perp_X \rangle].
$$

Because

$$\begin{aligned}
\mathsf{it}(f) &= [\langle \mathsf{it}_{n+1}(f) \circ \eta_\Sigma^{(n)} : \mathsf{it}_n(f) \circ \underline{n}_{\Sigma_n} \rightsquigarrow \mathsf{it}_{n+1}(f) \circ \underline{n+1}_{\Sigma_{n+1}} \rangle], \\
\mathsf{it}_{n+1}(f) \circ \eta_\Sigma^{(n)} &= f^n \circ \rho_X \circ L(f \circ \perp_X), \\
\mathsf{it}_n(f) \circ \underline{n}_{\Sigma_n} &= f^n \circ \perp_X. \qquad \square
\end{aligned}$$

We introduce the *passage-to-the-limit operator*:

Definition 6.6. ($\bigvee$) *For an $\boldsymbol{\omega}$-chain $\overline{f}$ in $\mathcal{C}_\sqsubseteq(A,B)$, define $\bigvee\overline{f} : A \to B$ as the composite*

$$A \cong 1 \times A \xrightarrow{\infty \times A} \overline{\omega} \times A \xrightarrow{[\overline{f}]} B. \qquad \square$$

Example 6.7. (Constant) *Let $f : A \to B$. Writing $\langle f \rangle$ for the constantly f $\boldsymbol{\omega}$-chain $\langle f \overset{\mathrm{id}_f}{\rightsquigarrow} f \overset{\mathrm{id}_f}{\rightsquigarrow} \ldots \rangle$ in $\mathcal{C}_\sqsubseteq(A,B)$, it follows that $\bigvee\langle f \rangle = f$ (see Example 6.3).* $\square$

Example 6.8. *For every endomorphism f on a pointed object X,*

$$\mathsf{fix}(f) \;=\; \bigvee\langle f^n \circ \rho_X \circ L(f \circ \perp_X) : f^n \circ \perp_X \rightsquigarrow f^{n+1} \circ \perp_X \rangle. \qquad \square$$

We have internalised arbitrary $\boldsymbol{\omega}$-chains; however, as we show below, it is enough to consider $\boldsymbol{\omega}$-chains of global elements.

Definition 6.9. *For an $\boldsymbol{\omega}$-chain $\overline{f} = \langle \alpha_n : f_n \rightsquigarrow f_{n+1} \rangle$ in $\mathcal{C}_\sqsubseteq(A,B)$, let $\ulcorner\overline{f}\urcorner$ be the $\boldsymbol{\omega}$-chain $\langle \lambda(\alpha_n) : \ulcorner f_n \urcorner \rightsquigarrow \ulcorner f_{n+1} \urcorner \rangle$ in $\mathcal{C}_\sqsubseteq(1, B^A)$.* $\square$

Proposition 6.10. *Let $\overline{f} = \langle \alpha_n : f_n \rightsquigarrow f_{n+1} \rangle$ be an $\boldsymbol{\omega}$-chain in $\mathcal{C}_\sqsubseteq(A,B)$. For $n \geq 0$, $\lambda\big([\overline{f}]_n\big) = [\ulcorner\overline{f}\urcorner]_n : \Sigma_n \to B^A$ and hence $\lambda[\overline{f}] = [\ulcorner\overline{f}\urcorner] : \overline{\omega} \to B^A$.* $\square$

Again, in order to discuss the properties of the $\bigvee$ operator it is enough to restrict attention to $\boldsymbol{\omega}$-chains of global elements:

Proposition 6.11. *For every $\boldsymbol{\omega}$-chain $\overline{f}$ in $\mathcal{C}_\sqsubseteq(A,B)$, $\ulcorner\bigvee\overline{f}\urcorner = \bigvee\ulcorner\overline{f}\urcorner : 1 \to B^A$.* $\square$

The operator $\bigvee$ provides *canonical* upper-bounds for $\boldsymbol{\omega}$-chains:

Proposition 6.12. (Canonical Upper Bounds) *Let $\overline{c} = \langle c_n \rangle$ be an $\boldsymbol{\omega}$-chain in $\mathcal{C}_\sqsubseteq(1,C)$. Then, $[\overline{c}] \circ \infty_n : c_n \rightsquigarrow \bigvee\overline{c} : 1 \to C$.*

Proof. Note that ∞_n has been defined in Example 5.3 (3).
We have that $[\overline{c}] \circ \infty_n \circ \perp = c_n$ because

$$\begin{tikzcd}
1 \arrow[drrrrrr, "c_n"] \arrow[dd, "\perp"'] \arrow[dr, "\eta_1^{(n)}"] & & & & & & \\
& \Sigma_n \arrow[rrrrr, "{[\overline{c}]_n}"'] \arrow[dd, "L^n\perp"'] \arrow[ddr, "L^n(\perp_{L\overline{\omega}})"] & & & & & C \\
\Sigma \arrow[dr, "\eta_\Sigma^{(n)}"'] & & & & & & \\
& \Sigma_{n+1} \arrow[r, "L^n(L\infty)"'] & L^n(L\overline{\omega}) \arrow[r, "L^n\sigma"'] & L^n\overline{\omega} & \cdots & L\overline{\omega} \arrow[r, "\sigma"'] & \overline{\omega} \arrow[uu, "{[\overline{c}]}"']
\end{tikzcd}$$

.

Moreover,

$$
\begin{aligned}
[\overline{c}] \circ \infty_n \circ \top &= [\overline{c}] \circ \sigma \circ \ldots \circ (L^n\sigma) \circ (L^{n+1}\infty) \circ \eta_\Sigma^{(n)} \circ \top \\
&= [\overline{c}] \circ \sigma \circ \ldots \circ (L^n\sigma) \circ (L^{n+1}\infty) \circ \eta_1^{(n+1)} \\
&= [\overline{c}] \circ \sigma \circ \ldots \circ (L^n\sigma) \circ \eta_{\overline{\omega}}^{(n+1)} \circ \infty \\
&= [\overline{c}] \circ \infty && \text{, by the invariance of } \infty. \quad \square
\end{aligned}
$$

Canonical upper bounds are preserved by composition:

Lemma 6.13.

