1 Introduction

The subject of this paper is how to construct proofs of accessibility for each of a system of ordinal notations for the predecessors of the Schütte−Feferman ordinal Γ_0. These proofs can be formalised (in a metamathematical sense) in a certain weak version of Martin−Löf type theory, although the algorithm for constructing them cannot.

The subject is historically part of proof theory, starting with Hilbert and Gentzen. In [3], Hilbert gives a construction of $\epsilon-$numbers, very like the one we describe here(??). The ordinal Γ_0 is closely connected with the analysis of predicativity made by Feferman and Schütte.

In our opinion, despite appearances, the subject is of interest for computer science. There is a connection between accessibility proofs and terminating interactive programs which we describe below in section 3.

First we define some of the terms just used (ordinal notations, accessibility, transfinite induction, type structure), and discuss three interpretations of accessibility. Then we show what proofs of accessibility have to do with interactive program.

Then we describe the proof.

Non−dependent lenses.

definitions Suppose S is a set (of 'notations'), and we have a function which for each $\alpha : S$ tells us whether it is zero, a successor (and of what predecessor), or a limit (and of what sequence/stream of predecessors); we have a 3−way classification in which 'zero' *etc.* are classifying labels. In other words, we have a function

$$pd : S \to 1 + S + (\mathbb{N} \to S)$$

(The '1' summand represents the class of notations which are zero, the $(\mathbb{N} \to \_)$ summand the class of limit notations, and the middle summand the class of successors.) I shall call such a thing a *branching structure* (??); in categorical language it is a coalgebra for the functor $\lambda X.\, 1 + X + (\mathbb{N} \to X)$ on the category of sets. The initial algebra for this functor, is usually known as the Brouwer ordinals, and is given by the following data.

$$\begin{array}{l} \Omega : \text{Set} \\ 0 : \Omega \\ (+1) : \Omega \to \Omega \\ \sqcup : (\mathbb{N} \to \Omega) \to \Omega \end{array}$$

As an initial algebra, Ω automatically has an inverse coalgebraic structure, and so Ω serves as an example of a branching structure.

In this paper by an *ordinal* we mean a Brouwer ordinal.

For any predicate P, using the coalgebraic structure we can define a predicate

$P^{\Box}$ which holds of α exactly when P holds of all predecessors of α.

$$
\begin{array}{rl}
P^{\Box}(\alpha) = \textbf{case } \alpha \textbf{ of} & \\
& 0 \quad\quad\; \to \{*\} \\
& \alpha' + 1 \to P(\alpha') \\
& \lambda \quad\quad\; \to (\, n : \mathbb{N}\,) \to P(\lambda[n])
\end{array}
$$

A predicate is *progressive* if $P' \subseteq P$, and an element of S is *accessible* if it is contained in all progressive predicates. The branching structure is *wellfounded*, or an *ordinal notation system* if all elements of S are accessible.

Interpretations of accessibility There is more than one way to interpret the universal quantification over the powerset of S in the definition of accessibility; we may interpret it

- using the 'impredicative' predicate quantification of second order logic. In second order logic we can simply define the accessibility predicate as the intersection of all progressive predicates (one of which is the accessibility relation itself).
- as schematic generality. We may interpret it as requiring that there are constructors

  $$\mathit{AccIn} : \mathit{Acc}^{\Box} \subseteq \mathit{Acc}$$

  for sets of the form $\mathit{Acc}\, s$ such that functions defined by the following *recursion scheme* are total. Given

  $$
  \begin{array}{lcl}
  P & : & (\, s : S\,) \to \mathit{Pow}(\mathit{Acc}\, s) \\
  p & : & (\, s : S, a : \mathit{Acc}^{\Box}\, s\,) \to \\
  & & \quad ((\, t : T\, s\,) \to P\,(s[t])\,(a\, t)) \to P\, s\,(\mathit{AccIn}\, s\, p)
  \end{array}
  $$

  we may introduce

  $$
  \begin{array}{lcl}
  R_{P,p} & : & (\, s : S, a : \mathit{Acc}\, s\,) \to P\, s\, a \\
  R_{P,p} & = & \lambda\, s, a.\, \textbf{case } a \textbf{ of} \\
  & & \quad\quad\quad \mathit{AccIn}\; ba \to p\; s\; ba\,(\lambda\, t.\, R_{P,p}(s[t])(ba\, t))
  \end{array}
  $$

  If we weaken this scheme to its 'iterative' version in which the induction predicate does not depend on the proof of accessibility, we get the following *iteration scheme*. If we have a predicate P over S, which by the Curry−Howard correspondence is a set−valued function on S, and a proof p that P is progressive, which amounts to having proofs $\langle p_0, p_s, p_l, \rangle$ of (respectively) $P(0)$, $(\,\alpha : S\,) \to P(\alpha) \to P(\alpha+1)$, and $((n : \mathbb{N}) \to P(\lambda[n])) \to P(\lambda)$ where λ is a limit, then from a proof $\bar{\alpha}$ that α is accessible, we can recover an element $R_{\bar{\alpha}}(P, p)$ of $P(\alpha)$, which satisfies the expected definitional equalities.

