Mediation is Inherently Parallel

Martin ESCARDO, Martin HOFMANN, Thomas STREICHER

1 Introduction

Generalities on exact real number computation

1.1 Preliminaries

Let $\mathcal{I}$ be the continuous Scott domain of nonempty closed intervals in $[0,1]$ ordered by reverse inclusion written as $\sqsubseteq$, i.e., $a \sqsubseteq b$ when $a \supseteq b$. We write $a \sqsubset b$ to mean $a \sqsubseteq b$ and $a \neq b$.

Any $a \in \mathcal{I}$ is of the form $[a_1, a_2]$ for some $0 \leq a_1 \leq a_2 \leq 1$. We have a least element $\perp \in \mathcal{I}$ given by $\perp = [0,1]$.

We write $|[a_1, a_2]|$ for the size of an interval given as $a_2 - a_1$. The elements $[a_1, a_2]$ with $a_1 < a_2$ are called partial real numbers. The elements of the form $[a_1, a_2]$ with $a_1 = a_2$ are total real numbers; we identify them with elements of $[0,1]$.

A rational interval is one with both endpoints rational numbers.

Two elements $a, b \in \mathcal{I}$ are *compatible* if they have a nonempty intersection, in other words there exists $c \in \mathcal{I}$ with $a \sqsubseteq c$ and $b \sqsubseteq c$. In this case the set-theoretic union of a and b is also an interval: the greatest lower bound or meet (with respect to $\sqsubseteq$) of a, b in $\mathcal{I}$ and will be written as $a \sqcap b$.

We denote Σ the domain $\Sigma = \{\perp, \top\}$ with the ordering $\perp \sqsubseteq \top$.

The "weak parallel or" function $\mathrm{wpor} : \Sigma \times \Sigma \to \Sigma$ is given by $\mathrm{wpor}(x, y) = \top$ iff $x = \top$ or $y = \top$.

Our goal is to define wpor from any continuous extension of mediation and certain undoubtedly sequential basic functions which we will now describe.

First, we need test functions from $\mathcal{I}$ to Σ. We can leave these largely unspecified and will simply assume that for any two intervals $a, b \in \mathcal{I}$ with $a \sqsubset b$ there is a test function $\mathrm{test}_{a,b} : \mathcal{I} \to \Sigma$ such that $\mathrm{test}_{a,b}(a) = \perp$ and $\mathrm{test}_{a,b} = \top$.

For any two rational intervals $p, q \in \mathcal{I}$ with $p \sqsubseteq q$ we have the path function $\mathrm{path}_{p,q} : \Sigma \to \mathcal{I}$ with $\mathrm{path}_{p,q}(\perp) = p$ and $\mathrm{path}_{p,q}(\top) = q$.

For rational interval q we let $\mathrm{cons}_q : \mathcal{I} \to \mathcal{I}$ be the unique increasing affine transformation sending $\perp = [0,1]$ to q. In fact, we have $\mathrm{cons}_q(x) = ax + b$ when $q = [a, a+b]$.

Definition 1.1 *Let f be a function. A function h (not necessarily unary) is called f-definable, if it can be obtained from f, the test functions* $\mathrm{test}_{a,b}$*, the path functions* $\mathrm{path}_{p,q}$*, the affine transformations* cons_q*, the constants $\perp, \top \in \Sigma$ using projections and substitution. In other words, the f-definable functions are the clone generated by f and the basic functions.*

We consider the function wpor as indisputably parallel and will use definability of wpor as definition of inherent parallelism.

Definition 1.2 *A function f is* inherently parallel *if the function* wpor *is f-definable.*

Definition 1.3 *A function $f : \mathcal{I} \times \mathcal{I} \to \mathcal{I}$ is called a continuous extension of mediation if f is continuous and for total elements $x, y \in \mathcal{I}$ the value $f(x, y)$ is also total and moreover*

$$f(x, y) = \frac{x + y}{2}$$

In particular, we have the maximal extension of mediation given by

$$[a_1, a_2] \oplus [b_1, b_2] = [(a_1 + a_2)/2, (b_1 + b_2)/2]$$

Another continuous extension of mediation is given by

$$f(a, b) = \begin{cases} a \sqcap b, \text{ if } a, b \text{ are compatible} \\ a \oplus b, \text{ otherwise} \end{cases}$$

The next section is devoted to shoing that any continuous extension of mediation is inherently parallel.

