PART II:

Scheduling

Scheduling: outline

1. Definitions

2. One dimensional linear schedules
   - Uniform equations and systems
   - Optimizing total execution time
   - Affine equations and systems

3. Multi-dimensional schedules

System of affine recurrence equations

Finite collection of equations

$$\forall z \in \mathcal{D}_V : V(z) = f_V[U(I(z)), ...]$$

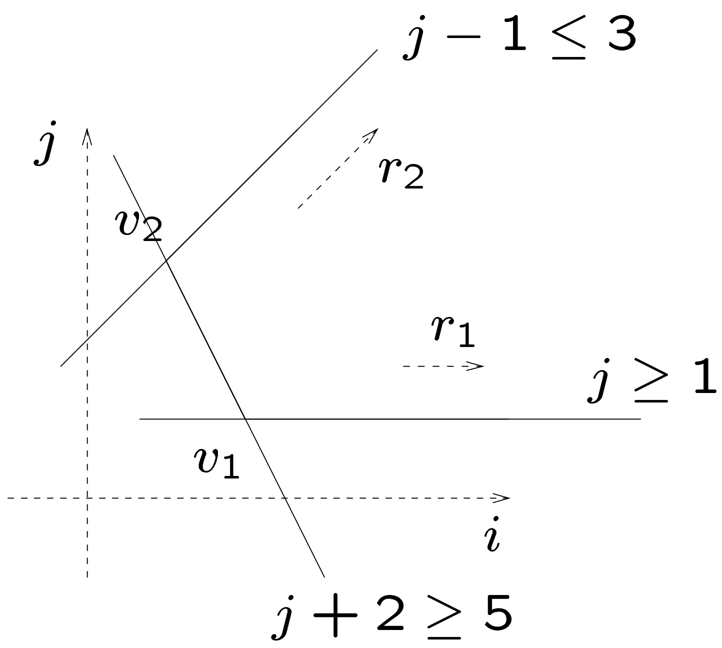
- $\mathcal{D}_V$: convex polyhedron included in $\mathbf{Z}^n$
- U, V: variables , i.e., fonctions from $\mathbf{Z}^n$ ($\mathbf{Z}^p$) to some set E
- $I(z)$: index (or dependence) function. Affine function from $\mathbf{Z}^n$ to $\mathbf{Z}^p$.
- f_V: strict function (its definition does not depend on z)

Example

$$\forall i,j,k \ : \ 1 \leq i,j \leq N, 0 \leq k \leq N, \quad C(i,j,k) = C(i,j,k-1) + a(i,k) \times b(k,j)$$

- $z = (i,j,k) \in \mathbf{Z}^3$: integral points of a $N \times N$ cube
- Variables : C, a and b
  - $C \ : \mathbf{Z}^3 \rightarrow \mathbf{real},$
  - $a, b \ : \mathbf{Z}^2 \rightarrow \mathbf{real}$
- index functions $I(z)$:
  - $I_C \ : \mathbf{Z}^3 \rightarrow \mathbf{Z}^3, \ i,j,k \rightarrow i,j,k-1,$
  - $I_a \ : \mathbf{Z}^3 \rightarrow \mathbf{Z}^2, \ i,j,k \rightarrow i,k$
- $f_C : \mathbf{real} \times \mathbf{real} \times \mathbf{real} \rightarrow \mathbf{real}, \ C,a,b \rightarrow C + a \times b$

Convex polyhedron



v_1, v_2 : vertices ; r_1, r_2 : extremal rays.

$$
\begin{aligned}
P &= \{(i,j) | j \ge 1, j + 2i \ge 5, j - i \le 3\} \\
&= \{x | x = \sum_{0 \le a \le 1} (av_1 + (1-a)v_2) + \sum_{b,c \ge 0} (br_1 + cr_2)\}
\end{aligned}
$$

Definitions

Convex combination of n points: $\sum_{i=1}^{n} \alpha_i x_i$, where $\alpha_i \geq 0$, $\sum_{i=1}^{n} \alpha_i = 1$

Convex polyhedron: set defined by a finite number of linear inequalities

Polytope: convex closure of a finite set of vectors. (bounded polyhedron)

Vertex: point which cannot be expressed as a convex combination of other points of the polyhedron

Ray: direction of a half-line contained in the polyhedron

Extremal ray: ray which cannot be expressed as a linear combination of other rays

Property (dual representation): a convex polyhedron is generated by the sum of convex combinations of his vertices, and of positive linear combinations of its extremal rays.

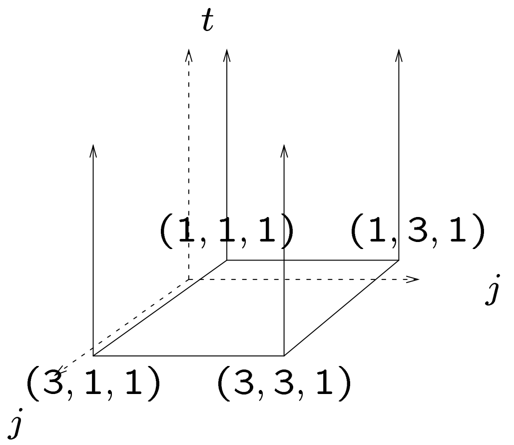
Uniform recurrence equations

$$\forall z \in \mathcal{D}_V : V(z) = f_V[U(z - d_1), \ldots]$$

d_i dependence vectors

All dependence functions have the same domain , and are translations

$$\begin{array}{l} 1 \leq i,j \leq N, t \geq 1 \rightarrow C(i,j,t) = C(i,j,t-1) + \\ \qquad A(i,j-1,t-1) \times B(i-1,j,t-1) \\ 1 \leq i,j \leq N, t \geq 1 \rightarrow A(i,j,t) = A(i,j-1,t-1) \\ 1 \leq i,j \leq N, t \geq 1 \rightarrow B(i,j,t) = B(i-1,j,t-1). \end{array}$$