1 *Let $\overline{a} = \langle \alpha_n : a_n \rightsquigarrow a_{n+1} \rangle$ be an $\boldsymbol{\omega}$-chain in $\mathcal{C}_\sqsubseteq(1, A)$ and let $\overline{b} = \langle \beta_n : b_n \rightsquigarrow b_{n+1} \rangle$ be an $\boldsymbol{\omega}$-chain in $\mathcal{C}_\sqsubseteq(1, B)$.*
Given $m : B \times A \to C$, let $\overline{c}$ be the ω-chain

$$\big\langle m \circ \langle \beta_n, \alpha_n \rangle : m \circ \langle b_n, a_n \rangle \rightsquigarrow m \circ \langle b_{n+1}, a_{n+1} \rangle \big\rangle$$

in $\mathcal{C}_\sqsubseteq(1, C)$.
Then, $[\overline{c}] = m \circ \big\langle [\overline{b}], [\overline{a}] \big\rangle$.

2 **(Continuity of composition)** *Let $\overline{f} = \langle \alpha_n : f_n \rightsquigarrow f_{n+1} \rangle$ and $\overline{g} = \langle \beta_n : g_n \rightsquigarrow g_{n+1} \rangle$ be $\boldsymbol{\omega}$-chains in $\mathcal{C}_\sqsubseteq(A, B)$ and $\mathcal{C}_\sqsubseteq(B, C)$ respectively.*
For $\overline{h} = \langle \beta_n \star \alpha_n : g_n \circ f_n \rightsquigarrow g_{n+1} \circ f_{n+1} \rangle$, $(\bigvee \overline{g}) \circ (\bigvee \overline{f}) = \bigvee \overline{h}$.

Proof.

(1) It is enough to show that $[\overline{c}]_i = m \circ \big\langle [\overline{b}]_i, [\overline{a}]_i \big\rangle$ for every $i \geq 0$. We proceed by induction on i. The case $i = 0$ reduces to $\overline{c}_0 = m \circ \langle b_0, a_0 \rangle$. For the case $i = j + 1$ ($j \geq 0$) we calculate:

$$
\begin{aligned}
[\overline{c}]_{j+1} &= \big[[\overline{c}]_j, m \circ \langle \beta_j, \alpha_j \rangle \big] && \text{, by Definition 6.1} \\
&= \big[m \circ \langle [\overline{b}]_j, [\overline{a}]_j \rangle, m \circ \langle \beta_j, \alpha_j \rangle \big] && \text{, by induction} \\
&= m \circ \big[\big\langle [\overline{b}]_j, [\overline{a}]_j \big\rangle, \big\langle \beta_j, \alpha_j \big\rangle \big] \\
&= m \circ \big\langle \big[[\overline{b}]_j, \beta_j \big], \big[[\overline{a}]_j, \alpha_j \big] \big\rangle \\
&= m \circ \big\langle [\overline{b}]_{j+1}, [\overline{a}]_{j+1} \big\rangle.
\end{aligned}
$$

(2) Let $M : C^B \times B^A \to C^A$ be the exponential transpose of

$$(C^B \times B^A) \times A \cong C^B \times (B^A \times A) \xrightarrow{C^B \times \mathsf{e}} C^B \times B \xrightarrow{\mathsf{e}} C.$$

Then,

$$
\begin{aligned}
\ulcorner (\bigvee \overline{g}) \circ (\bigvee \overline{f}) \urcorner &= M \circ \big\langle \ulcorner \bigvee \overline{g} \urcorner, \ulcorner \bigvee \overline{f} \urcorner \big\rangle \\
&= M \circ \big\langle [\ulcorner \overline{g} \urcorner] \circ \infty, [\ulcorner \overline{f} \urcorner] \circ \infty \big\rangle \\
&= M \circ \big\langle [\ulcorner \overline{g} \urcorner], [\ulcorner \overline{f} \urcorner] \big\rangle \circ \infty \\
&= [\ulcorner \overline{h} \urcorner] \circ \infty && \text{, by Lemma 6.13 (1)} \\
&= \ulcorner \bigvee \overline{h} \urcorner. && \square
\end{aligned}
$$

The operator $\bigvee$ is invariant under *shifting* $\boldsymbol{\omega}$-chains:

Convention. For an $\boldsymbol{\omega}$-chain $\overline{f} = \langle \alpha_n : f_n \rightsquigarrow f_{n+1} \rangle$ in $\mathcal{C}_{\sqsubseteq}(A,B)$ we write $\overline{f}^+$ for the $\boldsymbol{\omega}$-chain $\langle \alpha_{n+1} : f_{n+1} \rightsquigarrow f_{n+2} \rangle$ in $\mathcal{C}_{\sqsubseteq}(A,B)$.

Proposition 6.14. *For an $\boldsymbol{\omega}$-chain $\overline{c}$ in $\mathcal{C}_{\sqsubseteq}(1,X)$,*

1 $[\overline{c}^+] = [\overline{c}] \circ \mathsf{succ}$, *and*
2 **(Shift)** $\bigvee\overline{c}^+ = \bigvee\overline{c}$.

Proof.
(1) First we show by induction that for $n \geq 0$, $[\overline{c}^+]_n = [\overline{c}]_{n+1} \circ \eta_{\Sigma_n}$. Let $\overline{c} = \langle \alpha_n : x_n \rightsquigarrow x_{n+1} \rangle$. The case $n = 0$ reduces to the equality $[\overline{c}^+]_0 = x_1 = [\overline{c}]_1 \circ \top$. For the case $n = m+1$ ($m \geq 0$), by Definition 6.1, we have that $[\overline{c}^+]_{m+1} = [\overline{c}]_{m+2} \circ \eta_{\Sigma_{m+1}}$ if and only if

$$[\overline{c}^+]_m \quad = \quad [\overline{c}]_{m+2} \circ \eta_{\Sigma_{m+1}} \circ L^m\bot \tag{2}$$

and

$$\alpha_{m+1} \quad = \quad [\overline{c}]_{m+2} \circ \eta_{\Sigma_{m+1}} \circ \eta_{\Sigma}^{(m)}. \tag{3}$$

And (2) holds because

$$[\overline{c}^+]_m \;=\; [\overline{c}]_{m+1} \circ \eta_{\Sigma_m} \;=\; [\overline{c}]_{m+2} \circ L^{m+1}\bot \circ \eta_{\Sigma_m} \;=\; [\overline{c}]_{m+2} \circ \eta_{\Sigma_{m+1}} \circ L^m\bot$$

whilst (3) holds because

$$\alpha_{m+1} \;=\; [\overline{c}]_{m+2} \circ \eta_{\Sigma}^{(m+1)} \;=\; [\overline{c}]_{m+2} \circ \eta_{\Sigma_{m+1}} \circ \eta_{\Sigma}^{(m)}.$$

