Lifting as a KZ-doctrine

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Synopsis. In a cartesian closed category with an initial object and a dominance that classifies it, an *intensional* notion of approximation between maps —the *path relation* (c.f. link relation)— is defined. It is shown that if such a category admits *strict/upper-closed factorisations* then it preorder-enriches (as a cartesian closed category) with respect to the path relation. By imposing further axioms we can, on the one hand, endow maps and proofs of their approximations (viz. *paths*) with the 2-dimensional algebraic structure of a sesqui-category and, on the other, characterise lifting as a preorder-enriched lax colimit. As a consequence of the latter the *lifting* (or partial map classifier) *monad* becomes a *KZ-doctrine*.

1 Introduction

One of the concerns of *axiomatic domain theory*, and a particular concern of this paper, is the analysis of *notions of approximation* aiming at explaining and justifying (order-theoretic) properties of categories of domains. For example, in [Fio94c, Fio94a], while studying the interaction between partiality and order-enrichment we considered *contextual approximation* which, in the framework we were working in, coincided with the *specialisation preorder*. But in the applications carried out in [FP94, Fio94a] we had to work with an axiomatised notion of approximation, instead of the aforementioned one, for the following two reasons: first, the specialisation preorder is not appropriate in categories of domains and stable functions (see [Fio94c]) and, second, we do not know of non-order-theoretic axioms making the specialisation preorder ω-complete. To overcome these drawbacks another notion of approximation was to be considered. And, it was the second problem that motivated the *intensional* notion of approximation provided by the *path relation*. In fact, it is shown in [Fio94b] that under suitable axioms the path relation can be equipped with a *canonical passage-to-the-limit operator* appropriate for fixed-point computations; stronger axioms make this operator be given by lubs of ω-chains.

Incidentally, in the context of *synthetic domain theory* [Pho90, Hyl91, Tay91] the path relation (called there the *link relation*) was already considered (mainly as a technical tool) but was required to coincide with the specialisation preorder.

Here we study the path relation independently of the specialisation preorder. First, the path relation is shown to interact nicely with type constructors and, since in general it need *not* be transitive, an axiom under which it becomes a preorder is proposed. Afterwards, we study the interaction between the path relation and the *lifting monad* (sometimes called the *partial map classifier monad* —see, e.g. [Koc91]) arising from the *representability of partial maps*. In particular, we provide a (non-order-theoretic) axiomatic framework in which the lifting monad becomes a *KZ-doctrine* with respect to the path relation. We do this motivated by the fact that on posets lifting is the construction that freely adjoins suprema for all subsets with at most one element, whilst KZ-doctrines [Koc94] are equational 2-dimensional categorical algebras underlying the process of freely completing categories under a given class of colimits.

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The paper is organised as follows. In Section 2 we introduce the basic assumptions on our category of discourse. In the vein of [Fio94a], we equip it with a subcategory of *admissible monos* allowing us to consider a *notion of partiality* from which the *lifting monad* arises. In Section 3 we consider the category of Eilenberg-Moore algebras for the lifting monad as a category of *pointed objects* (as they come equipped with a *bottom element*) and *strict maps*. In Section 4 the *path relation* is introduced and various algebraic properties of *paths* are presented. In Section 5 we axiomatise *upper-closed* subsets as those which are orthogonal (in a parameterised or internal sense) to strict maps and establish their closure properties. In Section 6, we present our main axiom (*Axiom 1*); intuitively stating that every map $X \to A$ with X pointed factors as a strict map $X \multimap U$ with U a minimal upper-closed subset of A. The path relation is always reflexive but it might not be transitive (a counterexample to transitivity can be found in [Fio94b]). By introducing the above axiom, *transitivity* is achieved and, in fact, our category of discourse is seen to **Preo**-enrich (as a cartesian closed category) with respect to the path preorder. In Section 7 the relationship between approximation and upper-closure is established. In Section 8 we introduce *Axiom 2* (already present in [Fio94b]) and show how *lifting* can be characterised *as a lax colimit*. In Section 9, we endow our algebra of proofs-of-approximation (viz. *paths*) with the 2-dimensional algebraic structure of a sesqui-category. In Section 10 we introduce *Axiom 3* (also present in [Fio94b]) which together with Axiom 2 make our (now **Preo**-enriched) category of discourse have *cotensors* with the Sierpinski space. Extending the result of the previous section, we then characterise lifting as a lax **Preo**-colimit and as a corollary the *lifting monad becomes a KZ-doctrine*.

2 Lifting monad

We introduce the basic assumptions on our category of discourse $\mathcal{C}$ and define the lifting monad.

Definition 2.1 ([Mog86, Ros86]) A *domain structure* is a pair $(\mathcal{C}, \mathcal{D})$ consisting of a category (of *total* maps) $\mathcal{C}$ and a well-powered category (of *admissible* monos) $\mathcal{D}$, such that $\mathcal{D}$ is a full-on-objects subcategory of $\mathcal{C}$ all of whose morphisms are monos in $\mathcal{C}$, with the following closure property: every diagram $B \xrightarrow{f} A \overset{n}{\hookleftarrow} D$ with $f \in \mathcal{C}$ and $n \in \mathcal{D}$ has a pullback in $\mathcal{C}$ and if $B \overset{f^{-1}(n)}{\hookleftarrow} f^{-1}(D) \xrightarrow{n^* f} D$ is any such pullback then $f^{-1}(n) \in \mathcal{D}$. □

Convention. Admissible monos are denoted with $\hookrightarrow$.

Definition 2.2 The *category of partial maps* $p(\mathcal{C}, \mathcal{D})$ induced by a domain structure $(\mathcal{C}, \mathcal{D})$ has the same objects as $\mathcal{C}$; a partial map $[m, f] : A \rightharpoonup B$ is an equivalence class of spans $A \overset{m}{\hookleftarrow} D \xrightarrow{f} B$ in $\mathcal{D} \times \mathcal{K}$, where two spans $A \overset{m}{\hookleftarrow} D \xrightarrow{f} B$ and $A \overset{n}{\hookleftarrow} E \xrightarrow{g} B$ are equivalent if and only if $m = n \circ i$ and $f = g \circ i$ for some isomorphism $i : D \cong E$. Composition of partial maps is given as for relations, by pullback; identities have the form $[\mathrm{id}_A, \mathrm{id}_A]$. □

The category of total maps appears as a full-on-objects subcategory of the category of partial maps via the faithful *inclusion functor* $J : \mathcal{C} \to p(\mathcal{C}, \mathcal{D})$ sending a total map f to the partial map $[\mathrm{id}, f]$.

Assumption 1 We assume a domain structure $(\mathcal{C}, \mathcal{D})$ such that the inclusion functor $J : \mathcal{C} \to p(\mathcal{C}, \mathcal{D})$ has right adjoint L. □

Convention. We write $\chi : p(\mathcal{C}, \mathcal{D})(A, B) \cong \mathcal{C}(A, LB)$ for the *lifting transpose* operator provided by the adjunction $J \dashv L : p(\mathcal{C}, \mathcal{D}) \to \mathcal{C}$.

We recall from [Fio94a] that the unit η and the counit ε of $J \dashv L : p(\mathcal{C}, \mathcal{D}) \to \mathcal{C}$ are such that, for every $A \in |\, \mathcal{C}\, |$, $\eta_A : A \to LA$ is an admissible mono and $\varepsilon_A = [\eta_A, A] : LA \rightharpoonup A$. From Assumption 1 we obtain the *lifting monad* $\mathbb{L} = (LJ, \eta, \mu) : \mathcal{C} \to \mathcal{C}$ where $L(A \xrightarrow{f} B) = \chi[\eta_A, f]$, η is cartesian (i.e. the squares required to commute by naturality are pullbacks), and $\mu_A = \chi[\eta_A \circ \eta_{LA}, A]$. Moreover, $\mathbb{L}$ is a *commutative* monad [Koc70] with tensorial strength $\mathrm{t}' : L(A) \times B \to L(A \times B)$ given by $\chi[\eta_A \times B, A \times B]$.