Reduced dependence graph

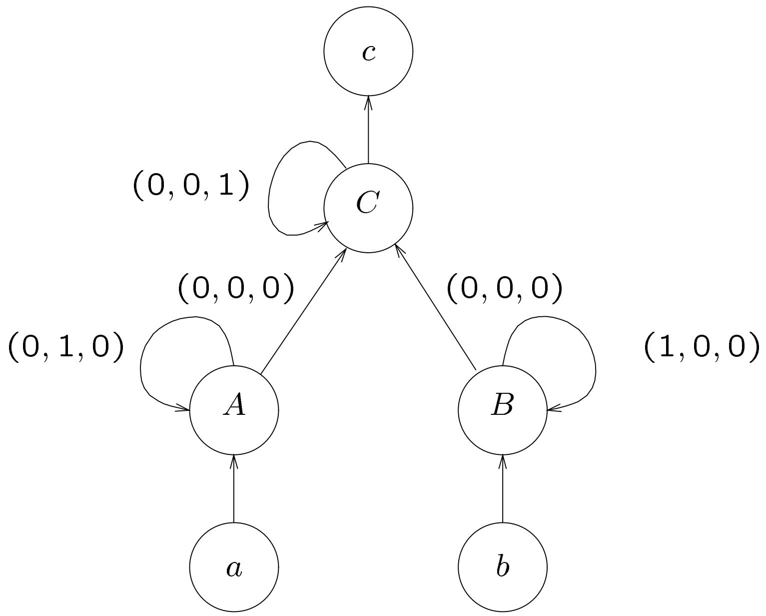
$$G = (\mathcal{V}, \mathcal{E})$$

where

- $\mathcal{V}$ is the set of variables
- there is an arc $e \in \mathcal{E}$, with origin U and extremity V if U is an argument of equation
$$V(z) = f(..., U(I(z)), ...)$$
- arcs are labeled with information related to the dependence function $(I(z))$

Example

$$1 \leq i,j,k \leq N \rightarrow C(i,j,k) = C(i,j,k-1) + A(i,j,k) \times B(i,j,k)$$
$$1 \leq i,j,k \leq N \rightarrow A(i,j,k) = A(i,j-1,k)$$
$$1 \leq i,j,k \leq N \rightarrow B(i,j,k) = B(i-1,j,k)$$



Dependence relation and graph

- $(V,x) \stackrel{1}{\leftarrow} (U,y)$ (depends strictly on) iff there exists an equation instance $V(x) = f[...U(y)...]$.
- $(V,x) \stackrel{t}{\leftarrow} (U,y)$ (depends in t steps on)
- $\leftarrow$: transitive closure of $\stackrel{1}{\leftarrow}$.

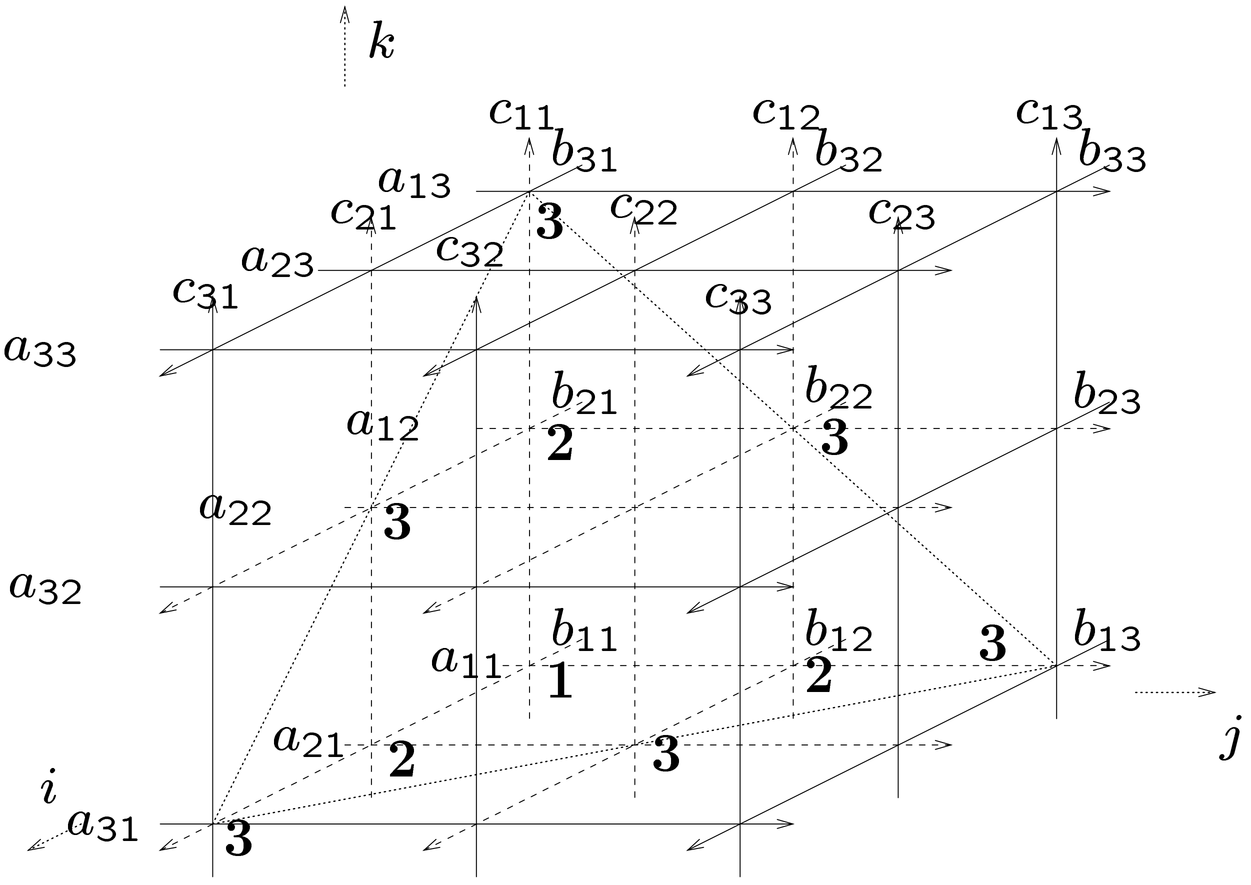
Extended dependence graph :