The result follows as, for $n \geq 0$, $[\overline{c}^+]_n = [\overline{c}] \circ \mathsf{s}_{n+1} \circ \eta_{\Sigma_n} = [\overline{c}] \circ \sigma \circ L(\mathsf{s}_n) \circ \eta_{\Sigma_n} = [\overline{c}] \circ \mathsf{succ} \circ \mathsf{s}_n$.
(2) $\bigvee\overline{c}^+ \;=\; [\overline{c}^+] \circ \infty \;=\; [\overline{c}] \circ \mathsf{succ} \circ \infty \;=\; [\overline{c}] \circ \infty \;=\; \bigvee\overline{c}$. □

Remark. It is interesting to note that from the characterisation of fix in Example 6.8, using (Constant), (Shift) and the fact that canonical upper bounds are preserved by composition, the *usual* algebraic argument for showing that fix is a fixed-point operator can be reproduced. That is, for every endomorphism f on a pointed object X,

$$\begin{aligned} f \circ \mathsf{fix}(f) \quad &= \quad \bigvee\langle \mathrm{id}_f : f \rightsquigarrow f \rangle_n \circ \bigvee\langle f^n \circ \rho_X \circ L(f \circ \bot_X) : f^n \circ \bot_X \rightsquigarrow f^{n+1} \circ \bot_X \rangle_n \\ &\qquad\qquad \text{, by (Constant) and Example 6.8} \\ &= \quad \bigvee\langle f^{n+1} \circ \rho_X \circ L(f \circ \bot_X) : f^{n+1} \circ \bot_X \rightsquigarrow f^{n+2} \circ \bot_X \rangle_n \\ &\qquad\qquad \text{, by continuity of composition} \\ &= \quad \mathsf{fix}(f) \\ &\qquad\qquad \text{, by (Shift) and Example 6.8.} \end{aligned}$$

Canonical upper bounds are preserved by L:

Lemma 6.15. *Let $\overline{f} = \langle \alpha_n : f_n \rightsquigarrow f_{n+1} \rangle$ be an $\boldsymbol{\omega}$-chain in $\mathcal{C}_{\sqsubseteq}(A,B)$, and write $L\overline{f}$ for the $\boldsymbol{\omega}$-chain $\langle L(\alpha_n) \circ \mathsf{t}' : Lf_n \rightsquigarrow Lf_{n+1} \rangle$ in $\mathcal{C}_{\sqsubseteq}(LA,LB)$. We have that*

1 $[L\overline{f}] = (L[\overline{f}]) \circ \mathsf{t}'$, *and*
2 **(Continuity of L)** $L(\bigvee\overline{f}) = \bigvee(L\overline{f})$.

Proof.
(1) Observe that $[L\overline{f}] = (L[\overline{f}]) \circ \mathsf{t}'$ if and only if, for $n \geq 0$,

$$\begin{aligned} [L\overline{f}]_n &= (L[\overline{f}]) \circ \mathsf{t}' \circ (\mathsf{s}_n \times LA) \\ &= (L[\overline{f}]) \circ L(\mathsf{s}_n \times A) \circ \mathsf{t}' \\ &= L([\overline{f}]_n) \circ \mathsf{t}'. \end{aligned} \tag{4}$$

We show (4) by induction on n. The case $n = 0$ reduces to the equality $[L\overline{f}]_0 = L(\alpha_0) \circ \mathsf{t}' = L([\overline{f}]_0) \circ \mathsf{t}'$. For the case $n = m + 1$ ($m \geq 0$), by Definition 6.1, we have that $[L\overline{f}]_{m+1} = L([\overline{f}]_{m+1}) \circ \mathsf{t}'$ if and only if

$$L([\overline{f}]_{m+1}) \circ \mathsf{t}' \circ (\eta_\Sigma^{(m)} \times LA) = L(\alpha_m) \circ \mathsf{t}' \tag{5}$$

and

$$L([\overline{f}]_{m+1}) \circ \mathsf{t}' \circ (L^m\bot \times LA) = [L\overline{f}]_m. \tag{6}$$

And (5) holds because

$$\begin{aligned} L([\overline{f}]_{m+1}) \circ \mathsf{t}' \circ (\eta_\Sigma^{(m)} \times LA) &= L([\overline{f}]_{m+1}) \circ L(\eta_\Sigma^{(m)} \times A) \circ \mathsf{t}' \\ &= L(\alpha_m) \circ \mathsf{t}' && \text{, by Definition 6.1} \end{aligned}$$

whilst (6) holds because

$$\begin{aligned} L([\overline{f}]_{m+1}) \circ \mathsf{t}' \circ (L^m\bot \times LA) &= L([\overline{f}]_{m+1}) \circ L(L^m\bot \times A) \circ \mathsf{t}' \\ &= L([\overline{f}]_m) \circ \mathsf{t}' && \text{, by Definition 6.1} \\ &= [L\overline{f}]_m && \text{, by induction.} \end{aligned}$$

(2) Follows from Lemma 6.15 (1). □

7. Unique proofs

We take a first step towards *externalising* Σ by identifying proofs of approximation between maps. As a consequence we are able to: show that colimits order-enrich (Proposition 7.3), and explore further algebraic properties of V and fix. Surprisingly, it follows that the path preorder is *antisymmetric*.

Proposition 7.1. *The following are equivalent:*

1 **(Unique proofs)** *If $\alpha, \beta : f \rightsquigarrow g : 1 \to C$ then $\alpha = \beta$.*
2 $1 \overset{\bot}{\hookrightarrow} \Sigma \overset{\top}{\longleftarrow} 1$ *are jointly epi.*
3 *The map*

$$\begin{aligned} \mathcal{C}(A, B^\Sigma) &\to \mathbf{Preo}\big(\mathbf{2}, \mathcal{C}_\sqsubseteq(A,B)\big) \\ \alpha &\mapsto \mathsf{e} \circ \langle \alpha, \bot \circ (A \to 1)\rangle \sqsubseteq \mathsf{e} \circ \langle \alpha, \top \circ (A \to 1)\rangle \end{aligned}$$

is a bijection. □

Axiom 4. (Unique proofs) $1 \overset{\bot}{\hookrightarrow} \Sigma \overset{\top}{\longleftarrow} 1$ *are jointly epi.* □

Convention. For $f \sqsubseteq g : A \to B$, we set $[f \sqsubseteq g] : f \rightsquigarrow g$.