2 Sequential computation in $\mathcal{I}$

We begin our study of continuous extensions of mediation with a design for a definition of wpor which works in particular for the maximal extension of mediation given by taking averages on endpoints.

Proposition 2.1 *Let f be a continuous extension of mediation. Either f is inherently parallel or for any intervals a, a', b, b' with $a \sqsubseteq a'$ and $b \sqsubseteq b'$ it holds that*

$$f(a, b) = f(a', b) \sqcap f(a, b')$$

Proof. By monotonicity we always have $f(a, b) \sqsubseteq f(a', b) \sqcap f(a, b')$. So assume that $f(a, b) \sqsubset f(a', b) \sqcap f(a, b')$. Then by continuity we can assume w.l.o.g. that a, a', b, b' are rational intervals and we have

$$\text{wpor}(x, y) = \text{test}_{f(a,b), f(a',b) \sqcap f(a,b')}(f(\text{path}_{a,a'}(x), \text{path}_{b,b'}(y))$$

$\square$

Notice that if f is the maximal extension of mediation then f is inherently parallel by the above reasoning with $a = b = \perp$ and $a' = b' = 1$.

The next result shows that unless a function is inherently parallel one can construct from it a unary non-stable function which moreover exhibits its non-stability at an a priori chosen pair of intervals.

Proposition 2.2 *Let f be a continuous extension of mediation. Either f is inherently parallel or for any two compatible intervals $a, b \in \mathcal{I}$ such that $a \sqcap b \sqsubset a$ and $a \sqcap b \sqsubset b$ there exists f-definable h such that*

$$h(a \sqcap b) \sqsubset h(a) \sqcap h(b)$$

Proof. Let $\lambda < 1$ be a constant such that $\max(|a|, |b|) < \lambda |a \sqcap b|$. Such λ exists by the assumption on a, b.

Let x be the lower endpoint of $a \sqcap b$ which w.l.o.g. we assume to be the lower endpoint of a as well. Let y be any rational point in $a \sqcap b \setminus a$.

Proposition 2.1 shows that unless f is inherently parallel we have

$$f(a \sqcap b, a \sqcap b) = f(a \sqcap b, y) \sqcap f(y, a \sqcap b) \tag{1}$$

Therefore, since $x \in f(a \sqcap b, a \sqcap b)$ we either have $x \in f(a \sqcap b, y)$ or $x \in f(y, a \sqcap b)$. W.l.o.g. we assume the former, i.e.,

$$x \in f(a \sqcap b, y) \tag{2}$$

We then claim that there exists a rational interval q such that

$$f(\mathrm{cons}_q(a \sqcap b), y) \sqsubset f(\mathrm{cons}_q(a), y) \sqcap f(\mathrm{cons}_q(b), y) \tag{3}$$

so we would be done with

$$h(x) = f(\mathrm{cons}_q(x), y) \tag{4}$$

If we want to exclude total elements from the f-definable ones we can of course replace y by a sufficiently close rational interval.

We prove the claim (3) by contradiction, so will assume that

$$f(\mathrm{cons}_q(a \sqcap b), y) = f(\mathrm{cons}_q(a), y) \sqcap f(\mathrm{cons}_q(b), y) \tag{5}$$

for each rational interval q.

We define inductively a sequence $(q_i)_{i \in \mathbb{N}}$ of rational intervals such that

$$x \in f(\mathrm{cons}_{q_i}(a \sqcap b), y) \tag{6}$$

and $\mathrm{cons}_{q_i}(a \sqcap b)$ converges to a total real number. This would give the desired contradiction as $x \notin f(z, y)$ for any total $z \in a \sqcap b$.