$$\Gamma = (\{(V,x)\} \cup \{\perp\}, \stackrel{1}{\leftarrow})$$

where $\perp$ represents the set of all points outside the domain.

NB: in graphs, arrows are traditionnaly oriented towards (V,x).

Example



k
c_{11}
c_{12}
c_{13}
b_{31}
b_{32}
b_{33}
a_{13}
c_{21}
c_{22}
c_{23}
a_{23}
c_{32}
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a_{32}
b_{11}
b_{12}
b_{13}
a_{11}
a_{21}
i
a_{31}
j
3
2
3
3
1
2
3
2
3
3

Schedule (timing function)

$t : \mathbf{V} \times \mathcal{D} \to \mathbf{N}$ such that

$$(V, x) \leftarrow (U, y) \Rightarrow t(V, x) > t(U, y).$$

Free schedule:

$$t(V, x) = \begin{cases} 1 \text{ si } (\nexists (U, y), y \in \mathcal{D}) \text{ s.t. } (V, x) \stackrel{1}{\leftarrow} (U, y)) \\ 1 + \max\{t(U, y) \,|\, (V, x) \stackrel{1}{\leftarrow} (U, y) \text{ et } y \in \mathcal{D}\}, \text{ otherwise.} \end{cases}$$

Property : If a system has a schedule, then it has a unique free schedule, denoted T_f.

We let $T_f(V, x) = +\infty$ when the schedule is undefined at point (V, x).

$T_f(V, x)$ is the length of the shortest path arriving at (V, x) in Γ.

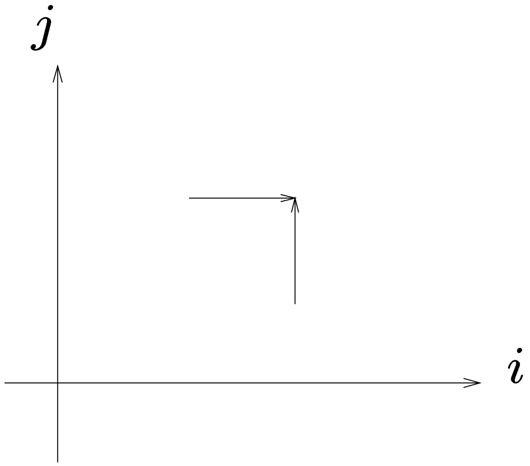
Scheduling and computability

Definition : a variable V is said to be *computable* – or *explicitly defined* – iff

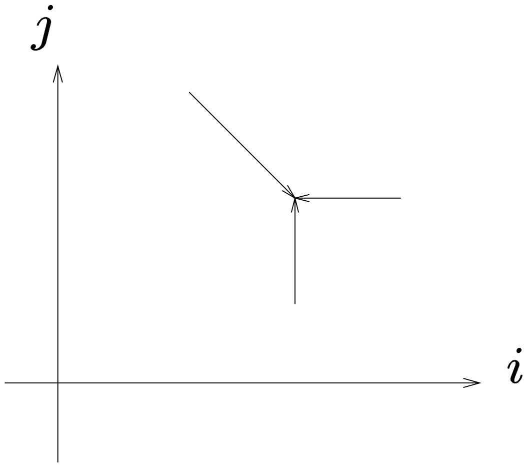
$$x \in \mathcal{D} \Rightarrow T_f(V, x) < +\infty.$$

Lemma . $T_f(V,x) = +\infty \Leftrightarrow \Gamma$ *has an infinite path arriving at* (V,x). i.e: there exists a cycle in the graph, or there exists an infinite (non cycle) path