  This is the 'inductive' interpretation, where one takes the notion of an inductive definition as basic, rather than as justified by an impredicative quantification. (In any case, the recursion scheme is not justified using impredicative quantification.)

It seems to be widely agreed that inductive definitions of this kind do not 'sin against predicativity', or involve impredicative quantification in an essential way. The generality involved is *schematic.*

Predicativity was in the first place relative to the natural numbers, whereas now the opinion seems to be that the class of inductive definitions we consider acceptable is something orthogonal to predicativity.

- 'predicative' quantification over a universe, or an set−indexed family of sets, closed under certain operations for forming sets. We can interpret quantification over predicates by relativisation to this universe, so that predicates are in fact functions from S into the index set of the universe. The accessibility predicate is then a *large* predicate, and so we cannot derive from the progressivity of a large predicate P an inhabitant of $P(\alpha)$ for an accessible α.

2 Cantor and Veblen hierarchies

By a hierarchy, I mean simply a binary function – the first argument is thought of as identifying a level in a hierarchy, the second argument as identifying a position in that level. If h is a hierarchy, let $\widetilde{h}$ denote the hierarchy with its arguments swapped. We will describe two hierarchies over the ordinals that we will call the *Cantor* and *Veblen* hierarchies, writing them as follows:

$$v_\alpha\,\beta, w_\alpha\,\beta$$

Notation. We write limit ordinals in the form $\lambda = \sqcup_n \lambda[n]$, and lift the notation to ordinal functions; thus if f_n is a sequence of ordinal functions, then $\sqcup_n f_n$ refers to their pointwise limit $\lambda\,\alpha.\ \sqcup_n\ (f_n\,\alpha)$. If f is an ordinal function, then f^n denotes its $n-$th finite iterate, and f^ω the $\omega-$th iteration $\sqcup_n f^n$.

Cantor hierarchy w. The definition is by transfinite recursion on the subscript argument:

$$\begin{array}{lcl} w_0 & = & (+1) \\ w_{\alpha+1} & = & (w_\alpha)^\omega \\ w_\lambda & = & \sqcup_n (w_{\lambda[n]}) \end{array}$$

By transfinite induction, one can see that $w_\alpha\,\beta = \beta + \omega^\alpha$.

Nabla function. To define the Veblen hierarchy, we need an auxiliary 'type 2' functional $\nabla : (\Omega \to \Omega) \to \Omega \to \Omega$, which enumerates the fixed−points of an idempotent inflationary weakly monotone continuous function (closure operator).

$$\begin{array}{lcl} \nabla f\, 0 & = & f\, 0 \\ \nabla f\, (\alpha+1) & = & f\, ((\nabla f \alpha)+1) \\ \nabla f\, \lambda & = & \sqcup_n (\nabla f\, (\lambda[n])) \end{array}$$

The idea is that the least fixed point of f will be $f\,0$ (it is a fixed point because f is idempotent, it the least since it is weakly monotone); if $f\,\alpha = \alpha$, then the least fixed point beyond α is at least $f(\alpha+1)$. By induction, one can see that $\nabla\, f\, \alpha = f^{(1+\alpha)}\, 0$.

Veblen hierarchy v. The definition of the Veblen hierarchy is by transfinite recursion on the subscript argument:

$$\begin{array}{lcl} v_0 & = & \nabla(\widetilde{w}_0)^\omega \\ v_{\alpha+1} & = & \nabla(v_\alpha)^\omega \\ v_\lambda & = & \nabla(\sqcup_n(v_{\lambda[n]})) \end{array}$$

By transfinite induction, one can see that v_α is normal (continuous and strictly monotone) and for non−zero subscripts exhausts in increasing order the fixed points of its predecessor or predecessors. This makes v_α the same as what has variously been written ϕ_α, or χ_α, and called the Veblen hierarchy (over ω−exponentiation). (Clearly, the step from w to v can be generalised.)