Assumption 2 We assume $\mathcal{C}$ has binary products and exponentials, and $p(\mathcal{C},\mathcal{D})$ has a zero object. $\square$

Convention. We write

- $\langle -, = \rangle : \mathcal{C}(C,A) \times \mathcal{C}(C,B) \cong \mathcal{C}(C, A \times B)$, and $\pi_1 : A \times B \to A$ and $\pi_2 : A \times B \to B$ for the *pairing* operator and the *projection* maps respectively;
- $\lambda : \mathcal{C}(C \times A, B) \cong \mathcal{C}(C, B^A)$ and $\mathsf{e} : A^B \times B \to A$ for the *currying* operator and the *evaluation* map respectively; and
- 0 for the zero object in $p(\mathcal{C},\mathcal{D})$.

It follows that $\mathcal{C}$ has an initial object 0 and a terminal object $1 = L0$.

Remark. If $m : D \rightarrowtail A$ and $n : E \rightarrowtail B$ are admissible then so is $m \times n : D \times E \rightarrowtail A \times B$ because $m \times n = (\chi[m, D \to 1] \times \chi[n, E \to 1])^{-1}(\eta_1 \times \eta_1)$.

Convention. For $f : A \to B$, we let $\ulcorner f \urcorner$ be $\lambda(1 \times A \cong A \xrightarrow{f} B) : 1 \to B^A$

Our leading examples are:

Example 2.3 1. $(\mathbf{Poset}, \boldsymbol{\Upsilon})$ where $\mathbf{Poset}$ is the category of small posets and monotone functions, and $\boldsymbol{\Upsilon}$ is its subcategory of all isomorphisms onto upper-closed subsets.
2. $(\mathbf{Cpo}, \boldsymbol{\Sigma})$ where $\mathbf{Cpo}$ is the category of small cpos (posets, possibly without a least element, closed under lubs of ω-chains) and continuous functions, and $\boldsymbol{\Sigma}$ is its subcategory of all isomorphisms onto Scott-open subsets. $\square$

3 Pointed objects and strict maps

We write $\mathcal{C}^{\mathbb{L}}$ for the category of Eilenberg-Moore algebras for the lifting monad. In the leading example $\mathbf{Poset}$ ($\mathbf{Cpo}$) we have that $\mathbf{Poset}^{\mathbb{L}}$ ($\mathbf{Cpo}^{\mathbb{L}}$) is the category of small posets (cpos) with a least element and strict monotone (continuous) functions. Thus the following:

Convention. An algebra for the lifting monad is called a *pointed object*. The algebra structure on a pointed object X is denoted by ρ_X but generally left implicit; in the particular case in which X is of the form LA, we take $\rho_X = \mu_A$. Algebra homomorphisms are called *strict* and are denoted with $\multimap$. When we write $X \multimap Y$, X and Y are assumed to be pointed.

We recall that the forgetful functor $\mathcal{C}^{\mathbb{L}} \to \mathcal{C}$ has a right adjoint determined by the following universal property: every $f : A \to X$ with X pointed has a unique *strict extension* $f^\# : LA \multimap X$ (given by the composite $\rho_X \circ Lf$) such that $(A \overset{\eta_A}{\hookrightarrow} LA \xrightarrow{f^\#} X) = f$.

Remark. 1 is a zero object in $\mathcal{C}^{\mathbb{L}}$ because $\mathcal{C}^{\mathbb{L}}(1,X) = \mathcal{C}^{\mathbb{L}}(L0,X) \cong \mathcal{C}(0,X) \cong 1$.

Pointed objects have least elements (c.f. Proposition 4.4 (4)):

Definition 3.1 (Bottom) For a pointed object X we define $\perp_X$ to be the unique strict map $1 \multimap X$. $\square$

It follows that $\perp_{LA} = \chi(1 \rightharpoonup 0 \to A)$. Moreover, for every $f : X \multimap Y$, $f \circ \perp_X = \perp_Y$ and hence $\perp_X = \rho_X \circ \perp_{LX}$.

Proposition 3.2 For every object A and every pointed object X,

1. X^A (with $\rho_{X^A} = \lambda(\rho_X \circ L(\mathsf{e}) \circ \mathsf{t}'))$ is a pointed object.
2. $\ulcorner A \to 1 \multimap X \urcorner : 1 \multimap X^A$. $\square$

4 Path relation

Our aim now is to define an intensional notion of approximation between maps in $\mathcal{C}$.

Convention. We write Σ_m for $L^m 1$, $\bot_m$ for $1 \circ\!\!\rightarrow \Sigma_m$ (omitting the subscript when $m = 1$), and $\top$ for $\eta_1 : 1 \hookrightarrow \Sigma$.

If we consider LA as the object obtained from A by freely adjoining $\bot_{LA}$ approximating any other element then, in the case $A = 1$, we have that $\bot : 1 \circ\!\!\rightarrow \Sigma$ approximates $\top : 1 \hookrightarrow \Sigma$. In addition, if we visualise this notion as the existence of a *canonical (directed) path* $\bot \rightsquigarrow \top$ from $\bot$ to $\top$ (thinking of such a path as a *proof* of the fact that $\bot$ approximates $\top$) then we can consider notions of approximations for arbitrary objects by stating that, for two elements x and y of C, x approximates y whenever the canonical path $\bot \rightsquigarrow \top$ can be transformed, via a map $\Sigma \to C$, into a path $x \rightsquigarrow y$ from x to y in C. Also, by regarding Σ_m as the object containing a canonical path of length m, we can generalise the notion to paths of arbitrary finite length. A formalisation of these intuitions leads to the following definitions:

Definition 4.1 (Top) For $m \geq 0$ we inductively define $\top_m : 1 \hookrightarrow \Sigma_m$ as follows: $\top_0 = 1$ and $\top_{m+1} = \eta_{\Sigma_m} \circ \top_m$. □

Definition 4.2 (Path relation) For $f, g : A \to B$ and $\alpha : \Sigma_m \times A \to B$ ($m \geq 0$), we set $\alpha : f \rightsquigarrow_m g : A \to B$ if and only if

$$\begin{tikzcd}
1 \times A \cong A \arrow[d, hook, "\top_m \times A"'] \arrow[dr, "g"] & \\
\Sigma_m \times A \arrow[r, "\alpha"] & B \\
1 \times A \cong A \arrow[u, "\bot_m \times A"] \arrow[ur, "f"'] &
\end{tikzcd}$$

Moreover, we define the *path relation* $f \sqsubseteq g : A \to B$ if and only if there exists $\alpha : \Sigma_m \times A \to B$ such that $\alpha : f \rightsquigarrow_m g$. □

Remark. Ideas similar to the above, though in the context of *undirected paths*, constitute the basics of *homotopy theory* (see e.g. [Bro88]).

Convention. When the subscript is omitted in Σ_m, $\bot_m$, $\top_m$ and $\rightsquigarrow_m$, it is assumed to be 1.

For testing approximation, paths of length one suffice:

Proposition 4.3 $f \sqsubseteq g$ if and only if $f \rightsquigarrow g$.