Examples

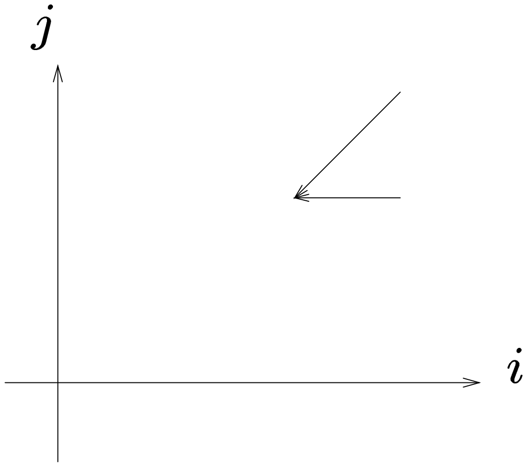


$V(i,j) = V(i-1,j) + V(i,j-1)$

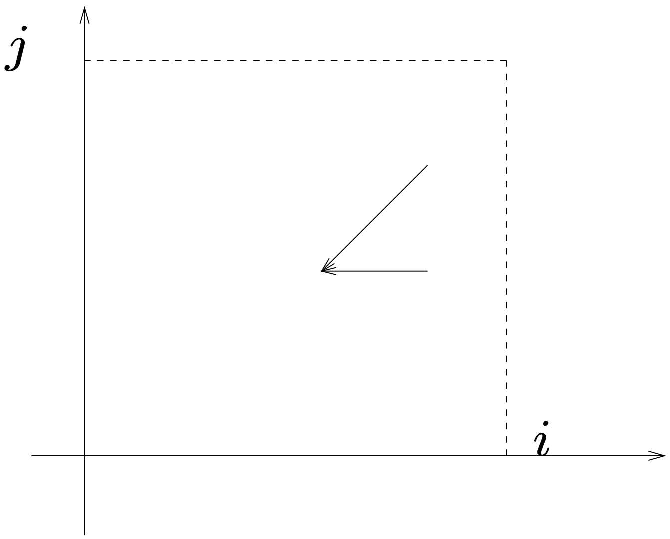


$V(i,j) = V(i-1,j) + V(i+1,j) + V(i,j+1)$

Influence of domain



$$i, j \geq 0 \rightarrow$$
$$V(i,j) = V(i+1, j+1) + V(i+1, j)$$

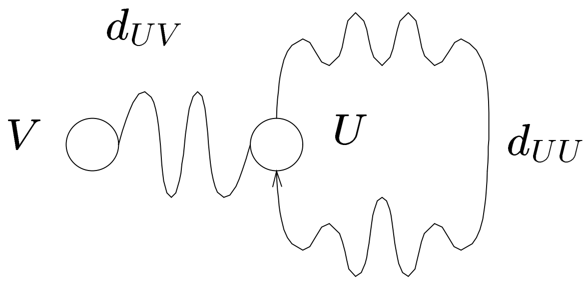


$$i, j \geq 0; i, j \leq N \rightarrow$$
$$V(i,j) = V(i+1, j+1) + V(i+1, j)$$

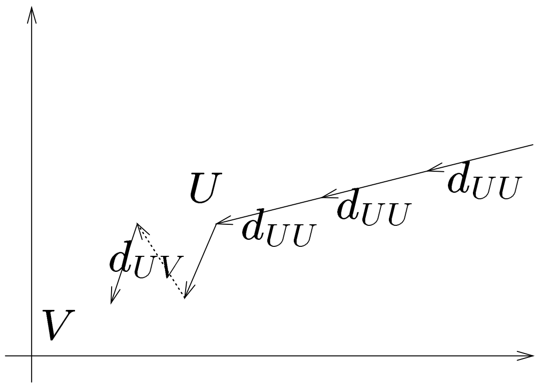
Computability of a uniform system

Notation : $d(\pi)$ weight of a path π in G, i.e. sum of the dependence vectors of its arcs.

Theorem 1 (Karp, Miller et Winograd, 1967) : Let G be a dependence graph on domain $\mathcal{D} = \mathbb{N}^n$. Variable V is computable iff G does not contain a path arriving at V whose origin U belongs to a non positive cycle



G



Γ

Case of a single equation

Theorem 2: Given one uniform recurrence equation

$$V(x) = f[V(x-d_1), ..., V(x-d_s)],$$

on domain $\mathcal{D} = \mathbf{N}^n$, the following conditions are equivalent:

1. $V(x)$ is computable ;

2. there exists no semi-positive vector $(u_1, ..., u_s)$ such that

$$\sum_{j=1}^{s} -u_j d_j \geq 0;$$

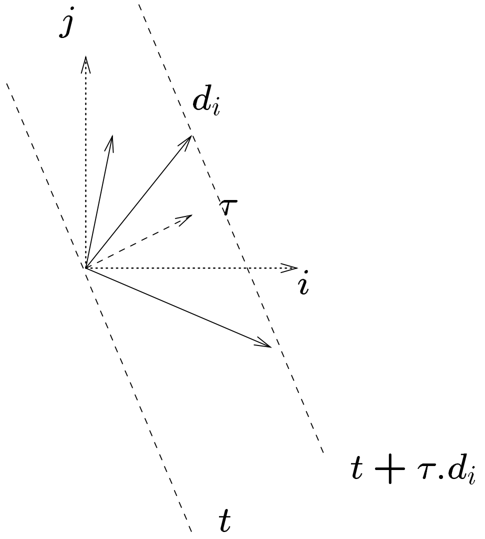
3. let $\tau = (\tau_1, ..., \tau_n)$ a vector of $\mathbf{Z}^n$. The system of inequalities $d_j.\tau \geq 1$, $1 \leq j \leq s$, and $\tau_i \geq 0$, $1 \leq i \leq n$ has a solution.

