A Note on a Model of Synthetic Domain Theory

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Abstract

A model of synthetic domain theory embedding the category ω-**Cpo** of ω-complete posets and ω-continuous functions as a reflective exponential ideal is introduced, and an internal characterisation of ω-**Cpo** within the model is presented. An expanded version of this note, not including the last section, appears in [4].

Introduction

This note is concerned with models of Synthetic Domain Theory (SDT) encompassing traditional categories of domains used in denotational semantics [7, 14], showing that the synthetic approach generalises the standard theory of domains and suggests new problems to it.

Consider a (locally small) category of domains **D** with a (small) dense generator **G** equipped with a Grothendieck topology. Assume further that every cover in **G** is effective epimorphic in **D**. Then, by Yoneda, **D** embeds fully and faithfully in the topos of sheaves on **G** for the canonical topology, which thus provides a set-theoretic universe for our original category of domains. In this note we explore such a situation for a traditional category of domains and, in particular, show that the Grothendieck topos so arising yields a model of SDT in which the category of domains has an internal characterisation.

We present a model of SDT embedding the category ω-**Cpo** of posets with least upper bounds of countable chains (hence called ω-complete) and ω-continuous functions (*i.e.* monotone functions which preserve these lubs) as a reflective exponential ideal. In turn, ω-**Cpo** embeds the category of domains in the model, *viz.* the full subcategory of replete objects [8, 13], again as a reflective exponential ideal. The replete ω-complete partial orders

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may offer a new interesting notion of domain to the standard theory but, at the moment, no explicit description of them is known.

1 The topos $\mathbf{H}$ of continuous $\overline{\omega}$-paths

We begin by fixing notation. We denote the finite cardinals as $0 = \emptyset, \ldots, n+1 = \{0, \ldots, n\}, \ldots$ and write N for the collection $\{0, \ldots, n, \ldots\}$ of finite cardinals. We write $\mathbf{n}$ for the ordinal $(n, \subseteq)$ and for a chain $x_0 \leq \ldots \leq x_{n-1}$ in a poset P we write $(x_0, \ldots, x_{n-1})$ for the monotone map $\mathbf{n} \to P$ sending i to x_i. We further write ω for the ordinal $(N, \subseteq)$, and $\overline{\omega}$ for its successor ordinal $0 \subseteq \ldots \subseteq n \subseteq \ldots \subseteq N$.

Consider the following full subcategories of ω-$\mathbf{Cpo}$:

- the monoid $\mathbf{L}$ of endomaps on $\overline{\omega}$,
- the category $\mathbf{F}$ of finite products of $\overline{\omega}$.

Remark 1.1 *In ω-$\mathbf{Cpo}$, $\overline{\omega}$ is a* dense generator. *Thus, the "Yoneda" functor $Y : \omega\text{-}\mathbf{Cpo} \to \mathbf{Set}^{\mathbf{L}^{\mathrm{op}}} : D \mapsto \omega\text{-}\mathbf{Cpo}(-, D)$ is full and faithful, and preserves limits and exponentials.*

Write $\mathrm{S}(\mathbf{C})$ for the idempotent splitting of a category $\mathbf{C}$, see [12, 6]. The splitting $\mathrm{S}(\mathbf{L})$ is equivalent to the full subcategory of ω-$\mathbf{Cpo}$ consisting of $\overline{\omega}$ and the non-empty finite ordinals.

Let $\mathbf{H}$ be the topos $\mathrm{sh}(\mathrm{S}(\mathbf{L}), \mathrm{can})$ of sheaves on $\mathrm{S}(\mathbf{L})$ for the canonical topology. It is useful to have various presentations of the topos.

Proposition 1.2 *The inclusions*

$$\begin{tikzcd} \mathbf{L} \arrow[r, hook] \arrow[d, hook] & \mathbf{F} \arrow[d, hook] \\ \mathrm{S}(\mathbf{L}) \arrow[r, hook] & \mathrm{S}(\mathbf{F}) \end{tikzcd}$$

induce equivalences

$$\begin{tikzcd} \mathrm{sh}(\mathbf{L}, \mathrm{can}) & \mathrm{sh}(\mathbf{F}, \mathrm{can}) \arrow[l, "\simeq"'] \\ \mathbf{H} \arrow[u, "\simeq"] & \mathrm{sh}(\mathrm{S}(\mathbf{F}), \mathrm{can}) \arrow[l, "\simeq"] \arrow[u, "\simeq"'] \end{tikzcd} .$$

Theorem 1.3 *The functor $Y : \omega\text{-}\mathbf{Cpo} \to \mathbf{H}$ presents ω-$\mathbf{Cpo}$ as a full reflective exponential ideal of $\mathbf{H}$.*

Henceforth we shall identify an ω-complete poset D with the sheaf $Y(D)$ of its continuous $\overline{\omega}$-paths.