PROOF: Because $f \rightsquigarrow_0 g$ if and only if $f = g$, and if $\alpha : f \rightsquigarrow_{m+1} g : A \to B$ then $\alpha \circ (L(\top_m) \times A) : f \rightsquigarrow g$. □

Paths enjoy the following algebraic properties (c.f. Section 9):

Proposition 4.4
1. $\mathrm{id}_f : f \rightsquigarrow_0 f : A \to B$ where $\mathrm{id}_f =^{def} (1 \times A \cong A \xrightarrow{f} B)$.
2. For $f : A' \to A$, $\alpha : h \rightsquigarrow_m h' : A \to B$, and $g : B \to B'$, $\mathrm{id}_g \star \alpha \star \mathrm{id}_f : g \circ h \circ f \rightsquigarrow_m g \circ h' \circ f : A' \to B'$ where $\mathrm{id}_g \star \alpha \star \mathrm{id}_f =^{def} g \circ \alpha \circ (\Sigma_m \times f)$.
3. If $\alpha : f \rightsquigarrow_m f' : A \to B$ and $\beta : g \rightsquigarrow_m g' : B \to C$ then $\beta \star \alpha : g \circ f \rightsquigarrow_m g' \circ f' : C \to B$ where $\beta \star \alpha =^{def} \beta \circ \langle \pi_1, \alpha \rangle$.
4. For every pointed object X and every $x : 1 \to X$, $\rho_X \circ (Lx) : \bot_X \rightsquigarrow x$.

5. If $\alpha : f \rightsquigarrow_m g : A \to B$ then $(L\alpha) \circ \mathsf{t}'' : Lf \rightsquigarrow_m Lg : LA \to LB$ where $\mathsf{t}'' =^{def} (L\langle \pi_2, \pi_1 \rangle) \circ \mathsf{t}' \circ \langle \pi_2, \pi_1 \rangle$.
6. $\alpha : f \rightsquigarrow_m f' : C \to A$ and $\beta : g \rightsquigarrow_m g' : C \to B$ if and only if $\langle \alpha, \beta \rangle : \langle f, g \rangle \rightsquigarrow_m \langle f', g' \rangle : C \to A \times B$.
7. $\alpha : f \rightsquigarrow_m g : A \times B \to C$ if and only if $\lambda(\alpha \circ a) : \lambda(f) \rightsquigarrow_m \lambda(g) : A \to C^B$ where a is the canonical isomorphism $(\Sigma_m \times A) \times B \cong \Sigma_m \times (A \times B)$.
8. Let $A \xrightarrow{\amalg_1} A + B \xleftarrow{\amalg_2} B$ be a coproduct diagram with associated natural isomorphism $[\,\_\,, =] : \mathcal{C}(A, -) \times \mathcal{C}(B, -) \cong \mathcal{C}(A + B, -)$.
   Then, $\alpha : f \rightsquigarrow_m f' : A \to C$ and $\beta : g \rightsquigarrow_m g' : B \to C$ if and only if $\gamma : [f, g] \rightsquigarrow_m [f', g'] : A + B \to C$ where γ is the unique mediating morphism between $\Sigma_m \times A \xrightarrow{\Sigma_m \times \amalg_1} \Sigma_m \times (A + B) \xleftarrow{\Sigma_m \times \amalg_2} \Sigma_m \times B$ and $\Sigma_m \times A \xrightarrow{\alpha} C \xleftarrow{\beta} \Sigma_m \times B$. □

Corollary 4.5 1. $(A \to 1 \circ\!\!\!\rightarrow X) \rightsquigarrow f : A \to X$.
2. $\alpha : f \rightsquigarrow_m g : A \to B$ if and only if $\lambda(\alpha) : \ulcorner f \urcorner \rightsquigarrow_m \ulcorner g \urcorner : 1 \to B^A$. □

Example 4.6 Using Corollary 4.5 (2) one can readily see that:

- in **Poset** and **Cpo** the path relation is the *pointwise order*; whilst,
- in $\mathbf{Poset}_\wedge$ ($\mathbf{Cpo}_\wedge$), the category of posets (cpos) with (continuous) pullbacks and pullback-preserving monotone (continuous) functions, the path relation is the *stable order*. □

5 Upper-closed monos

To express Axiom 1 a notion of upper-closure needs to be incorporated. In this section, we axiomatise upper-closed monos as those orthogonal (in a *parameterised* sense) to strict maps, and establish their closure properties.

Definition 5.1 A mono $U \rightarrowtail A$ is *upper-closed* if every commuting square

$$\begin{tikzcd} X \times C \arrow[r, "h \times C"] \arrow[d] & Y \times C \arrow[d] \\ U \arrow[r, tail] & A \end{tikzcd}$$

with $h : X \circ\!\!\!\rightarrow Y$, has a (necessarily unique) fill-in. □

Convention. Upper-closed monos are denoted with $\rhd\!\!\!\rightarrow$.

Remark. It is well-known that orthogonality in the above parameterised sense coincides with orthogonality in an *internal* sense as a mono $m : U \rightarrowtail A$ is upper-closed if and only if for every $h : X \circ\!\!\!\rightarrow Y$, the commuting square

$$\begin{tikzcd} U^Y \arrow[r, "U^h"] \arrow[d, tail, "m^Y"'] & U^X \arrow[d, tail, "m^X"] \\ A^Y \arrow[r, "A^h"'] & A^X \end{tikzcd}$$

is a pullback.

Justifying our terminology, in the leading examples the usual definition of upper-closure and the above categorical one coincide:

Proposition 5.2 In **Poset** (**Cpo**), $m : U \rightarrowtail A$ is upper-closed if and only if for every $u \in U$ and $a \in A$ with $m(u) \leq a$, there exists (a necessarily unique) $u' \in U$ such that $u \leq u'$ and $m(u') = a$.

Convention. We write $\mathbf{1}$ for the discrete poset $\{0\}$, and $\mathbf{2}$ for the Sierpinski space $\{0 \leq 1\}$.

Proof: ($\Rightarrow$) Let $m : U \rightarrowtail A$ be upper-closed and let $u \in U$ and $a \in A$ be such that $m(u) \leq a$. Since the square

$$\begin{tikzcd} \mathbf{1} \arrow[r, "{[0 \mapsto 0]}"] \arrow[d, "{[0 \mapsto u]}"'] & \mathbf{2} \arrow[d, "{[0 \mapsto m(u), 1 \mapsto a]}"] \\ U \arrow[r, tail, "m"'] & A \end{tikzcd}$$

commutes, there exists a unique fill-in $p : \mathbf{2} \to U$ such that $u = p(0) \leq p(1)$ and $m(p1) = a$. Thus, taking $u' = p(1)$ we are done.

($\Leftarrow$) Let $(X \times C \xrightarrow{f} U \overset{m}{\rightarrowtail} A) = (X \times C \xrightarrow{h \times C} Y \times C \xrightarrow{g} A)$ with $h : X \multimap Y$.

For every $(y, c) \in Y \times C$, since $m(f(\perp, c)) = g(\perp, c) \leq g(y, c)$ it follows that there exists a unique $u_{(y,c)} \in U$ such that $f(\perp, c) \leq u_{(y,c)}$ and $m(u_{(y,c)}) = g(y, c)$. Thus, the mapping sending $(y, c) \in Y \times C$ to $u_{(y,c)} \in U$ is a function from $Y \times C$ to U. Moreover it is monotone (continuous) because so is g and for every $u, u' \in U$, $u \leq u' \Leftrightarrow m(u) \leq m(u')$. Thus, $Y \times C \to U : (y, c) \mapsto u_{(y,c)}$ is the required fill-in. □

Various closure properties of upper-closed monos are summarised below. In particular, because of our parameterised definition, closure under exponentiation is guaranteed.