4. for all $x \in \mathbf{N}^n$, the following linear programs have an common optimal solution $m(x)$:

$$I \begin{cases} x - \sum_{j=1}^{s} u_j d_j \geq 0, \\ u_j \geq 0, 1 \leq j \leq s, \\ \max \sum_{i=1}^{s} u_j \end{cases} \qquad II \begin{cases} \tau_i \geq 0, 1 \leq i \leq n \\ d_j.\tau \geq 1, 1 \leq j \leq s, \\ \min x.\tau \quad . \end{cases}$$

Remarks

- geometric interpretation of condition *3*: all dependence vectors strictly contained in a half-space

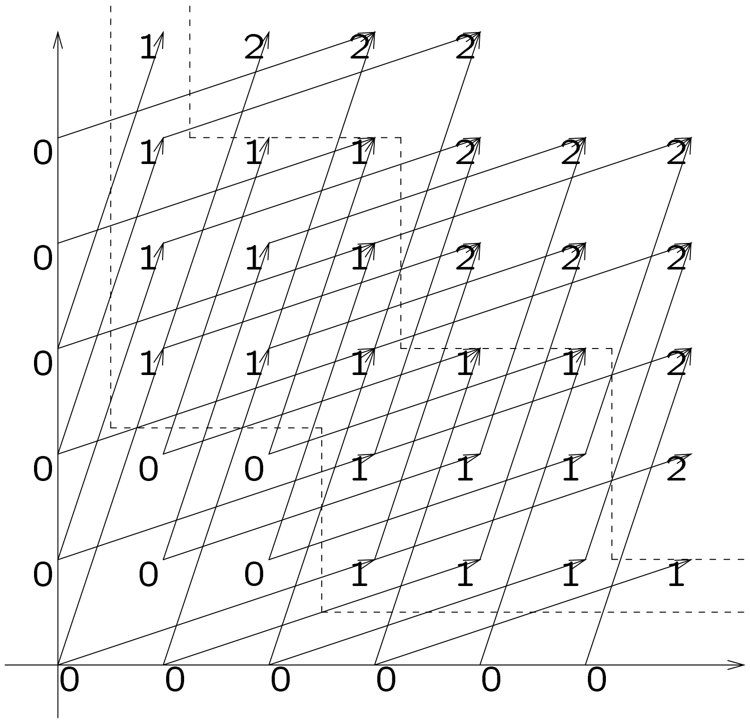


- condition *4*:
  - $m(x)$ is the length of the longest path arriving at x in graph Γ.
  - $m(x)$ is a schedule which can be computed pointwisely using linear programming

Remarks

- $m(x)$ is as fast as the free schedule T_f, except at the boundaries of the domain
- the case of a system of equations is more complicated

Example



$$i, j \geq 0 \rightarrow V(i,j) = V(i-1, j-3) + V(i-3, j-1)$$

Linear schedules

Corollary : any solution τ of system in condition *3* in Theorem I gives a schedule $t(x) = \tau.x$ which is linear in x.

Example:

$$i, j \geq 0 \rightarrow V(i, j) = V(i-1, j-3) + V(i-3, j-1)$$

Dependence vectors : $(1, 3)$ and $(3, 1)$

$$\tau_1 + 3\tau_2 \geq 1$$
$$3\tau_1 + \tau_2 \geq 1$$

Rational solution : $(1/4, 1/4)$.

Integral solution : $(1, 0)$ or $(0, 1)$ or $(1, 1)$.

Application to matrix multiplication

Equations

$$1 \leq i,j,k \leq 3 \rightarrow C(i,j,k) = C(i,j,k-1) + A(i,j,k) \times B(i,j,k)$$
$$1 \leq i,j,k \leq 3 \rightarrow A(i,j,k) = A(i,j-1,k)$$
$$1 \leq i,j,k \leq 3 \rightarrow B(i,j,k) = B(i-1,j,k)$$

Dependence vectors : $(0,0,1)^t, (0,1,0)^t, (1,0,0)^t$

Linear program II :

$$\begin{array}{rcl} (0,0,1).\tau & \geq & 1 \\ (0,1,0).\tau & \geq & 1 \\ (1,0,0).\tau & \geq & 1, \end{array} \quad \text{i.e.} \quad \begin{array}{rcl} \tau_3 & \geq & 1 \\ \tau_2 & \geq & 1 \\ \tau_1 & \geq & 1, \end{array}$$

Solution :

$$\tau = (1,1,1)^t \Rightarrow t(i,j,k) = i + j + k$$

Minimum latency linear schedule

Assumptions:

- equation's domain is polytope $\mathcal{D} = \{z | Az \leq b\}$.
- uniform equations

Schedule prototype $T_l(z) = \lfloor \tau z + \alpha \rfloor$ (τ may be rational)

Total execution time

$$\max_{z_1, z_2 \in \mathcal{D}} T_l(z_1) - T_l(z_2) \quad = \quad \max_{(Az_1 \leq b, Az_2 \leq b)} (\lfloor \tau z_1 \rfloor - \lfloor \tau z_2 \rfloor)$$

z_1 and z_2 integral.