In the vernacular. The following well−known ordinals ("in the vernacular") are written in this ("official") notation as follows:

$$\begin{array}{lcl} \textit{vernacular} & & \textit{official} \\ 1 & = & w_0\, 0 \\ 2 & = & w_0(w_0\, 0) \\ \alpha+1 & = & w_0\, \alpha \\ \alpha+n & = & (w_0)^n\, \alpha \\ \alpha+\omega & = & w_1\, \alpha \\ \alpha+\omega.n & = & (w_1)^n\, \alpha \\ \alpha+\omega^2 & = & w_2\, \alpha \\ \alpha+\omega^n & = & w_n\, \alpha \\ \alpha+\omega^\omega & = & w_\omega\, \alpha \\ \beta+\omega^\alpha & = & w_\alpha\, \beta \\ \beta+\omega^\alpha.n & = & (w_\alpha)^n\, \beta \\ \omega^{\alpha_1}.n_1+\ldots+\omega^{\alpha_k}.n_k & = & ((w_{\alpha_k})^{n_k}\cdots(w_{\alpha_1})^{n_1})\, 0 \\ \epsilon_0 & = & v_0\, 0 \end{array}$$

The ordinal ϵ_0 is the limit of the sequence $0, w_0\, 0, w_{w_0\, 0}\, 0, \ldots, w_{n+1} = w_{w_n}\, 0$, and in fact can be written $v_0\, 0$.

normal forms. We can define a relation of equality

For $\alpha < \beta$, we have (wrt non−trivial equality) $w_\beta \cdot w_\alpha = w_\beta$ and $v_\alpha \cdot v_\beta = v_\beta$.

The rôle of ϵ_0 in proof theory if first order arithmetic is wellknown. Hilbert ([3]). Hilbert and Bernays [4]. Ackermann ([1]). Gentzen ([2]). Reverse mathematics/combinatorial equivalents? (ACA_0.)

The ordinal Γ_0 is the limit of the sequence $0, v_0\,0, v_{v_0\,0}\,0, \ldots, v_{n+1} = v_{v_n}\,0$. To obtain a notation for Γ_0, we could introduce a third hierarchy, based on $\widetilde{v}_0$ as v is based on $\widetilde{w}_0$, or use more generally a three−place Veblen function, or still more generally an n−place Veblen function, *etc.* . We choose to stop short of Γ_0.

The ordinal Γ_0 and the Veblen hierarchy has pervaded proof theoretical investigations of a variety of a−priori very different formal systems: *long list...* autonomous ramified analysis (Feferman Schütte), Feferman's system with Δ^1_1−CA or CR (iterated?), infinitary propositional calculus (Tait, Feferman, Martin−Löf, ...), Martin−Löf type theory with an external sequence of universes (here), various Friedman−Simpson type things (ATR_0 ?), check in Avigad, check relation with Kruskal theorem: some version of Kruskal theorem has strength Γ_0 I think. Any other ways Γ_0 crops up? Rewriting?

It is sometimes known as 'the ordinal of predicativity', but to be completely explicit, one should probably call it 'the ordinal of predicativity relative to the natural numbers'. Doubts can be raised even given this relativisation (which is really to a notion of inductive definition); one can ask what becomes impredicative if we have a superuniverse which is closed under an internal (and so iterable) next−universe operator, or a superuniverse operator (and so an external sequence of super−universes).

3 Programs and ordinals

We can define a type of terminating interactive programs which send data of type *Op* to their environment and read data of type *Op*, in strict alternation, until a final output.

$$\mathit{Proc}_{(\mathit{Inp},\mathit{Op})} = \mu X.\, \mathit{Op} \times (1 + (\mathit{Inp} \to X))$$

The programs begin with an output, then either terminate, or read an input and go back to the output phase. We can think of the last output as a termination code. All programs eventually terminate, as long as their environments continue to supply inputs, because we have asked for the least fixed point.

By a concrete datatype I mean a datatype encodable in the natural numbers. (Something fancier can be said.. one can allow dependent types so long as the function type is not used.)

In general we can regard an ordinal as a process which produces and consumes concrete data, by means of the following isomorphism between Ω and

$Proc_{(\mathbb{N},\mathbb{N})}$.

$$\begin{array}{lcl}
f & : & \Omega \to Proc_{(\mathbb{N},\mathbb{N})} \\
f\,0 & = & (0,0) \\
f\,(\alpha+1) & = & \textbf{let}\;\;(n,x) = f\,\alpha \\
 & & \textbf{in}\;\;\;(n+1,x) \\
f\,\lambda & = & (0, \lambda\, n.\, f\,(\lambda[n]))
\end{array}$$

The process is *terminating*, meaning that eventually it will yield its last output, and stop. The natural numbers play the role here of a universal datatype, into which all other first order (concrete) datatypes (expressible without functions) can be encoded. Many other datatypes would do, such as the type of finite strings of bits.

The fact that we can prove in a given system the accessibility of certain notations for ordinals, but not for their supremum gives an interesting analysis of the interactive complexity of the provably terminating interactive programs that can be written in that system. This will affect, for example, the kind of deadlocks it will be able to find (it may be that a program is deadlock free for environments of complexity below some ordinal), the kind of observations it will be able to make, and perhaps the interactive complexity of programs that do not necessarily terminate (though this concept is hard to analyse).

References

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