Proposition 5.3
1. Upper-closed monos are closed under isomorphisms, composition and pullbacks (whenever they exist).
2. For every A, $0 \rightarrowtail A$ is upper-closed.
3. If $m : U \rightarrowtail A$ and $n : V \rightarrowtail B$ are upper-closed then so is $m \times n : U \times V \rightarrowtail A \times B$.
4. If $m : U \rightarrowtail A$ is upper-closed then so is $m^B : U^B \rightarrowtail A^B$.
5. If $U \rightarrowtail A \rightarrowtail B$ is upper-closed then so is $U \rightarrowtail A$. □

Convention. For monos $D \rightarrowtail A$ and $E \rightarrowtail A$ we write $D \subseteq_A E$ when there exists $D \to E$ such that $(D \rightarrowtail A) = (D \to E \rightarrowtail A)$, and we write $D \simeq_A E$ when $D \subseteq_A E \subseteq_A D$. (Subscripts are omitted whenever they are understood from the context.)

Remark. It follows from Proposition 5.3 (1) that if $\mathcal{C}$ has pullbacks of upper-closed monos under arbitrary maps then $(\mathcal{C}, \mathcal{U})$, with $\mathcal{U}$ the category of all upper-closed monos in $\mathcal{C}$, is a domain structure.

In the leading examples, admissible monos are upper-closed. This situation is characterised below:

Proposition 5.4 The following are equivalent:

1. Admissible monos are upper-closed.
2. $\top$ is upper-closed.
3. For every admissible mono $m : D \hookrightarrow Y \times C$ with Y pointed, if $\perp_Y \times C \subseteq m$ then $m \simeq Y \times C$.

Proof:

(1) $\Leftrightarrow$ (2) By Proposition 5.3 (1).

(2) $\Rightarrow$ (3) Let $\perp_Y \times C \subseteq m$. Then, the commutativity of

$$\begin{tikzcd}
1 \times C \arrow[rr, "\perp_Y \times C"] \arrow[d] & & Y \times C \arrow[dd, "{\chi[m, D \to 1]}"] \\
D \arrow[urr, hook, "m"'] \arrow[d] & & \\
1 \arrow[rr, "\top"'] & & \Sigma
\end{tikzcd}$$

implies $(Y \times C \to 1 \overset{\top}{\hookrightarrow} \Sigma) = \chi[m, D \to 1]$ and hence $m \simeq Y \times C$.

(3) $\Rightarrow$ (2) Assuming

$$\begin{tikzcd}
X \times C \arrow[r, "h \times C"] \arrow[d] & Y \times C \arrow[d, "g"] \\
1 \arrow[r, hook, "\top"'] & \Sigma
\end{tikzcd}$$

with $h : X \mathrel{\circ\!\!\rightarrow} Y$, we need show that $g = (Y \times C \to 1 \overset{\top}{\hookrightarrow} \Sigma)$, or equivalently that $g^{-1}(\top) \simeq Y \times C$.

Observing that

$$(h \times C)^{-1}(g^{-1}(\top)) \simeq \big(g \circ (h \times C)\big)^{-1}(\top) \simeq \big(\top \circ (X \times C \to 1)\big)^{-1}(\top) \simeq (X \times C \to 1)^{-1}(1) \simeq X \times C$$

we have that $h \times C$ factors through $g^{-1}(\top)$. So that $\perp_Y \times C = (h \circ \perp_X) \times C = (h \times C) \circ (\perp_X \times C)$ factors through $g^{-1}(\top)$ and hence we are done. $\square$

6 The strict/upper-closed factorisation

We require the following:

Axiom 1 (Strict/upper-closed factorisation) Every morphism with source a pointed object factors as a strict map followed by an upper-closed mono. $\square$

In the leading examples, every map $f : X \to A$ with X pointed has a factorisation as above, namely as $X \mathrel{\circ\!\!\rightarrow} \uparrow \big(f(\perp_X)\big) \mathrel{\rhd\!\!\rightarrow} A$ where $\uparrow \big(f(\perp_X)\big)$ is the poset with underlying set $\{a \in A \mid f(\perp_X) \leq a\}$ ordered as in A. Thus the following:

Convention. For $f : X \to A$ with X a pointed object, we write $X \mathrel{\circ\!\!\rightarrow} \uparrow(f) \mathrel{\rhd\!\!\rightarrow} A$ for a factorisation of f according to the strict/upper-closed factorisation axiom.

Proposition 6.1 1. If $(X \mathrel{\circ\!\!\rightarrow} Y \rightarrowtail A) = (X \to U \mathrel{\rhd\!\!\rightarrow} A)$ then $Y \subseteq U$.
2. For $f : X \mathrel{\circ\!\!\rightarrow} Y$, $\uparrow(f) \simeq Y$.
3. Let X and Y be pointed objects. For $f : X \to Y$ and $g : Y \to B$, $\uparrow(g \circ f) \subseteq \uparrow(g)$; and moreover, if f is strict then $\uparrow(g \circ f) \supseteq \uparrow(g)$. $\square$

The key to establishing the transitivity of the path order is the following:

Proposition 6.2 The pullback

$$
\begin{tikzcd}
X \arrow[r, "h"] \arrow[d, hook, "\eta_X"'] & Y \arrow[d, hook, "\eta_Y"] \\
LX \arrow[r, "Lh"'] & LY
\end{tikzcd}
\tag{1}
$$

is a weak pushout.

PROOF: Let $(X \overset{\eta_X}{\hookrightarrow} LX \overset{f}{\longrightarrow} C) = (X \overset{h}{\circ\!\!\rightarrow} Y \overset{g}{\longrightarrow} C)$.

Since $\uparrow(g) \simeq \uparrow(g \circ h) \simeq \uparrow(f \circ \eta_X) \subseteq \uparrow(f)$ it follows that

$$(X \overset{\eta_X}{\hookrightarrow} LX \overset{f'}{\circ\!\!\rightarrow} \uparrow(f)) \;=\; (X \overset{h}{\circ\!\!\rightarrow} Y \overset{g'}{\circ\!\!\rightarrow} \uparrow(g) \rhd\!\!\rightarrow \uparrow(f)).$$

So, letting $LY \circ\!\!\rightarrow \uparrow(f)$ be the unique strict extension of $Y \overset{g'}{\circ\!\!\rightarrow} \uparrow(g) \rhd\!\!\rightarrow \uparrow(f)$ we have that $LY \circ\!\!\rightarrow \uparrow(f) \rhd\!\!\rightarrow C$ mediates between $LX \overset{Lh}{\circ\!\!\rightarrow} LY \overset{\eta_Y}{\hookleftarrow} Y$ and $LX \overset{f}{\longrightarrow} C \overset{g}{\longleftarrow} Y$ because

$$(Y \overset{\eta_Y}{\hookrightarrow} LY \circ\!\!\rightarrow \uparrow(f) \rhd\!\!\rightarrow C) \;=\; (Y \overset{g'}{\circ\!\!\rightarrow} \uparrow(g) \rhd\!\!\rightarrow \uparrow(f) \rhd\!\!\rightarrow C) \;=\; (Y \overset{g'}{\circ\!\!\rightarrow} \uparrow(g) \rhd\!\!\rightarrow C) \;=\; g$$