Corollary 7.2. *The map*

$$\mathcal{C}(A, B^{\overline{\omega}}) \to \mathbf{Preo}\big(\boldsymbol{\omega}, \mathcal{C}_{\sqsubseteq}(A,B)\big) : \alpha \mapsto \big\langle \mathsf{e} \circ \langle \alpha, \underline{n}_{\overline{\omega}} \circ (A \to 1)\rangle \big\rangle_{n \in |\boldsymbol{\omega}|}$$

is a bijection. □

Axiom 4 forces $\rightsquigarrow$ and $\sqsubseteq$ to coincide. More importantly, it allows us to conclude:

Proposition 7.3. *Colimits in* $\mathcal{C}$ **Preo**-*enrich.*

Proof. Let $\Delta : \mathcal{G} \to \mathcal{C}$ and let $\amalg : \Delta \dot{\to} C$ be colimiting in $\mathcal{C}$. For $f, g : C \to D$, we show that if $f \circ \amalg \sqsubseteq g \circ \amalg : \Delta \dot{\to} D$ then $f \sqsubseteq g : C \to D$.
Assume $f \circ \amalg \sqsubseteq g \circ \amalg : \Delta \dot{\to} D$. For every $e : M \to N$ in $\mathcal{G}$, we have that

$$\begin{aligned}[]
[f \circ \amalg_N \sqsubseteq g \circ \amalg_N] \circ (\Sigma \times \Delta e) \circ (\bot \times \Delta M) &= f \circ \amalg_N \circ \pi_2 \circ (1 \times \Delta e) \\
&= f \circ \amalg_N \circ \Delta e \circ \pi_2 \\
&= f \circ \amalg_M \circ \pi_2
\end{aligned}$$

and

$$\begin{aligned}[]
[f \circ \amalg_N \sqsubseteq g \circ \amalg_N] \circ (\Sigma \times \Delta e) \circ (\top \times \Delta M) &= g \circ \amalg_N \circ \pi_2 \circ (1 \times \Delta e) \\
&= g \circ \amalg_N \circ \Delta e \circ \pi_2 \\
&= g \circ \amalg_M \circ \pi_2 .
\end{aligned}$$

Thus, by the unique-proofs axiom, $[f \circ \amalg_N \sqsubseteq g \circ \amalg_N] \circ (\Sigma \times \Delta e) = [f \circ \amalg_M \sqsubseteq g \circ \amalg_M]$ and so $\Gamma \stackrel{\text{def}}{=} \langle [f \circ \amalg_M \sqsubseteq g \circ \amalg_M] \rangle_M$ is a cone $\Sigma \times \Delta \dot{\to} D$.
As $\Sigma \times \_$ preserves colimits, it follows that $\Sigma \times \amalg : \Sigma \times \Delta \dot{\to} \Sigma \times C$ is colimiting. Then, for $\alpha : \Sigma \times C \to D$ the unique mediating morphism between $\Sigma \times \amalg$ and Γ we have that $\alpha : f \rightsquigarrow g$ as $1 \times \amalg$ is colimiting, and $\alpha \circ (\bot \times C) \circ (1 \times \amalg) = f \circ \pi_2 \circ (1 \times \amalg)$ and $\alpha \circ (\top \times C) \circ (1 \times \amalg) = g \circ \pi_2 \circ (1 \times \amalg)$. □

Typical applications of the above proposition (see Corollary 8.3) are given in the corollary below.

Convention. For $f \sqsubseteq g \sqsubseteq h : A \to B$ we write $[f \sqsubseteq g \sqsubseteq h]$ for the unique mediating morphism $\big[[f \sqsubseteq g], [g \sqsubseteq h]\big] : \Sigma_2 \times A \to X$ from (1) to the cone $\Sigma \times A \xrightarrow{[f \sqsubseteq g]} X \xleftarrow{[g \sqsubseteq h]} \Sigma \times A$.

Corollary 7.4.

1. *For* $x \sqsubseteq y \sqsubseteq z$ *and* $x' \sqsubseteq y' \sqsubseteq z'$ *in* $\mathcal{C}_{\sqsubseteq}(1, C)$, *if* $[x \sqsubseteq y] \sqsubseteq [x' \sqsubseteq y']$ *and* $[y \sqsubseteq z] \sqsubseteq [y' \sqsubseteq z']$ *in* $\mathcal{C}_{\sqsubseteq}(\Sigma, C)$ *then* $[x \sqsubseteq y \sqsubseteq z] \sqsubseteq [x' \sqsubseteq y' \sqsubseteq z'] : \Sigma_2 \to C$.
2. *For* $\boldsymbol{\omega}$-*chains* $\overline{f}$ *and* $\overline{g}$ *in* $\mathcal{C}_{\sqsubseteq}(A, B)$, *if* $[\overline{f}]_n \sqsubseteq [\overline{g}]_n : \Sigma_n \times A \to B$ $(n \geq 0)$ *then* $[\overline{f}] \sqsubseteq [\overline{g}] : \overline{\omega} \times A \to B$. □

In addition to (Constant) and (Shift), further algebraic properties of V, needed to establish the antisymmetry of the path preorder, are studied below.

Proposition 7.5. *Let* $\langle f_n \rangle$ *be an* ω-*chain in* $\mathcal{C}_{\sqsubseteq}(\overline{\omega}, C)$. *Then,*

$$[\langle f_n \rangle] \circ \langle \infty, \infty \rangle \quad = \quad [\langle f_n \circ \infty \rangle] \circ \infty .$$

Proof.

$$
\begin{aligned}
[\langle f_n\rangle]\circ\langle\infty,\infty\rangle &= (1 \xrightarrow{\infty} \overline{\omega} \cong 1\times\overline{\omega} \xrightarrow{\infty\times\overline{\omega}} \overline{\omega}\times\overline{\omega} \xrightarrow{[\langle f_n\rangle]} C) &&= (\bigvee\langle f_n\rangle)\circ\infty \\
&= (\bigvee\langle f_n\rangle)\circ(\bigvee\langle\infty\rangle) &&= \bigvee\langle f_n\circ\infty\rangle \\
&= [\langle f_n\circ\infty\rangle]\circ\infty. && \square
\end{aligned}
$$

Corollary 7.6. *For $n, m \geq 0$ let $c_{n,m} : 1 \to C$ be such that $\langle c_{n,m}\rangle_m$ is an $\boldsymbol{\omega}$-chain in $\mathcal{C}_{\sqsubseteq}(1, C)$ and $[c_{n,m} \sqsubseteq c_{n,m+1}] \sqsubseteq [c_{n+1,m} \sqsubseteq c_{n+1,m+1}] : \Sigma \to C$. Then,*

1. **(Interchange)** *$\langle c_{n,m}\rangle_n$, $\big\langle \bigvee\langle c_{n,m}\rangle_m\big\rangle_n$ and $\big\langle\bigvee\langle c_{n,m}\rangle_n\big\rangle_m$ are $\boldsymbol{\omega}$-chains in $\mathcal{C}_{\sqsubseteq}(1, C)$, and $\bigvee\big\langle\bigvee\langle c_{n,m}\rangle_m\big\rangle_n = \bigvee\big\langle\bigvee\langle c_{n,m}\rangle_n\big\rangle_m$.*
2. **(Diagonal)** *$\langle c_{m,m}\rangle_m$ is an $\boldsymbol{\omega}$-chain in $C_{\sqsubseteq}(1, C)$, and $\bigvee\big\langle\bigvee\langle c_{n,m}\rangle_m\big\rangle_n = \bigvee\langle c_{m,m}\rangle_m$.*

Proof.