We put $q_0 = \perp$; then $f(\mathrm{cons}_{q_0}(a \sqcap b), y) = f(a \sqcap b, y)$ and $x \in f(\mathrm{cons}_{q_0}(a \sqcap b), y)$ by assumption (2). If q_i has already been constructed then by (6) we know that $x \in f(\mathrm{cons}_{q_i}(a \sqcap b), y)$ and assumption (5) shows that either $x \in f(\mathrm{cons}_{q_i}(a), y)$ or $f(\mathrm{cons}_{q_i}(b), y)$ using the fact that affine transformations preserve meets. W.l.o.g. we may assume the former and let q_{i+1} be a rational interval satisfying

- $\mathrm{cons}_{q_{i+1}}(a \sqcap b) \sqsubseteq \mathrm{cons}_{q_i}(a)$
- $|\mathrm{cons}_{q_{i+1}}(a \sqcap b)| \leq \lambda |\mathrm{cons}_{q_i}(a \sqcap b)|$

Notice that if a, b are rational intervals, we can simply choose q_i as the unique rational interval making the first part of the specification an equality. Otherwise we slightly enlarge it so that the second part still holds.

By construction (2) holds for q_{i+1} and, since $\lambda < 1$ the sequence of intervals $\mathrm{cons}_{q_i}(a \sqcap b)$ shrinks and converges against a total number $z \in a \sqcap b$.

Now $f(z, y) = (z + y)/2$ is a total number different from x. Therefore, by continuity of f we must have

$$x \notin \bigsqcup_i f(\mathrm{cons}_{q_i}(a \sqcap b, y)$$

hence $x \notin f(\mathrm{cons}_{q_i}(a \sqcap b), y)$ contradicting (6). $\square$

Notice that if f is the maximal extension of mediation then $f(a \sqcap b, y) = f(a, y) \sqcap f(b, y)$ for any two compatible a, b and arbitrary y so that the case distinction in the statement of the preceding proposition is reasonable. However, it is still the case that a unary non-stable function can be defined from this f, but not following the design (4).

Now we will establish a relationship between inherent parallelity and a naive generalisation of sequentiality in the sense of Vuillemin [].

Proposition 2.3 *Let f be a continuous extension of mediation. Either f is inherently parallel or for any two intervals u, v one of the following two cases applies.*

- $f(u, v) = f(u, v')$ *for each* $v \sqsubseteq v'$,
- $f(u, v) = f(u', v)$ *for each* $u \sqsubseteq u'$

Proof. Suppose for a contradiction that we are given intervals $u, u', v, v' \in \mathcal{I}$ with $u \sqsubseteq u'$ and $v \sqsubseteq v'$ such that

- $f(u, v) \sqsubset f(u', v)$
- $f(u, v) \sqsubset f(u, v')$

It requires a little thought to see that this is indeed the negation of the second disjunct of the statement.

Let us put $a = f(u', v)$ and $b = f(u, v')$. By Prop. 2.1 we have $f(u, v) = a \sqcap b$ unless f is inherently parallel. So, in this case $a \sqcap b \sqsubset a$ and $a \sqcap b \sqsubset b$, hence by Proposition 2.2 we can find f-definable h such that

$$h(a \sqcap b) \sqsubset h(a) \sqcap h(b)$$

It follows that

$$\mathrm{wpor}(x, y) = \mathrm{test}_{h(a \sqcap b), h(a) \sqcap h(b)}(h(f(\mathrm{path}_{u,u'}(x), \mathrm{path}_{v,v'}(y))))$$

$\square$

We have now laid the groundwork to state and prove our main result.