and

$$
\begin{aligned}
(X \overset{\eta_X}{\hookrightarrow} LX \overset{Lh}{\circ\!\!\rightarrow} LY \circ\!\!\rightarrow \uparrow(f) \rhd\!\!\rightarrow C) &= (X \overset{h}{\circ\!\!\rightarrow} Y \overset{\eta_Y}{\hookrightarrow} LY \circ\!\!\rightarrow \uparrow(f) \rhd\!\!\rightarrow C) \\
&= (X \overset{h}{\circ\!\!\rightarrow} Y \overset{g'}{\circ\!\!\rightarrow} \uparrow(g) \rhd\!\!\rightarrow C) \\
&= (X \overset{\eta_X}{\hookrightarrow} LX \overset{f'}{\circ\!\!\rightarrow} \uparrow(f) \rhd\!\!\rightarrow C) \\
\Rightarrow\; (X \overset{\eta_X}{\hookrightarrow} LX \overset{Lh}{\circ\!\!\rightarrow} LY \circ\!\!\rightarrow \uparrow(f)) &= (X \overset{\eta_X}{\hookrightarrow} LX \overset{f'}{\circ\!\!\rightarrow} \uparrow(f)) \\
\Rightarrow\; (LX \overset{Lh}{\circ\!\!\rightarrow} LY \circ\!\!\rightarrow \uparrow(f)) &= (LX \overset{f'}{\circ\!\!\rightarrow} \uparrow(f)) \\
\Rightarrow\; (LX \overset{Lh}{\circ\!\!\rightarrow} LY \circ\!\!\rightarrow \uparrow(f) \rhd\!\!\rightarrow C) &= (LX \overset{f'}{\circ\!\!\rightarrow} \uparrow(f) \rhd\!\!\rightarrow C) = f.
\end{aligned}
$$

□

Definition 6.3 Let **Preo** be the category of small preorders and monotone functions. □

Theorem 6.4 The category $\mathcal{C}$ equipped with $\sqsubseteq$ (henceforth denoted $\mathcal{C}_{\sqsubseteq}$) is a **Preo**-cartesian-closed-category.

PROOF: By Proposition 4.4 (1) the path relation is reflexive. To see that it is transitive first observe that, by Proposition 6.2 and the fact that left adjoints preserve weak colimits, the pullback

$$
\begin{tikzcd}
1 \times A \arrow[r, "\bot \times A"] \arrow[d, hook, "\top \times A"'] & \Sigma \times A \arrow[d, hook, "\eta_\Sigma \times A"] \\
\Sigma \times A \arrow[r, "L\bot \times A"'] & \Sigma_2 \times A
\end{tikzcd}
\tag{2}
$$

is a weak pushout. Now, if $f \sqsubseteq g \sqsubseteq h : A \to B$ then there exist $\alpha : f \rightsquigarrow g$ and $\beta : g \rightsquigarrow h$ making $\Sigma \times A \overset{\alpha}{\longrightarrow} B \overset{\beta}{\longleftarrow} \Sigma \times A$ a cone for (2) and any mediating morphism $\gamma : \Sigma_2 \times A \to B$ makes $\gamma : f \rightsquigarrow_2 h$.

Finally, by Proposition 4.4 (3), the composition of morphisms is a monotone operation with respect to the path preorder. And, by Proposition 4.4 (6–7), the cartesian closed structure is **Preo**-enriched. □

By Proposition 4.4 (5) we have:

Proposition 6.5 $\mathbb{L}$ determines a **Preo**-monad on $\mathcal{C}_{\sqsubseteq}$ and hence the category of Eilenberg-Moore algebras $\mathcal{C}^{\mathbb{L}}$ and the Kleisli category $\mathcal{C}_{\mathbb{L}} \cong p(\mathcal{C},\mathcal{D})$ **Preo**-enrich w.r.t. $\sqsubseteq$. $\square$

Proposition 6.6 Coproducts in $\mathcal{C}_{\sqsubseteq}$ are **Preo**-enriched. $\square$

7 Approximation and upper-closure

Axiom 1 allows a characterisation of approximation in terms of upper-closure: for two elements x and y, we have that x approximates y if and only if the minimal upper-closed containing x includes the minimal upper-closed containing y. More precisely,

Proposition 7.1 The following are equivalent:

1. $x \rightsquigarrow y : 1 \to A$.
2. $\uparrow(x) \supseteq \uparrow(y)$.
3. y factors through $\uparrow(x) \rightarrowtriangle A$.

PROOF:
(1) $\Rightarrow$ (2) Let $\alpha : x \rightsquigarrow y$. Then,

$$\begin{tikzcd}
1 \arrow[r, circ] \arrow[d, circ] & \Sigma \arrow[d, "\alpha \circ \langle \Sigma, \Sigma \to 1 \rangle"] \arrow[dl, dotted, "\alpha'"'] \\
\uparrow(x) \arrow[r, rightarrowtriangle] & A
\end{tikzcd}$$

and hence

$$\begin{tikzcd}
1 \arrow[r, circ] \arrow[d, hook, "\top"'] & \uparrow(y) \arrow[dd, triangle] \arrow[ddl, dotted] \\
\Sigma \arrow[d, "\alpha'"'] & \\
\uparrow(x) \arrow[r, rightarrowtriangle] & A
\end{tikzcd} .$$

(2) $\Rightarrow$ (1) Let $y' = (1 \multimap \uparrow(y) \rightarrowtriangle \uparrow(x))$ and let $\alpha = (\Sigma \overset{Ly'}{\circ\!\!\longrightarrow} L(\uparrow x) \xrightarrow{\rho_{\uparrow(x)}} \uparrow(x) \rightarrowtriangle A)$. Then, $\alpha : x \rightsquigarrow y$.

(2) $\Leftrightarrow$ (3) Straightforward. $\square$

Corollary 7.2 $(A \to 1 \xrightarrow{b} B) \rightsquigarrow f$ if and only if f factors through $\uparrow(b) \rightarrowtriangle B$.

PROOF: For every $y : 1 \to B$ we write $\mathsf{K}y$ for the composite $(A \to 1 \xrightarrow{y} B)$.

Then,

$$
\begin{aligned}
&\mathsf{K}b \rightsquigarrow f : A \to B \\
&\quad \Leftrightarrow \ulcorner \mathsf{K}b \urcorner \rightsquigarrow \ulcorner f \urcorner : 1 \to B^A \\
&\quad \Leftrightarrow \ulcorner f \urcorner \text{ factors through } \uparrow (\ulcorner \mathsf{K}b \urcorner) \rhd\!\!\!\rightarrow B^A \\
&\quad \Leftrightarrow \ulcorner f \urcorner \text{ factors through } (\uparrow (b) \rhd\!\!\!\rightarrow B)^A : (\uparrow b)^A \rhd\!\!\!\rightarrow B^A \\
&\qquad\qquad \text{, because } \ulcorner \mathsf{K}b \urcorner = (1 \overset{\ulcorner \mathsf{K}b' \urcorner}{\circ\!\!\!\longrightarrow} (\uparrow b)^A \overset{m^A}{\rhd\!\!\!\rightarrow} B^A) \text{ whenever } b = (1 \overset{b'}{\circ\!\!\!\longrightarrow} \uparrow (b) \overset{m}{\rhd\!\!\!\rightarrow} B) \\
&\quad \Leftrightarrow f \text{ factors through } \uparrow (b) \rhd\!\!\!\rightarrow B. \qquad \square
\end{aligned}
$$

8 Lifting as a lax colimit

In this section, by introducing Axiom 2, we are able to provide a characterisation of lifting as a lax colimit.