(1) Observe that

$$\big[\big\langle[\langle c_{n,m}\rangle_m]\big\rangle_n\big] \quad = \quad \big[\big\langle[\langle c_{n,m}\rangle_n]\big\rangle_m\big]\circ\langle\pi_2,\pi_1\rangle \tag{7}$$

because for $i, j \geq 0$,

$$
\begin{aligned}
\big[\big\langle[\langle c_{n,m}\rangle_m]\big\rangle_n\big]\circ\langle \underline{i},\underline{j}\rangle &= \big[\big\langle[\langle c_{n,m}\rangle_m]\big\rangle_n\big]\circ(\underline{i}\times\overline{\omega})\circ\langle 1,\underline{j}\rangle &&= [\langle c_{i,m}\rangle_m]\circ\underline{j} \\
&= c_{i,j} &&= [\langle c_{n,j}\rangle_n]\circ\underline{i} \\
&= \big[\big\langle[\langle c_{n,m}\rangle_n]\big\rangle_m\big]\circ(\underline{j}\times\overline{\omega})\circ\langle 1,\underline{i}\rangle &&= \big[\big\langle[\langle c_{n,m}\rangle_n]\big\rangle_m\big]\circ\langle\underline{j},\underline{i}\rangle. \\
&= \big[\big\langle[\langle c_{n,m}\rangle_n]\big\rangle_m\big]\circ\langle\pi_2,\pi_1\rangle\circ\langle\underline{i},\underline{j}\rangle.
\end{aligned}
$$

Then,

$$
\begin{aligned}
\bigvee\big\langle\bigvee\langle c_{n,m}\rangle_m\big\rangle_n &= \big[\big\langle[\langle c_{n,m}\rangle_m]\circ\infty\big\rangle_n\big]\circ\infty \\
&= \big[\big\langle[\langle c_{n,m}\rangle_m]\big\rangle_n\big]\circ\langle\infty,\infty\rangle && \text{, by Proposition 7.5} \\
&= \big[\big\langle[\langle c_{n,m}\rangle_n]\big\rangle_m\big]\circ\langle\pi_2,\pi_1\rangle\circ\langle\infty,\infty\rangle && \text{, by (7)} \\
&= \big[\big\langle[\langle c_{n,m}\rangle_n]\big\rangle_m\big]\circ\langle\infty,\infty\rangle \\
&= \big[\big\langle[\langle c_{n,m}\rangle_n]\circ\infty\big\rangle_m\big]\circ\infty && \text{, by Proposition 7.5} \\
&= \bigvee\big\langle\bigvee\langle c_{n,m}\rangle_n\big\rangle_m.
\end{aligned}
$$

(2) Let

$$c'_{n,m} \quad = \quad \begin{cases} c_{m,m} & \text{, if } 0 \leq m \leq n \\ c_{n,m} & \text{, if } m > n \geq 0 \end{cases}$$

Then,

$$
\begin{aligned}
\bigvee\langle\bigvee\langle c_{n,m}\rangle_m\rangle_n &= \bigvee\langle\bigvee\langle c'_{n,m}\rangle_m\rangle_n && \text{, by (Shift)} \\
&= \bigvee\langle\bigvee\langle c'_{n,m}\rangle_n\rangle_m && \text{, by (Interchange)} \\
&= \bigvee\langle c_{m,m}\rangle_m && \text{, by (Shift) and (Constant).} \qquad \square
\end{aligned}
$$

Corollary 7.7.

1. **(Finality)** *Let $\overline{a} = \langle a_n\rangle$ and $\overline{b} = \langle b_n\rangle$ be $\boldsymbol{\omega}$-chains in $\mathcal{C}_{\sqsubseteq}(1, C)$ such that $\forall\, n.\exists\, m.\, a_n \sqsubseteq b_m$ and $\forall\, m.\,\exists\, n.\, b_m \sqsubseteq a_n$. Then, $\bigvee\overline{a} = \bigvee\overline{b}$.*

2 **(Dinaturality)** *Let X and Y be pointed objects. For $f : X \to Y$ and $g : Y \to X$,* $\mathsf{fix}(f \circ g) = f \circ \mathsf{fix}(g \circ f)$.

Proof.
(1) It suffices to show that for every $\boldsymbol{\omega}$-chain $\langle c_n \rangle$ in $\mathcal{C}_\sqsubseteq(1, C)$ and every *unbounded* monotone endofunction f on $\boldsymbol{\omega}$, $\bigvee\langle c_n \rangle = \bigvee\langle c_{f(n)} \rangle$.
Let $c'_{n,m} = c_{\mathsf{min}(f(n),m)}$. Then, for $n \geq 0$,

$$\langle c'_{n,m} \rangle \quad = \quad \langle c_0, \ldots, c_{f(n)-1}, c_{f(n)}, \ldots, c_{f(n)}, \ldots \rangle$$

and

$$\begin{aligned} \bigvee\langle c'_{n,m} \rangle_m &= \bigvee\langle c_{f(n)} \rangle_m && \text{, by (Shift)} \\ &= c_{f(n)} && \text{, by (Constant).} \end{aligned}$$

Finally, since $[c'_{n,m} \sqsubseteq c'_{n,m+1}] \sqsubseteq [c'_{n+1,m} \sqsubseteq c'_{n+1,m+1}]$ (by construction, using Proposition 5.6 (1) and the fact that $x \sqsubseteq y \sqsubseteq z$ implies $[x \sqsubseteq x] \sqsubseteq [y \sqsubseteq z]$) it follows that

$$\begin{aligned} \bigvee\langle c_m \rangle_m &= \bigvee\langle c'_{m,m} \rangle_m \\ &= \bigvee\langle \bigvee\langle c'_{n,m} \rangle_m \rangle_n && \text{, by (Diagonal)} \\ &= \bigvee\langle c_{f(n)} \rangle_n . \end{aligned}$$

(2) By (Finality) using the characterisation of fix in Example 6.8. □

As a surprising consequence we have:

Theorem 7.8. *$\mathcal{C}_\sqsubseteq$ is* **Poset***-cartesian-closed.*

Proof. By Theorem 5.11 we need only show that the path preorder is antisymmetric. Assuming $f \sqsubseteq g \sqsubseteq f : A \to B$, by (Finality) we have that $\bigvee\langle f \rangle = \bigvee\langle g \rangle$ and, by (Constant), we are done. □

8. Least upper bounds of ω-chains

We introduce Axiom 5 which together with Axiom 4 make Σ and $\overline{\varpi}$ behave externally as **2** and $\boldsymbol{\omega}$ respectively (see Proposition 8.1 (3) and Corollary 8.3). We are thus in position to establish the *enrichment theorem*.