Problem: Find out the optimal solution:

$$\min_{\tau D \geq 1} \max_{z_1, z_2 \in \mathcal{D}} (\lfloor \tau z_1 \rfloor - \lfloor \tau z_2 \rfloor)$$

Example

Equation

$$\left\{ \begin{array}{l} i \geq 0, j \geq 0, \\ N \leq i + j \leq 2N \end{array} \right\} \rightarrow V(i,j) = f[V(i-1, j+3), V(i-2, j-4), V(i-2, j)]$$

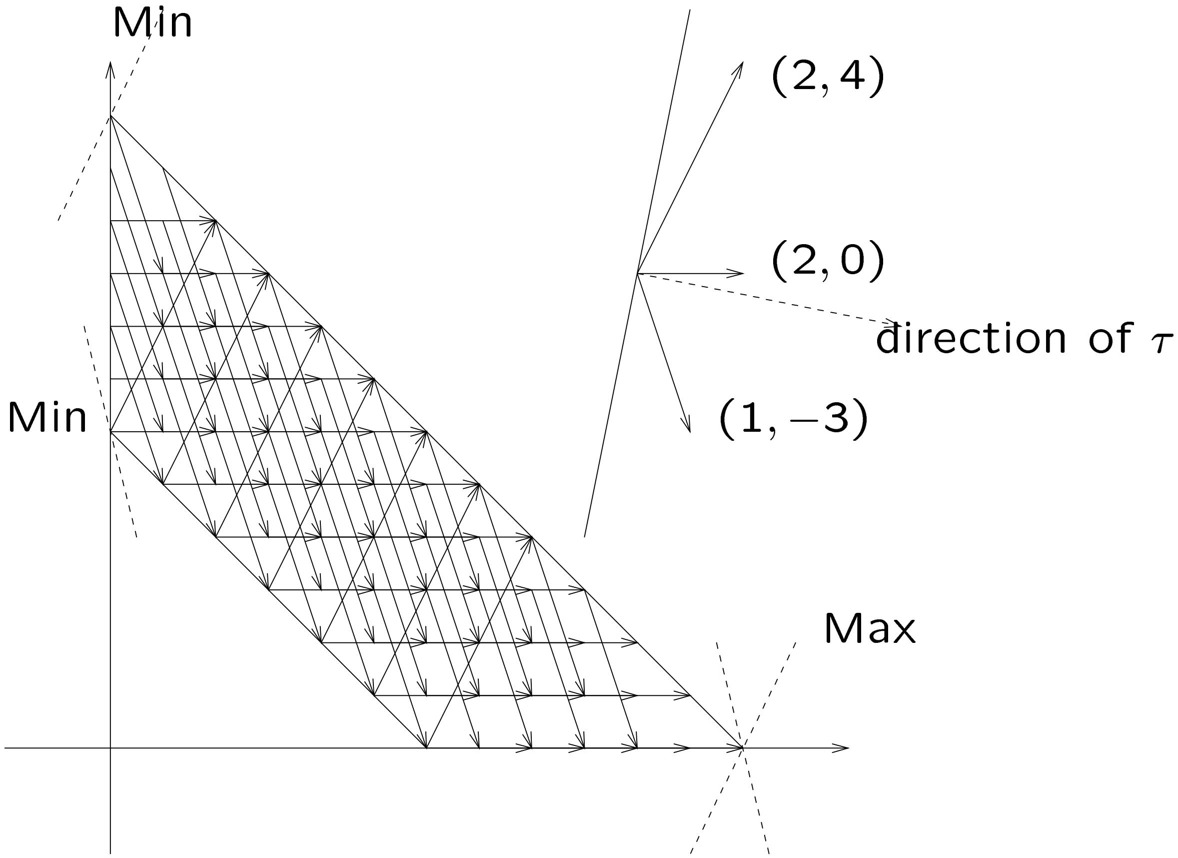
Matrices

$$A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \\ 1 & 1 \\ -1 & -1 \end{pmatrix} \quad b = \begin{pmatrix} 0 \\ 0 \\ 2N \\ -N \end{pmatrix}$$

Dependence vectors

$$\begin{pmatrix} 1 \\ -3 \end{pmatrix} \begin{pmatrix} 2 \\ 4 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

Example



Solution using linear programming

Method: Let

$$A' = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}, \; c = (\tau, -\tau), \; b' = \begin{pmatrix} b \\ b \end{pmatrix}, \; z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} .$$

$$\begin{cases} Az_1 \leq b \\ Az_2 \leq b \\ \max \tau(z_1 - z_2) \end{cases} \quad \text{becomes} \quad \begin{cases} A'z \leq b' \\ \max c.z \end{cases} .$$

Duality

$$\begin{cases} y \geq 0 \\ yA' = c \\ \min yb' \end{cases} .$$

Final problem

Letting $y = (\tau_1, \tau_2)$, and adding dependence constraints $\tau D \geq 1$:

$$\begin{cases} \tau D \geq 1 \\ \tau_1 A = \tau \\ \tau_2 A = -\tau \\ \tau_1 \geq 0 \\ \tau_2 \geq 0 \\ \min\,(\tau_1 + \tau_2)b\ . \end{cases}$$

Or

$$\begin{cases} (\tau_1 + \tau_2)A = 0 \\ \tau_1 AD \geq 1 \\ \tau_1 \geq 0 \\ \tau_2 \geq 0 \\ \min\,(\tau_1 + \tau_2)b \end{cases}$$

Back to example

Constraint $(\tau_1 + \tau_2)A = 0$:

$$-(\tau_{11} + \tau_{21}) + (\tau_{13} + \tau_{23}) - (\tau_{14} + \tau_{24}) = 0$$
$$-(\tau_{12} + \tau_{22}) + (\tau_{13} + \tau_{23}) - (\tau_{14} + \tau_{24}) = 0$$

Constraint $\tau_1 AD \geq 1$:

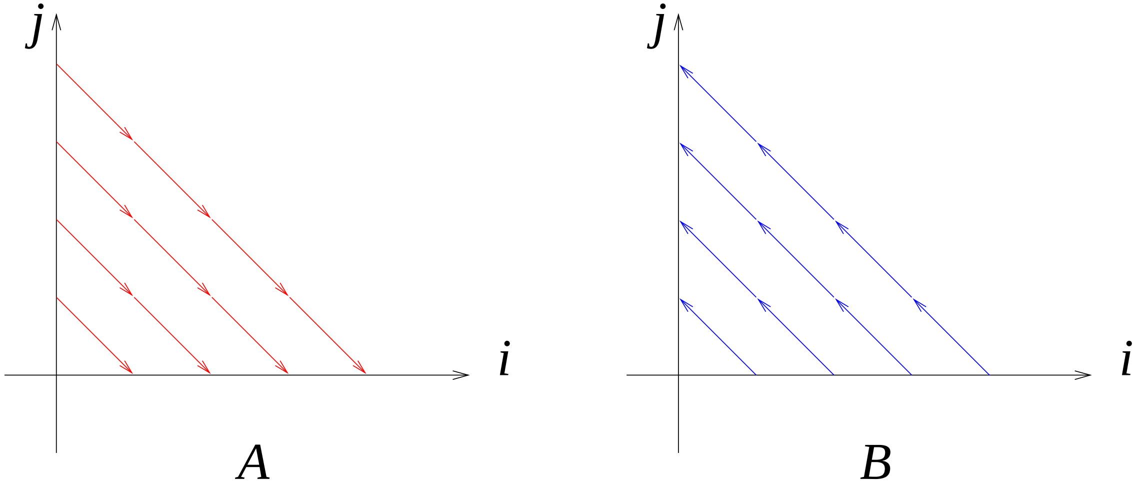
$$-2\tau_{11} - 4\tau_{12} + 6\tau_{13} - 6\tau_{14} \geq 1$$
$$2(-\tau_{11} + \tau_{13} - \tau_{14}) \geq 1$$
$$-\tau_{11} + 3\tau_{12} - 2\tau_{13} + 2\tau_{14} \geq 1$$

Objective function:

$$Min - \tau_{14} + 2(\tau_{13} + \tau_{23}) - \tau_{24}$$
$$\tau = (-\tau_{11} + \tau_{13} - \tau_{14}, -\tau_{12} + \tau_{13} - \tau_{14})^t$$

Solution : $\tau = (\frac{7}{10}, \frac{-1}{10})$

Limitation of simple linear schedules



$$\begin{aligned} A(i,j) &= f[A(i-1,j+1)] \\ B(i,j) &= g[B(i+1,j-1)] \end{aligned}$$

$$\tau_1 - \tau_2 \geq 1 \text{ and } \tau_2 - \tau_1 \geq 1$$

Need to schedule each variable separately !

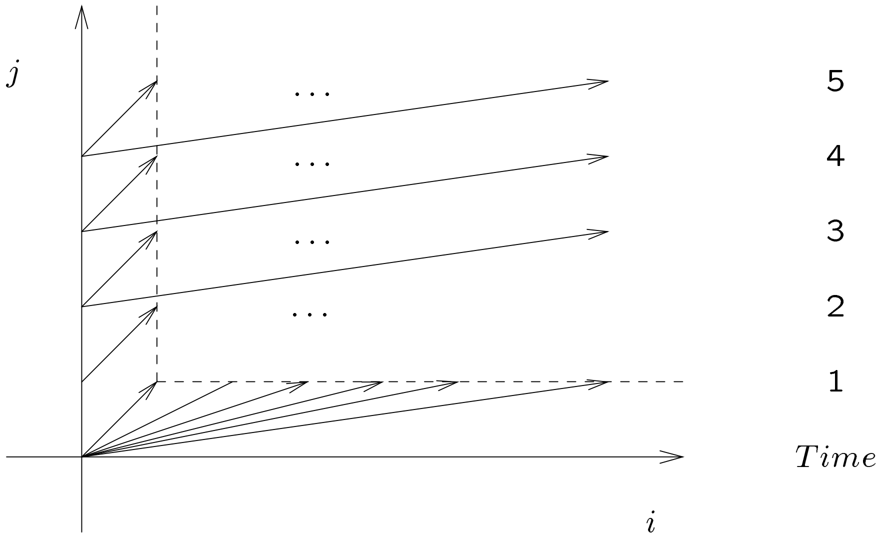
Affine recurrences

- Equations : $\forall z \in \mathcal{D} : V(z) = f_V[U(I(z)), ...]$.
- $I(z) = Az + B$.
- Arcs e of dependence graph labeled with $(\mathcal{D}, I)$.
- Schedule prototype : $t_V(z) = \tau_V^1 z_1 + ... + \tau_V^n z_n + \alpha_V$.
- Domain may be infinite and/or parametrized
- Variables with different dimensions $\Rightarrow$ no notion of dependence vector
- Schedule may not even exist !

Example

$$1 \le i, j \le n, n > 0 \quad \rightarrow \quad U(i,j,n) = U(0, j-1, n) + V(i,j,n)$$
$$1 \le i, j \le n, n > 0 \quad \rightarrow \quad V(i,j,n) = V(i, j-1, n)$$

Extended dependence graph of variable U



Domain of dependence vectors is a polyhedron !

The "vertex method"

Idea: check the properties at "extreme points" of domain.