We recall the definition of lax colimit. Let $\mathcal{A}_{\leq}$ be a **Preo**-category and let $\mathcal{A}$ be its underlying ordinary category. Let $\Delta : \mathcal{G} \to \mathcal{A}$. A lax cone for Δ with vertex A, $\amalg : \Delta \xrightarrow[\geq]{\cdot} A$, is an indexed family of morphisms $\amalg_M : \Delta M \to A$ ($M \in |\,\mathcal{G}\,|$) in $\mathcal{A}$ such that, for $e : M \to N$ in $\mathcal{G}$, $\amalg_M \geq \amalg_N \circ \Delta e$. We write $\mathrm{LaxCone}(\Delta, A)$ ($\mathrm{LaxCone}_{\leq}(\Delta, A)$) for the collection of lax cones for Δ with vertex A (preordered pointwise).

Consider $\mathcal{G}$ small. A lax cone $\amalg : \Delta \xrightarrow[\geq]{\cdot} A$ is colimiting if

$$\_ \circ \amalg : \mathcal{A}(A, \_) \cong \mathrm{LaxCone}(\Delta, \_) : \mathcal{A} \to \mathbf{Set}$$

and it is **Preo**-colimiting if

$$\_ \circ \amalg : \mathcal{A}_{\leq}(A, \_) \cong \mathrm{LaxCone}_{\leq}(\Delta, \_) : \mathcal{A} \to \mathbf{Preo}.$$

Remark. (Weak) lax colimits are preserved by left adjoints.

We have:

Proposition 8.1 For every $A \in |\,\mathcal{C}\,|$, the lax cone

$$
\begin{tikzcd}
A \arrow[rr] \arrow[dr, hook, "\eta_A"'] & \arrow[d, phantom, "\sqsupseteq"] & 1 \arrow[dl, "\perp_{LA}"] \\
& LA &
\end{tikzcd}
\tag{3}
$$

is weak colimiting.

PROOF: Let $(A \xrightarrow{f} B) \sqsupseteq (A \to 1 \xrightarrow{b} B)$. Then, by Corollary 7.2, $f = (A \to \uparrow (b) \rhd\!\!\!\rightarrow B)$. So, letting $LA \circ\!\!\!\rightarrow \uparrow (b)$ be the unique strict extension of $A \to \uparrow (b)$, we have that $LA \circ\!\!\!\rightarrow \uparrow (b) \rhd\!\!\!\rightarrow B$ mediates between $A \overset{\eta_A}{\hookrightarrow} LA \overset{\perp_{LA}}{\longleftarrow\!\!\!\circ} 1$ and $A \xrightarrow{f} B \xleftarrow{b} 1$ because

$$(A \overset{\eta_A}{\hookrightarrow} LA \circ\!\!\!\rightarrow \uparrow (b) \rhd\!\!\!\rightarrow B) \;=\; (A \to \uparrow (b) \rhd\!\!\!\rightarrow B) \;=\; f$$

and

$$(1 \circ\!\!\!\rightarrow LA \circ\!\!\!\rightarrow \uparrow (b) \rhd\!\!\!\rightarrow B) \;=\; (1 \circ\!\!\!\rightarrow \uparrow (b) \rhd\!\!\!\rightarrow B) \;=\; b. \qquad \square$$

Axiom 2 For every $A \in |\,\mathcal{C}\,|$, $A \overset{\eta_A}{\hookrightarrow} LA \overset{\perp_{LA}}{\longleftarrow\!\!\!\circ} 1$ are jointly epi. $\square$

Corollary 8.2 1. The lax cone (3) is colimiting.

2. *Convention.* For $n \geq 0$, we inductively define $\eta_C^{(n)} : C \hookrightarrow L^n C$ as $\eta_C^{(0)} = C$ and $\eta_C^{(n+1)} = \eta_{L^n C} \circ \eta_C^{(n)} = \eta_{LC}^{(n)} \circ \eta_C$.

   For $h : X \circ\!\!\rightarrow Y$, the pullbacks

$$\begin{tikzcd}
X \times A \arrow[r, "h \times A"] \arrow[d, hook, "\eta_X^{(n)} \times A"'] & Y \times A \arrow[d, hook, "\eta_Y^{(n)} \times A"] \\
L^n X \times A \arrow[r, "L^n h \times A"'] & L^n Y \times A
\end{tikzcd}$$

   are pushouts. □

The following characterisation of Axiom 2 for the case $A = 1$ (especially Proposition 8.3 (2) stating that there is at most one path of length one between any two maps) should help in understanding the axiom.

Proposition 8.3 The following are equivalent:

1. $1 \overset{\top}{\hookrightarrow} \Sigma \overset{\perp}{\longleftarrow\!\!\circ} 1$ are jointly epi.
2. If $\alpha, \beta : f \rightsquigarrow g : A \to B$ then $\alpha = \beta$.
3. The map $\mathcal{C}(A, B^{\Sigma}) \to \mathbf{Preo}(\mathbf{2}, \mathcal{C}_{\sqsubseteq}(A, B)) : \alpha \mapsto \langle B^{\perp} \circ \alpha \sqsubseteq B^{\top} \circ \alpha \rangle$ is a bijection. □

Proposition 8.4 Colimits in $\mathcal{C}_{\sqsubseteq}$ are **Preo**-enriched. □

9 An algebra of paths

Using versions of Corollary 8.2 (2) we define a composition operation for paths:

Definition 9.1 For $f, g : A \to B$ and $\alpha : \Sigma_m \times A \to B$ we set $\alpha : f \mathrel{\rightsquigarrow\mkern-14mu\rightsquigarrow} g$ if and only if $\alpha : f \rightsquigarrow_m g$.

And given $f \overset{\alpha}{\mathrel{\rightsquigarrow\mkern-14mu\rightsquigarrow}} g \overset{\beta}{\mathrel{\rightsquigarrow\mkern-14mu\rightsquigarrow}} h : A \to B$ we define $\beta \bullet \alpha : f \mathrel{\rightsquigarrow\mkern-14mu\rightsquigarrow} h$ as the unique mediating morphism in the diagram

$$\begin{tikzcd}
1 \times A \arrow[r, "\perp_n \times A"] \arrow[d, hook, "\top_m \times A"'] & \Sigma_n \times A \arrow[d, hook, "\eta_{\Sigma_n}^{(m)} \times A"'] \arrow[ddr, bend left, "\beta"] & \\
\Sigma_m \times A \arrow[r, "L^m \perp_n \times A"] \arrow[drr, bend right, "\alpha"'] & \Sigma_{m+n} \times A \arrow[dr, dotted, "\beta \bullet \alpha"'] & \\
 & & B
\end{tikzcd}$$

. □

It follows that $\mathcal{C}$ is the underlying category of a sesqui-category with paths as 2-cells:

Proposition 9.2 1. For every $A, B \in |\,\mathcal{C}\,|$, $\mathcal{C}(A, B)$ with $\mathrel{\rightsquigarrow\mkern-14mu\rightsquigarrow}$, $\bullet$ and id_f (defined in Proposition 4.4 (1)) is a category.

2. $\mathcal{C}$ with $\mathrel{\rightsquigarrow\mkern-14mu\rightsquigarrow}$, $\bullet$, id_f and $\mathrm{id}_g \star \alpha \star \mathrm{id}_f$ (defined in Proposition 4.4 (2)) is a sesqui-category (i.e. a category enriched over **CAT**, the category of locally small categories and functors, with the *funny* symmetric monoidal closed structure —see e.g. [Str92]). □

10 Lifting as a KZ-doctrine

We recall the definition of KZ-doctrine (when the 2-categorical structure is quasi-posetal):

Definition 10.1 ([Koc94]) A *KZ-doctrine* on a **Preo**-category $\mathcal{A}_\leq$ consists of a **Preo**-monad $\mathbb{T} = (T, \eta, \mu) : \mathcal{A}_\leq \to \mathcal{A}_\leq$ such that, for every $A \in |\mathcal{A}_\leq|$, $T\eta_A \leq \eta_{TA}$. □

The last axiom (Axiom 3) which together with Axiom 2 makes $\mathcal{C}_\sqsubseteq$ have cotensors with the Sierpinski space is introduced. From it we deduce that lax colimits are **Preo**-enriched, and from the **Preo**-enrichment of the lax colimit (3) the inequality needed to make $\mathbb{L}$ into a KZ-doctrine follows.