Theorem 2.1 *Let f be a continuous extension of mediation. Then f is inherently parallel.*

Proof. Define the subsets $\mathcal{L}, \mathcal{R}$ of $\mathcal{I} \times \mathcal{I}$ by

$$\begin{array}{lcl} (a, b) \in \mathcal{L} & \Leftrightarrow & \forall b' \sqsupseteq b.\ f(a, b) = f(a, b') \\ (a, b) \in \mathcal{R} & \Leftrightarrow & \forall a' \sqsupseteq a.\ f(a, b) = f(a', b) \end{array}$$

Of course, $\mathcal{L}$ consists of all arguments (a, b) where improving the second argument doesn't improve output, i.e. "f looks at its left argument, and analogously for $\mathcal{R}$.

Lemma 2.1 *The sets $\mathcal{L}$ and $\mathcal{R}$ are each closed under under directed suprema.*

Proof. Suppose that $(a_i, b_i)_{i\in I}$ is a directed family in $\mathcal{L}$. Suppose $b' \sqsupseteq \bigsqcup_{i\in I} b_i$. Then for all a_i we have $f(a_i, b') = f(a_i, b_i)$. Accordingly, we have

$$f(\bigsqcup_{i\in I} a_i, \bigsqcup_{i\in I} b_i) = \bigsqcup_{i\in I} f(a_i, b_i) = \bigsqcup_{i\in I} f(a_i, b') = f(\bigsqcup_{i\in I} a_i, b') \tag{7}$$

and, therefore, $(\bigsqcup_{i\in I} a_i, \bigsqcup_{i\in I} b_i) \in \mathcal{L}$. The proof for $\mathcal{R}$ is symmetric. □

Now assume for a contradiction that f is not inherently parallel. Then Proposition 2.3 states that $\mathcal{L} \cup \mathcal{R} = \mathcal{I} \times \mathcal{I}$.

W.l.o.g. assume that $(\perp, \perp) \in \mathcal{L}$

Then for all total $x \in [0,1]$ we have $(x, \perp) \notin \mathcal{L}$ as otherwise $f(x, y) = f(x, \perp)$ for all total $y \in [0,1]$ which is impossible as $f(x, y_1) \neq f(x, y_2)$ for different $y_1, y_2 \in [0,1]$. In particular, we have that $(0, \perp) \notin \mathcal{L}$. Pick any $\epsilon > 0$. Since by Lemma 2.1 $\mathcal{L}$ is closed under directed suprema this means that already $(a, \perp) \notin \mathcal{L}$ for some non-total element $a \in \mathcal{I}$ with $|a| \leq \epsilon$. Since, $\mathcal{L} \cup \mathcal{R} = \mathcal{I} \times \mathcal{I}$ we must therefore have $(a, \perp) \in \mathcal{R}$.

By a similar argument as above we have for all total $y \in [0,1]$ that the pair $(a, y) \notin \mathcal{R}$ and that, therefore, for each such y we can find a non-total element b_y containg y. such that $(a, b_y) \notin \mathcal{R}$, hence $(a, b_y \in \mathcal{L}$. Then, by compactness of $[0,1]$, there exist finitely many intervals $b_1, \ldots, b_n$ such that

(1) $b_1 \cup \cdots \cup b_n = [0,1]$ and

(2) $(a, b_i) \in \mathcal{L}$ for $i = 1, \ldots, n$.

It follows from (2) that whenever $z \in b_i \cap b_j$ then $f(a, b_i) = f(a, z) = f(a, b_j)$.

By iterating this argument using (1) we get that $f(a, b_i) = f(a, b_j)$ for any two i, j. Therefore, $f(a, 0) = f(a, 1)$.

Summing up, for each $\epsilon > 0$ we can find an interval a containing 0 and with $|a| \leq \epsilon$ such that $f(a, 0) = f(a, 1)$, in particular $1/2 \in f(a, 0)$. Continuity of f then gives $1/2 \in f(0, 0)$ which contradicts the assumption that f extends mediation. □

3 Discussion

PCF without pif defines all the basic functions. So does Abbas' language. Possible ways out. Dropping extensionality?

Don't need all rational path functions, suffices to have them for a "dense" set, e.g. dyadic intervals.

Argument should work for functions other than mediation as well.

Should be possible to define Plotkin's parallel or as well. Perhaps even pif.