Proposition 8.1. *The following are equivalent:*

1 **(Interpolation Property)** *The commuting square*

$$\begin{tikzcd} \Sigma \arrow[r, "L\top"] \arrow[d, "L\top"'] & \Sigma_2 \arrow[d, "{\langle L(\Sigma \to 1), \mu_1 \rangle}"] \\ \Sigma_2 \arrow[r, "{\langle \mu_1, L(\Sigma \to 1) \rangle}"'] & \Sigma \times \Sigma \end{tikzcd} \qquad (8)$$

is a pushout.

Remark. *The maps in the above diagram can be better understood as the* proofs $L\top = [\underline{0} \sqsubseteq \underline{2}]$, $\langle \mu_1, L(\Sigma \to 1)\rangle = \big[\langle \bot, \bot\rangle \sqsubseteq \langle \bot, \top\rangle \sqsubseteq \langle \top, \top\rangle\big]$, *and* $\langle L(\Sigma \to 1), \mu_1\rangle = \big[\langle \bot, \bot\rangle \sqsubseteq \langle \top, \bot\rangle \sqsubseteq \langle \top, \top\rangle\big]$.

2 **(Diamond property)** *If*

$$\begin{tikzcd} & y' & \\ x' \arrow[ur, phantom, "\sqsubseteq" sloped] & & y \arrow[ul, phantom, "\sqsupseteq" sloped] \\ & x \arrow[ul, phantom, "\sqsupseteq" sloped] \arrow[ur, phantom, "\sqsubseteq" sloped] & \end{tikzcd} \tag{9}$$

in $\mathcal{C}_\sqsubseteq(1, C)$ *then* $[x \sqsubseteq x'] \sqsubseteq [y \sqsubseteq y'] : \Sigma \to C$.

3 **($\mathcal{C}_\sqsubseteq$ has Poset-cotensors with 2)** *The monotone map*

$$\begin{array}{rcl} \mathcal{C}_\sqsubseteq(A, B^\Sigma) & \to & \mathbf{Poset}_\sqsubseteq\big(\mathbf{2}, \mathcal{C}_\sqsubseteq(A, B)\big) \\ \alpha & \mapsto & \mathsf{e} \circ \langle \alpha, \bot \circ (A \to 1)\rangle \sqsubseteq \mathsf{e} \circ \langle \alpha, \top \circ (A \to 1)\rangle \end{array}$$

is an isomorphism of preorders.

4 *(Tay94)* $(\Sigma, \sqsubseteq_\Sigma)$ *where* $\sqsubseteq_\Sigma \stackrel{\text{def}}{=} \langle \Sigma^\bot, \Sigma^\top\rangle : \Sigma^\Sigma \rightarrowtail \Sigma \times \Sigma$ *is a distributive lattice for which* $\bot$ *and* $\top$ *are respectively the least and greatest elements.* $\square$

Axiom 5. *The commuting square (8) is a pushout.* $\square$

Lemma 8.2. *For* $\boldsymbol{\omega}$*-chains* $\overline{f} = \langle f_n\rangle$ *and* $\overline{g} = \langle g_n\rangle$ *in* $\mathcal{C}_\sqsubseteq(A, B)$,

$$\forall\, n \geq 0.\ f_n \sqsubseteq g_n : A \to B \quad \textit{implies} \quad [\overline{f}] \sqsubseteq [\overline{g}] : \overline{\omega} \times A \to B. \tag{10}$$

Proof. Assuming the antecedent of (10), it follows by induction on n (using Axiom 5 and Proposition 7.3 —see Corollary 7.4 (1)) that $[\overline{f}]_n \sqsubseteq [\overline{g}]_n$ $(n \geq 0)$, and hence $[\overline{f}] \sqsubseteq [\overline{g}]$ (by Corollary 7.4 (2)). $\square$

Corollary 8.3.

1 **($\mathcal{C}_\sqsubseteq$ has Poset-cotensors with $\boldsymbol{\omega}$)** *The monotone map*

$$\mathsf{ext} : \mathcal{C}_\sqsubseteq(A, B^{\overline{\omega}}) \to \mathbf{Poset}_\sqsubseteq\big(\boldsymbol{\omega}, \mathcal{C}_\sqsubseteq(A, B)\big) : \alpha \mapsto \big\langle \mathsf{e} \circ \langle \alpha, \underline{n}_{\overline{\omega}} \circ (A \to 1)\rangle\big\rangle_{n \in |\boldsymbol{\omega}|}$$

is an isomorphism of preorders.

2 **(Monotonicity of $\bigvee$)** *The passage-to-the-limit operator* $\bigvee$ *is monotone.*

Proof.

(1) By Corollary 7.2 and Lemma 8.2.

(2) By Lemma 8.2 because composition is monotone. $\square$

At this stage, it is possible to give a direct proof that the passage-to-the-limit operator computes lubs: let $\langle f_n\rangle$ be an $\boldsymbol{\omega}$-chain, then

— for $i \geq 0$,

$$\begin{array}{rcll} f_i & = & \bigvee\langle f_i\rangle_n & \text{, by (Constant)} \\ & \sqsubseteq & \bigvee\langle f_{i+n}\rangle_n & \text{, by monotonicity of } \bigvee \\ & = & \bigvee\langle f_n\rangle_n & \text{, by (Shift);} \end{array}$$

— for $f_n \sqsubseteq f$ $(n \geq 0)$,

$$\begin{array}{rcll} \mathsf{V}\langle f_n\rangle & \sqsubseteq & \mathsf{V}\langle f\rangle & \text{, by monotonicity of } \mathsf{V} \\ & = & f & \text{, by (Constant).} \end{array}$$

However, our axiomatisation yields a stronger result, viz. that the objects of $\mathcal{C}_{\sqsubseteq}$ are *internally* complete (in the sense of Lemma 8.4); and from this, externalising via the Yoneda embedding, we will give another proof of the enrichment theorem. We have decided to proceed in this way to explore the consequences of the axioms and to show that they provide insight in understanding domains. Indeed, the internal-completeness property, which holds in **Cpo** but fails in $\mathbf{Cpo}_\wedge$, exhibits a crucial difference between continuity and stability.