To state Axiom 3 we need the following:

Definition 10.2 Let $\overline{0} = (1 \xrightarrow{\perp} \Sigma \xrightarrow{L\perp} \Sigma_2)$, $\overline{1} = (1 \xhookrightarrow{\top} \Sigma \xrightarrow{L\perp} \Sigma_2) = (1 \xrightarrow{\perp} \Sigma \xhookrightarrow{\eta_\Sigma} \Sigma_2)$ and $\overline{2} = (1 \xhookrightarrow{\top} \Sigma \xhookrightarrow{\eta_\Sigma} \Sigma_2)$. □

Convention. For $f \sqsubseteq g : A \to B$, we set $[f \sqsubseteq g] : f \rightsquigarrow g$. And, for $f \sqsubseteq g \sqsubseteq h : A \to B$ we write $[f \sqsubseteq g \sqsubseteq h]$ for $[g \sqsubseteq h] \bullet [f \sqsubseteq g]$ (see Definition 9.1).

Axiom 3 The commuting square

$$\begin{tikzcd}
\Sigma \arrow[r, "{[\overline{0} \sqsubseteq \overline{2}]}"] \arrow[d, "{[\overline{0} \sqsubseteq \overline{2}]}"'] & \Sigma_2 \arrow[d, "{[\langle\perp,\perp\rangle \sqsubseteq \langle\top,\perp\rangle \sqsubseteq \langle\top,\top\rangle]}"] \\
\Sigma_2 \arrow[r, "{[\langle\perp,\perp\rangle \sqsubseteq \langle\perp,\top\rangle \sqsubseteq \langle\top,\top\rangle]}"'] & \Sigma \times \Sigma
\end{tikzcd} \tag{4}$$

is a pushout. □

More illuminating than the axiom is the following characterisation:

Proposition 10.3 The following are equivalent:

1. Axiom 3.
2. **(Diamond property)** If

$$\begin{tikzcd}
 & g' & \\
f' \arrow[ur, Rightarrow, no head] & & g \arrow[ul, Rightarrow, no head] \\
 & f \arrow[ul, Rightarrow, no head] \arrow[ur, Rightarrow, no head] &
\end{tikzcd} \tag{5}$$

   in $\mathcal{C}_\sqsubseteq(A, B)$ then $[f \sqsubseteq f'] \sqsubseteq [g \sqsubseteq g'] : \Sigma \times A \to B$.
3. The monotone map $\mathcal{C}_\sqsubseteq(A, B^\Sigma) \to \mathbf{Preo}_\sqsubseteq(\mathbf{2}, \mathcal{C}_\sqsubseteq(A, B)) : \alpha \mapsto \langle B^\perp \circ \alpha \sqsubseteq B^\top \circ \alpha\rangle$ is an isomorphism of preorders. (Hence, $\mathcal{C}_\sqsubseteq$ has cotensors with the Sierpinski space $\mathbf{2}$.)

PROOF:

(1) ⇒ (2) Assume (5). Set $C = B^A$, $x = \ulcorner f \urcorner$, $x' = \ulcorner f' \urcorner$, $y = \ulcorner g \urcorner$, $y' = \ulcorner g' \urcorner$, and let $\mu : \Sigma \times \Sigma \to C$ be the unique mediating morphism from the pushout (4) to the cone $\Sigma_2 \xrightarrow{[x \sqsubseteq x' \sqsubseteq y']} C \xleftarrow{[x \sqsubseteq y \sqsubseteq y']} \Sigma_2$.

Then, since $\mu \circ \langle \top \circ (\Sigma \to 1), \Sigma\rangle \circ \top = y'$ and $\mu \circ \langle \top \circ (\Sigma \to 1), \Sigma\rangle \circ \perp = y$ it follows that $\mu \circ \langle \top \circ (\Sigma \to 1), \Sigma\rangle = [y \sqsubseteq y']$. Also, $\mu \circ \langle \perp \circ (\Sigma \to 1), \Sigma\rangle \circ \top = x'$ and $\mu \circ \langle \perp \circ (\Sigma \to 1), \Sigma\rangle \circ \perp = x$ so $\mu \circ \langle \perp \circ (\Sigma \to 1), \Sigma\rangle = [x \sqsubseteq x']$. Hence, $\mu : [x \sqsubseteq x'] \rightsquigarrow [y \sqsubseteq y']$ and thus $[f \sqsubseteq f'] \sqsubseteq [g \sqsubseteq g']$.

(2) $\Rightarrow$ (1) If $\mu : \Sigma \times \Sigma \to C$ mediates between the pushout (4) and a cone $\Sigma_2 \xrightarrow{f} C \xleftarrow{g} \Sigma_2$ for it, then $\mu : [f \circ \overline{0} \sqsubseteq f \circ \overline{1}] \rightsquigarrow [g \circ \overline{1} \sqsubseteq g \circ \overline{2}]$. This shows the uniqueness of the mediating morphism. For existence consider

$$\begin{tikzcd}[column sep=small]
& f \circ \overline{2} = g \circ \overline{2} & \\
f \circ \overline{1} \arrow[ur, phantom, sloped, "\sqsubseteq"] & & g \circ \overline{1} \arrow[ul, phantom, sloped, "\sqsupseteq"] \\
& f \circ \overline{0} = g \circ \overline{0} \arrow[ul, phantom, sloped, "\sqsupseteq"] \arrow[ur, phantom, sloped, "\sqsubseteq"] &
\end{tikzcd}.$$

Then, by the diamond property, $[f \circ \overline{0} \sqsubseteq f \circ \overline{1}] \sqsubseteq [g \circ \overline{1} \sqsubseteq g \circ \overline{2}]$. Finally, take μ to be the unique such that $\mu : [f \circ \overline{0} \sqsubseteq f \circ \overline{1}] \rightsquigarrow [g \circ \overline{1} \sqsubseteq g \circ \overline{2}]$.

(2) $\Rightarrow$ (3) It is enough to show that the mapping

$$\mathbf{Preo}_{\sqsubseteq}(\mathbf{2}, \mathcal{C}_{\sqsubseteq}(A, B)) \to \mathcal{C}_{\sqsubseteq}(\Sigma, B^A) : \ \langle f \sqsubseteq f' \rangle \mapsto [\ulcorner f \urcorner \sqsubseteq \ulcorner f' \urcorner]$$

is monotone.

Let $f \sqsubseteq f'$ and $g \sqsubseteq g'$ be such that $f \sqsubseteq g$ and $f' \sqsubseteq g'$. Then,

$$\begin{tikzcd}[column sep=small]
& \ulcorner g' \urcorner & \\
\ulcorner f' \urcorner \arrow[ur, phantom, sloped, "\sqsubseteq"] & & \ulcorner g \urcorner \arrow[ul, phantom, sloped, "\sqsupseteq"] \\
& \ulcorner f \urcorner \arrow[ul, phantom, sloped, "\sqsupseteq"] \arrow[ur, phantom, sloped, "\sqsubseteq"] &
\end{tikzcd}$$

and, by the diamond property, $[\ulcorner f \urcorner \sqsubseteq \ulcorner f' \urcorner] \sqsubseteq [\ulcorner g \urcorner \sqsubseteq \ulcorner g' \urcorner]$.