2 H as a model of SDT

We write 1 for the terminal object $Y(\mathbf{1})$, Σ for the sheaf of continuous $\overline{\omega}$-paths $Y(\mathbf{2})$ associated to the Sierpinski space $\mathbf{2}$, and $\top: 1 \to \Sigma$ for $Y(1: \mathbf{1} \to \mathbf{2})$.

In $\mathbf{H}$, the map $\top: 1 \to \Sigma$ is a dominance, see [11]. It follows from *loc.cit.* that it induces a notion of composable partial map in $\mathbf{H}$, with a representation. We shall denote by L the action of the representation functor on $\mathbf{H}$ which underlies the so-called *lifting* monad. This abstract construction extends a well-known behaviour on the category ω-$\mathbf{Cpo}$. Indeed, the notion of partial map is that expected from partial computations, see [2]. In ω-$\mathbf{Cpo}$ one can consider partial maps defined on a Scott-open subset. A *Scott-open* subset is a monomorphism in ω-$\mathbf{Cpo}$ which reflects the order and whose image is upward closed and inaccessible by lubs of ω-chains. The representing functor adds an element below the given order.

The lifting monad migrates to other subcategories of $\mathbf{H}$.

Proposition 2.1 *The functor $L: \mathbf{H} \to \mathbf{H}$ restricts to the endofunctor on the subcategory ω-$\mathbf{Cpo}$ which represents partial maps defined on Scott-open subsets. Moreover, the functor L restricts to the sites defining $\mathbf{H}$. In other words there are functors such that the diagram*

$$\begin{tikzcd}
\mathrm{S}(\mathbf{L}) \arrow[r, hook] \arrow[d, "L"'] & \mathrm{S}(\mathbf{F}) \arrow[r, hook] \arrow[d] & \omega\text{-}\mathbf{Cpo} \arrow[r, hook, "Y"] \arrow[d] & \mathbf{H} \arrow[d, "L"] \\
\mathrm{S}(\mathbf{L}) \arrow[r, hook] & \mathrm{S}(\mathbf{F}) \arrow[r, hook] & \omega\text{-}\mathbf{Cpo} \arrow[r, hook, "Y"'] & \mathbf{H}
\end{tikzcd}$$

commutes up to isomorphism. As the inclusions are full, all restrictions induce monads.

Another functor representing a different kind of partial map can be defined on $\mathbf{H}$: consider the notion of Scott-closed subset as just the complement of a Scott-open subset. Explicitly, a monomorphism in ω-$\mathbf{Cpo}$ is *Scott-closed* if it reflects the order and its image is closed downward and under lubs of ω-chains. The representing functor adds an element above the given order. This notion of partiality is that induced by the dominance $\perp: 1 \to \Sigma$, defined as $Y(0: \mathbf{1} \to \mathbf{2})$, in $\mathbf{H}$.

So to see $\mathbf{H}$ as a model of the axioms of synthetic domain theory as in [8] we take the *disjoint* dominances $\top: 1 \to \Sigma$ and $\perp: 1 \to \Sigma$.

The map $\Sigma^{[\perp,\top]}: \Sigma^{\Sigma} \to \Sigma^{1+1}$ is monic in $\mathbf{H}$ and the object Σ is a suplattice with countable joins $\Sigma^{N} \to \Sigma$ in $\mathbf{H}$, because both properties hold in ω-$\mathbf{Cpo}$.

Writing c for the canonical map from the initial L-algebra to the final L-coalgebra [5], we have that Σ^c is iso and so the main infinitary axiom of SDT holds.

Theorem 2.2 *The topos* $\mathbf{H}$ *with the dominance* $\top\colon 1 \to \Sigma$ *is a model of the axioms of synthetic domain theory as in [8].*

Remark 2.3 *A slightly different collection of axioms, also denoting models of synthetic domain theory, appeared in the literature, see [13]. By the results in* loc.cit., *it follows that* $\mathbf{H}$ *with* Σ *is also such a model.*

3 An internal characterisation of ω-Cpo

Recall from [9, 10] that an object F is called *complete* when F^c is iso. Now, consider the following notion (used implicitly in [3]): an object F is said to be *path-transitive* when, for t the unique mediating map in

$$\begin{tikzcd}
1 \arrow[r, "\perp"] \arrow[d, "\top"'] & \Sigma \arrow[d] \arrow[ddr, bend left, "\eta_\Sigma"] & \\
\Sigma \arrow[r] \arrow[drr, bend right, "L\perp"'] & \Sigma \vee \Sigma \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dr, dotted, "t"] & \\
& & L\Sigma
\end{tikzcd}$$

where η denotes the unit of the lifting monad, F^t is iso.

Theorem 3.1 ω-**Cpo** *is equivalent to the full subcategory of* $\mathbf{H}$ *consisting of the complete, path-transitive,* $\neg\neg$*-separated objects.*

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