Lemma 8.4. (Internal completeness) *For every $C \in |\,\mathcal{C}_{\sqsubseteq}\,|$, we have the following adjoint situation $C^\infty \dashv \lambda(C \times \overline{\omega} \xrightarrow{\pi_1} C) : C \to C^{\overline{\omega}}$ in $\mathcal{C}_{\sqsubseteq}$.*

Proof. The diagram

$$\begin{tikzcd}
C \arrow[rrrr, "\lambda(\pi_1)"] \arrow[d, "\cong"] & & & & C^{\overline{\omega}} \arrow[d, "\cong"'] \arrow[dddll, bend left, "C^\infty"] \\
C \times 1 \arrow[rrrr, "\lambda(\pi_1) \times 1"] \arrow[dr, "C \times \infty"] \arrow[ddrr, bend right, "\pi_1"'] & & & & C^{\overline{\omega}} \times 1 \arrow[dl, "C^{\overline{\omega}} \times \infty"'] \\
& C \times \overline{\omega} \arrow[rr, "\lambda(\pi_1) \times \overline{\omega}"'] \arrow[dr, "\pi_1"'] & & C^{\overline{\omega}} \times \overline{\omega} \arrow[dl, "\mathsf{e}"] & \\
& & C & &
\end{tikzcd}$$

shows that $C^\infty \circ \lambda(\pi_1) = C$.
Finally we need show that $C^{\overline{\omega}} \sqsubseteq \lambda(\pi_1) \circ C^\infty$; or equivalently that

$$(C^{\overline{\omega}} \times \overline{\omega} \xrightarrow{\mathsf{e}} C) \quad \sqsubseteq \quad (C^{\overline{\omega}} \times \overline{\omega} \xrightarrow{\pi_1} C^{\overline{\omega}} \xrightarrow{C^\infty} C);$$

or equivalently, as $C^{\overline{\omega}} \times \_$ preserves **Poset**-colimits, that for $n \geq 0$,

$$\begin{array}{cl} & (C^{\overline{\omega}} \times \Sigma_n \xrightarrow{C^{\overline{\omega}} \times \mathsf{s}_n} C^{\overline{\omega}} \times \overline{\omega} \xrightarrow{\mathsf{e}} C) \\ \sqsubseteq & (C^{\overline{\omega}} \times \Sigma_n \xrightarrow{C^{\overline{\omega}} \times \mathsf{s}_n} C^{\overline{\omega}} \times \overline{\omega} \xrightarrow{\pi_1} C^{\overline{\omega}} \xrightarrow{C^\infty} C) \\ = & (C^{\overline{\omega}} \times \Sigma_n \xrightarrow{C^{\overline{\omega}} \times \left(\infty \circ (\Sigma_n \to 1)\right)} C^{\overline{\omega}} \times \overline{\omega} \xrightarrow{\mathsf{e}} C). \end{array}$$

Since

$$\begin{array}{rcll} \mathsf{s}_n & = & [\langle \underline{i} \rangle_i]_n & \text{, by Example 6.4} \\ & \sqsubseteq & [\langle \infty \rangle]_n & \text{, by Corollary 8.3 (1) and Example 5.3 (3)} \\ & = & \infty \circ (\Sigma_n \to 1) & \text{, by Example 6.3} \end{array}$$

we are done. □

Theorem 8.5. (Enrichment theorem) *The* **Poset**-*category* $\mathcal{C}_{\sqsubseteq}$ *is* **Cpo**-*cartesian-closed, and* L *determines a* **Cpo**-*endofunctor on it.*

Proof. As for every $A \in |\,\mathcal{C}\,|$, $\mathcal{C}_{\sqsubseteq}(A, \_) : \mathcal{C}_{\sqsubseteq} \to \mathbf{Poset}_{\sqsubseteq}$ preserves adjunctions (as it is a **Poset**-functor) we have, by Lemma 8.4 and Corollary 8.3 (1), the following adjoint situation:

$$\begin{tikzcd} \mathcal{C}_{\sqsubseteq}(A,B) \arrow[r, bend right, "\lambda(\pi_1)\circ\_"'] \arrow[r, phantom, "\perp"] & \mathcal{C}_{\sqsubseteq}(A,B^{\overline{\omega}}) \arrow[l, bend right, "B^{\infty}\circ\_"'] \end{tikzcd} \underset{\mathsf{ext}}{\cong} \mathbf{Poset}_{\sqsubseteq}\big(\boldsymbol{\omega}, \mathcal{C}_{\sqsubseteq}(A,B)\big) \;;$$

and since $\mathsf{ext}(\lambda(\pi_1)\circ\_) : f \mapsto \langle f \rangle$, it follows that every hom-poset is $\boldsymbol{\omega}$-complete.
Composition is continuous by Proposition 5.6 (2) and Lemma 6.13.
L determines a **Cpo**-functor by Theorem 5.11 and Lemma 6.15 (2). □

It follows that $\mathbb{L}$ is a **Cpo**-monad and hence:

Corollary 8.6. $\mathcal{C}^{\mathbb{L}}$ *and* $\mathcal{C}_{\mathbb{L}}$ **Cpo**-*enrich with respect to* $\sqsubseteq$. □

9. Domain-theoretic categories

A category satisfying Assumptions 1–3 in which Axioms 1–5 hold is said to be *domain theoretic.*

In domain-theoretic categories, $\overline{\omega}$ complies Freyd's axiomatisation (Fre91):

Lemma 9.1. *In a domain-theoretic category,* $\sigma : L\overline{\omega} \to \overline{\omega}$ *is a* free L-algebra *(i.e. an initial* L*-algebra whose inverse is a final* L*-coalgebra —see (Fre91)), and hence* $\infty : 1 \to \overline{\omega}$ *is the unique coalgebra homomorphism* $(1, \top) \to (\overline{\omega}, \sigma^{-1})$.