(3) $\Rightarrow$ (2) Because (2) is equivalent to the monotonicity of the mapping

$$\mathbf{Preo}_{\sqsubseteq}(\mathbf{2}, \mathcal{C}_{\sqsubseteq}(A, B)) \to \mathcal{C}_{\sqsubseteq}(\Sigma \times A, B) : \langle f \sqsubseteq f' \rangle \mapsto [f \sqsubseteq f'].$$ $\square$

Remark. Axiom 3 does not hold in categories of domains and stable functions. In fact the diamond property exhibits a crucial difference between the pointwise and the stable order; for example, the following proposition does not hold in $\mathbf{Poset}_\wedge$ and $\mathbf{Cpo}_\wedge$ (c.f. [Fio94b]).

Proposition 10.4 Lax colimits in $\mathcal{C}_{\sqsubseteq}$ are **Preo**-enriched.

PROOF: Let $\Delta : \mathcal{G} \to \mathcal{C}$ and let $\amalg : \Delta \xrightarrow[\geq]{\cdot} C$ be colimiting in $\mathcal{C}$. For $f, g : C \to D$, we show that if $f \circ \amalg \sqsubseteq g \circ \amalg : \Delta \xrightarrow[\geq]{\cdot} D$ then $f \sqsubseteq g : C \to D$.

Assume $f \circ \amalg \sqsubseteq g \circ \amalg : \Delta \xrightarrow[\geq]{\cdot} D$. For every $e : M \to N$ in $\mathcal{G}$, we have that

$$\begin{tikzcd}[column sep=small]
& g \circ \amalg_M & \\
f \circ \amalg_M \arrow[ur, phantom, sloped, "\sqsubseteq"] & & g \circ \amalg_N \circ \Delta e \arrow[ul, phantom, sloped, "\sqsupseteq"] \\
& f \circ \amalg_N \circ \Delta e \arrow[ul, phantom, sloped, "\sqsupseteq"] \arrow[ur, phantom, sloped, "\sqsubseteq"] &
\end{tikzcd}$$

so that

$$\begin{aligned}[f \circ \mathrm{II}_M \sqsubseteq g \circ \mathrm{II}_M] &\sqsupseteq [f \circ \mathrm{II}_N \circ \Delta e \sqsubseteq g \circ \mathrm{II}_N \circ \Delta e] \text{ , by the diamond property} \\ &= [f \circ \mathrm{II}_N \sqsubseteq g \circ \mathrm{II}_N] \circ \Delta e\end{aligned}$$

and hence $\Gamma =^{def} \langle [f \circ \mathrm{II}_M \sqsubseteq g \circ \mathrm{II}_M] \rangle_M : \Delta \times \Sigma \overset{\cdot}{\underset{\geq}{\rightarrow}} D$.

As $\Sigma \times \_$ preserves lax colimits, it follows that $\Sigma \times \mathrm{II} : \Sigma \times \Delta \overset{\cdot}{\underset{\geq}{\rightarrow}} \Sigma \times C$ is lax colimiting.

Finally, if $\alpha : \Sigma \times C \to D$ is the unique mediating morphism between $\Sigma \times \mathrm{II} : \Sigma \times \Delta \overset{\cdot}{\underset{\geq}{\rightarrow}} \Sigma \times C$ and $\Gamma : \Sigma \times \Delta \overset{\cdot}{\underset{\geq}{\rightarrow}} D$ we have that $\alpha : f \rightsquigarrow g$ because:

$$\begin{tikzcd}
 & & D \\
\Delta \cong 1 \times \Delta \arrow[urr, bend left, "f \circ \mathrm{II}"] \arrow[d, shift right=4, "\mathrm{II}"'] \arrow[d, "1 \times \mathrm{II}"] \arrow[r, "\perp \times \Delta"] & \Sigma \times \Delta \arrow[ur, "\Gamma"] \arrow[d, "\Sigma \times \mathrm{II}"'] & \\
C \cong 1 \times C \arrow[r, "\perp \times C"'] & \Sigma \times C \arrow[uur, bend right, "\alpha"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & & D \\
\Delta \cong 1 \times \Delta \arrow[urr, bend left, "g \circ \mathrm{II}"] \arrow[d, shift right=4, "\mathrm{II}"'] \arrow[d, "1 \times \mathrm{II}"] \arrow[r, "\top \times \Delta"] & \Sigma \times \Delta \arrow[ur, "\Gamma"] \arrow[d, "\Sigma \times \mathrm{II}"'] & \\
C \cong 1 \times C \arrow[r, "\top \times C"'] & \Sigma \times C \arrow[uur, bend right, "\alpha"'] &
\end{tikzcd}$$

and hence $\alpha \circ (\perp \times C) = f \circ \pi_1$ whilst $\alpha \circ (\top \times C) = g \circ \pi_1$. $\square$

Corollary 10.5 The lifting monad $\mathbb{L}$ is a KZ-doctrine on $\mathcal{C}_{\sqsubseteq}$.

Proof: Because $\mathbb{L}$ is a **Preo**-monad and, for every $A \in |\mathcal{C}|$, since $(L\eta_A) \circ \eta_A = \eta_{LA} \circ \eta_A$ and $(L\eta_A) \circ \perp_{LA} = \perp_{L^2A} \sqsubseteq \eta_{LA} \circ \perp_{LA}$ it follows, using that the lax cone (3) is **Preo**-colimiting, that $L\eta_A \sqsubseteq \eta_{LA}$. $\square$

From [Koc94] we know that: for every pointed object X, $\rho_X \dashv_r \eta_X$ (i.e. algebra structures are reflections to units) because $\rho_X \circ \eta_X = X$ and $\eta_X \circ \rho_X = L(\rho_X) \circ \eta_X \sqsupseteq L(\rho_X) \circ L\eta_X = L(\rho_X \circ \eta_X) = LX$; moreover, for every object A, $L\eta_A \dashv_{cor} \mu_A \dashv_r \eta_{LA}$.

It follows that the two equivalent statements in the following proposition hold:

Proposition 10.6 Pointed objects have a unique algebra structure (up to path equivalence) if and only if isomorphisms are strict (up to path equivalence).

Proof:

($\Rightarrow$) Let X and Y be pointed objects, and let $f : X \cong Y : g$ in $\mathcal{C}$. We show that $g \circ \rho_Y \circ Lf$ provides an algebra structure on X and hence $\rho_X \equiv g \circ \rho_Y \circ Lf$; so that, by Proposition 4.4 (3), $f \circ \rho_X \equiv \rho_Y \circ Lf$.

Writing x for $g \circ \rho_Y \circ Lf$, we have that

$$x \circ \eta_X \;=\; g \circ \rho_Y \circ (Lf) \circ \eta_X \;=\; g \circ \rho_Y \circ \eta_Y \circ f \;=\; g \circ f \;=\; X$$

and that

$$\begin{aligned} x \circ Lx &= g \circ \rho_Y \circ (Lf) \circ L(g \circ \rho_Y \circ Lf) = g \circ \rho_Y \circ (L\rho_Y) \circ (L^2 f) \\ &= g \circ \rho_Y \circ \mu_Y \circ (L^2 f) \qquad\qquad = g \circ \rho_Y \circ (Lf) \circ \mu_X \\ &= x \circ \mu_X . \end{aligned}$$

($\Leftarrow$) Because identities are strict. $\square$

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