Theorem: τ_V, $\alpha_V, V \in \mathbf{V}$ define an affine schedule iff:

for all arc $(V, U, \mathcal{D}, I)$ of the dependence graph, for all vertex σ of $\mathcal{D}$,

$$\tau_V.\sigma - \tau_U.(I(\sigma)) + \alpha_V - \alpha_U > 0$$

Example

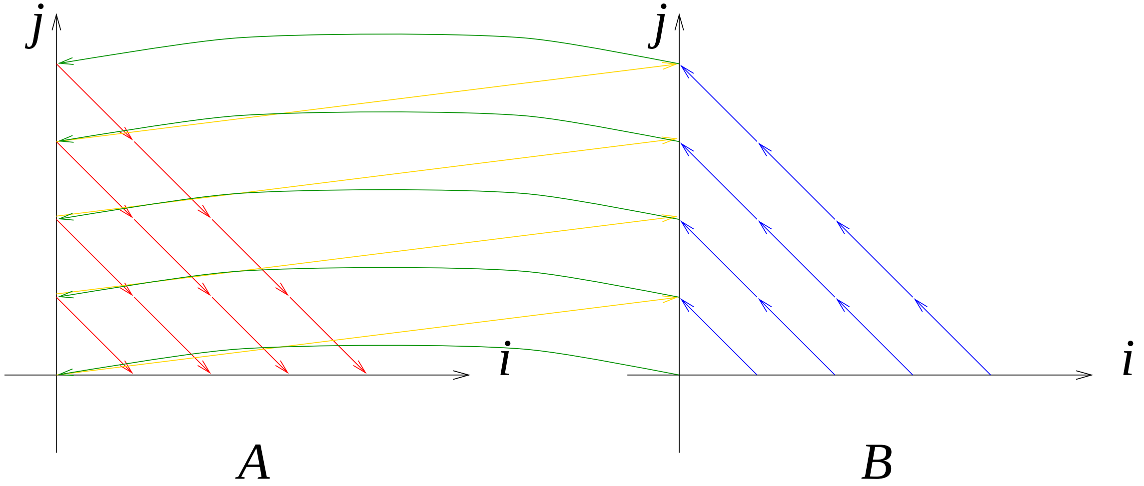
Inequalities (after some cleaning..)

$$\begin{array}{rcl} \tau_V^1 \geq 0 & & \\ \tau_U^1 + \tau_U^2 & > & 0 \\ \alpha_U - \alpha_V & > & 0 \\ \tau_V^2 & > & 0 \end{array}$$

Solution :

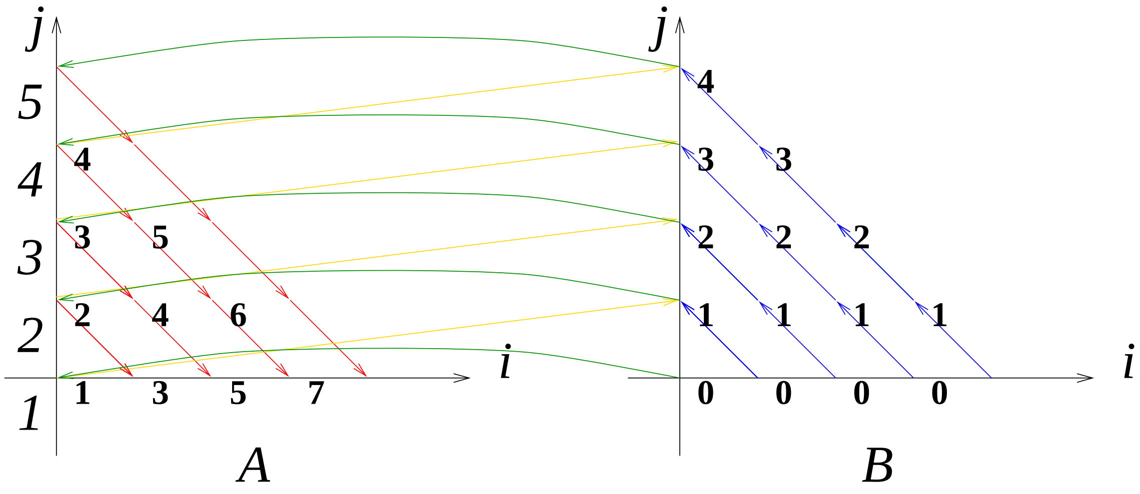
$$t_U(i,j,n) = j \text{ and } t_V(i,j,n) = j-1$$

Multidimensional schedules



$$
\begin{aligned}
A(i,j) &= f[A(i-1,j+1), B(i,j)] \\
B(i,j) &= g[B(i+1,j-1), A(i-1,j)]
\end{aligned}
$$

Multidimensional schedules



Schedule:

$$t_A^1(i,j) = t_B^2(i,j) = i + j$$
$$t_B^2(i,j) = j$$
$$t_A^2(i,j) = 2i + j + 1$$

In summary

1. Equations:
   (a) Uniform vs affine
   (b) same domain vs different domains
   (c) polytope domains vs infinite domains
   (d) parameterized vs non parameterized
2. Schedules:
   (a) linear
   (b) affine
   (c) affine by variable
   (d) piecewise affine
   (e) multi-dimensional

References (see companion survey)

- Computation of recurrence equations: [KMW67, GJ87, SQ93]
- Uniform recurrence equations: [KMW67, Rao85,Qui87]
- Multi-dimensional schedules: [Rao85, Roy88, Fea92c, DV94, DV95]
- Affine recurrence equations and schedules: [DI86, RFP86, RF90, QV89, MQRS90]
- Piecewise affine schedules: [RMK92, Fea92b]
- Optimality of schedules: [KMW67, SF91, DKR91, DR94a, Fea92b, DR95, DV95]