Proof. Since $\langle L^n\bot : \Sigma_n \to \Sigma_{n+1}\rangle$ is an $\boldsymbol{\omega}$-chain of embeddings (because $\bot : 1 \to \Sigma$ is an embedding and L is a **Cpo**-functor) with colimiting cone $\langle \mathsf{s}_n \rangle : \langle L^n\bot : \Sigma_n \to \Sigma_{n+1}\rangle \stackrel{\cdot}{\to} \overline{\omega}$, by the limit/colimit-coincidence theorem (SP82), it follows that the cone $\langle L\mathsf{s}_n \rangle : \langle L^{n+1}\bot : \Sigma_{n+1} \to \Sigma_{n+2}\rangle \stackrel{\cdot}{\to} L\overline{\omega}$ is colimiting because L is a **Cpo**-functor. As $\sigma : L\overline{\omega} \to \overline{\omega}$ is the unique mediating morphism $\langle L\mathsf{s}_n\rangle \to \langle \mathsf{s}_{n+1}\rangle$, it is an isomorphism and hence and initial L-algebra by Proposition 4.8 (3).
Again by the limit/colimit-coincidence theorem (SP82), the cone $\langle {\mathsf{s}_n}^R \rangle : \overline{\omega} \stackrel{\cdot}{\to} \langle L^n(1 \leftarrow \Sigma) : \Sigma_n \leftarrow \Sigma_{n+1}\rangle$, where ${\mathsf{s}_0}^R = (\overline{\omega} \to 1)$ and ${\mathsf{s}_{n+1}}^R = L({\mathsf{s}_n}^R)\circ\sigma^{-1}$ $(n \geq 0)$, is limiting as so is $\langle L{\mathsf{s}_n}^R\rangle : L\overline{\omega} \stackrel{\cdot}{\to} \langle L^{n+1}(1 \leftarrow \Sigma) : \Sigma_{n+1} \leftarrow \Sigma_{n+2}\rangle$ because L is a **Cpo**-functor. Then, by (the dual of) the basic lemma (SP82), the coalgebra $\sigma^{-1} : \overline{\omega} \to L\overline{\omega}$ is final because it is the unique mediating morphism $\langle {\mathsf{s}_{n+1}}^R\rangle \to \langle L{\mathsf{s}_n}^R\rangle$. □

Examples of domain-theoretic categories are: $[{\mathcal{W}_{\leq}}^{\mathrm{op}}, \mathbf{Cpo}_{\sqsubseteq}]$, for $\mathcal{W}_{\leq}$ a small **Cpo**-category, with the pointwise lifting monad.

The rest of the categories in Example 1.2 are not examples:

— **Set**, **Preo** and **Poset** fail Axiom 2.

— $\mathbf{Poset}_\wedge$ fails Axioms 2 and 5.
— $\mathbf{Cpo}_\wedge$ fails Axiom 5. (Though, in the light of the following section, this is not unsatisfactory.)

10. Stability

Though indirectly, it is possible to incorporate stability into our axiomatic framework. However, as we argue below, it is best to abandon **Cpo**-enrichment.

The analysis of Sections 6 and 7 suggested considering the following:

Definition 10.1. *Let* $\mathsf{V}\mathbf{Poset}_\wedge$ *be the category with:*

— *objects:* (P, V) *with* $P \in |\ \mathbf{Poset}_\wedge\ |$ *and* $\mathsf{V} : P^{\boldsymbol{\omega}} \to P$ *in* $\mathbf{Poset}_\wedge$ *satisfying*

$$\begin{array}{lll} \textit{(Constant)} & \mathsf{V}\langle x\rangle_n = x & (x \in |\ P\ |); \\ \textit{(Diagonal)} & \mathsf{V}\big\langle \mathsf{V}\langle x_{n,m}\rangle_m\big\rangle_n = \mathsf{V}\langle x_{m,m}\rangle_m & (x : \boldsymbol{\omega} \times \boldsymbol{\omega} \to P \textit{ in } \mathbf{Poset}_\wedge); \\ \textit{(Shift)} & \mathsf{V}\langle x_{n+1}\rangle = \mathsf{V}\langle x_n\rangle & (x : \boldsymbol{\omega} \to P \textit{ in } \mathbf{Poset}_\wedge); \end{array}$$

and

— *morphisms:* $f : (P, \mathsf{V}_P) \to (Q, \mathsf{V}_Q)$ *with* $f : P \to Q$ *in* $\mathbf{Poset}_\wedge$ *satisfying*

$$\textit{(Continuity)} \quad f(\mathsf{V}_P\langle x_n\rangle) = \mathsf{V}_Q\langle f(x_n)\rangle \quad (x : \boldsymbol{\omega} \to P \textit{ in } \mathbf{Poset}_\wedge). \qquad \square$$

By construction, $\mathbf{Cpo}_\wedge$ is a full subcategory of $\mathsf{V}\mathbf{Poset}_\wedge$. Moreover, $\mathsf{V}\mathbf{Poset}_\wedge$ satisfies Axioms 1–4 but *not* Axiom 5, does *not* **Cpo**-enrich with respect to the path order, and further:

Proposition 10.2. *There is no* **Cpo***-enrichment on* $\mathsf{V}\mathbf{Poset}_\wedge$ *in which* $\perp$ *approximates* $\top$ *in the hom of global elements of* Σ. $\square$

Outstandingly, $(\mathsf{V}\mathbf{Poset}_\wedge)^{\mathbb{L}}$ together with the lifting comonad is a model of intuitionistic linear type theory (in the sense of (BBHdP93)) with sums and products in which the solution of domain equations is possible. Moreover, notice that in $\mathsf{V}\mathbf{Poset}_\wedge$ we have broaden the notion of *domain* by allowing, for example, the posets

$$0 \to 1 \to \cdots \to i \to \cdots \to * \to \infty$$

$$0 \to 1 \to \cdots \to i \to \cdots \begin{array}{l} \nearrow \infty \\ \searrow * \end{array}$$

equipped with a passage-to-the-limit operator V given by

$$\mathsf{V}\langle x_n\rangle \quad = \quad \begin{cases} x & \text{, if } x_n = x \text{ for all } n \text{ greater than some } n_0 \\ \infty & \text{, otherwise} \end{cases}$$

We conclude that **Cpo**-enrichment is not the natural setting for stability.

Remark. An abstract approach to "formal cpos" generalising the construction of Definition 10.1 has been considered by the author and Gordon Plotkin —see (FJM$^+$96, § Axiomatic Domain Theory). For example, this construction applied to **Poset** (instead of $\mathbf{Poset}_\wedge$) yields $\mathsf{V}\mathbf{Poset} \cong \mathbf{Cpo}$.

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Some diagrams have been drawn with Paul Taylor's commutative-diagrams package.