Hancock's Proof

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Lenses and the Veblen hierarchy

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Ω: set of formal ordinals

$$\begin{array}{lcl} 0 & : & \Omega \\ (+1) & : & \Omega \to \Omega \\ \sqcup & : & (\mathbb{N} \to \Omega) \to \Omega \end{array}$$

λ is a limit $\sqcup_n(\lambda[n])$.

$$B : \mathbb{P}\,\Omega \to \mathbb{P}\,\Omega$$

$$\begin{array}{lcl} B\,X\,0 & = & \{*\} \\ B\,X\,(\alpha+1) & = & X\,\alpha \\ B\,X\,\lambda & = & \forall\, n : \mathbb{N}.\, X(\lambda[n]) \end{array}$$

X is *progressive* if $B\,X \subseteq X$.
α is *accessible* if $X\,\alpha$ for all progressive X.

Ordinals: Background

- A very natural arithmetic, essentially of fixed points.
  - Cantor (1880): $+$, $\times$, 2^α, ..
  - Veblen (1900).

  Used by:
  - Gentzen (1930), Ackermann, ..
  - Schütte, Feferman (1950), Tait, ..
- 'Paradigmatic' of interactive programs which communicate values in first-order (concrete) datatypes. (To be pursued.)

Cantor hierarchy

$$w_\alpha\beta = \beta + \omega^\alpha$$

$$\begin{aligned} w_0 &= (+1) \\ w_{\alpha+1} &= (w_\alpha)^\omega \\ w_\lambda &= \sqcup_n w_{\lambda[n]} \end{aligned}$$

$$w_\alpha\beta = I_\alpha\, I_\omega\,(+1)\,\beta$$

Gentzen transform

$$X_G\,\alpha = w_\alpha : X \to X$$

- progressive $X \Rightarrow$ progressive X_G.
- $(\lambda\,\alpha.\,w_\alpha 0) : X_G \to X$ for progressive X.

$\lambda\,X.\,X_G : \mathbb{P}\,\Omega \to \mathbb{P}\,\Omega$ is a lens for
$\lambda\,\alpha.\,\omega^\alpha : \Omega \to \Omega$

Gentzen transformer and accessibility

$$\frac{X_G\,\alpha \qquad X\,\beta}{X\,(w_\alpha\beta)}$$

$$\frac{Acc\,\alpha \qquad Acc\,\beta}{Acc\,(w_\alpha\beta)}$$

$$\frac{\dfrac{Prog\,X \vdash Prog\,X_G \qquad Prog\,X_G \vdash X_G\,\alpha}{Prog\,X \vdash X_G\,\alpha} \qquad Prog\,X \vdash X\,\beta}{Prog\,X \vdash X\,(w_\alpha\beta)}$$

Veblen hierarchy: $v_\alpha\,\beta = \phi_\alpha\beta$

$$\begin{aligned} v_0\,\beta &= \nabla\overbrace{(w_{\cdot}0)^\omega}^{(0+\omega^{\cdot})^\omega} \\ v_{\alpha+1} &= \nabla(v_\alpha)^\omega \\ v_\lambda &= \nabla(\sqcup_n v_{\lambda[n]}) \end{aligned}$$

$$\nabla : (\Omega \to \Omega) \to \Omega \to \Omega$$

Enumerates range of a closure operator.

$$\nabla f\,\alpha = (f \cdot (+1))^\alpha (f\,0)$$

$$\begin{aligned} \nabla f\,0 &= f\,0 \\ \nabla f\,(\alpha+1) &= f\,(\nabla f\,\alpha + 1) \\ (\nabla f\,\lambda)[n] &= \nabla f(\lambda[n]) \end{aligned}$$

The problem

To show that expressions formable using the Cantor and Veblen functions are accessible, using only a universe operator, and (higher type) primitive recursion on the natural numbers.

The solution is an 'external' program, not formalisable (presumably) without a superuniverse of some form.

The universe operator corresponds to certain '*autonomy*' features in the systems studied by Feferman and Schütte.

Basic facts, definitions

- predicates $X : \mathbb{P}\,\Omega$: partial ordered by $\subseteq$, infs $(\cap_i X_i)$
- predicate transformers $F : \mathbb{P}\,\Omega \to \mathbb{P}\,\Omega$: partially ordered by $\sqsubseteq$, infs $(\sqcap_i F_i)$
  - $F \sqsubseteq G$: pointwise. $F\,X \subseteq G\,X$ for all X.
  - $\sqcap_i F_i$: pointwise. $\cap_i(F_i\,X)$ at X.
- $B : \mathbb{P}\,\Omega \to \mathbb{P}\,\Omega$ is monotone if $B\,X \subseteq B\,Y$ for $X \subseteq Y$.
  $X : \mathbb{P}\,\Omega$ is progressive (*w.r.t.* B) if $B\,X \subseteq X$.
- For monotone B, progressive predicates closed under intersections.

Lenses: the basic kinds

- A *lens* for $f : \Omega \to \Omega$ is a predicate transformer pt that preserves progressivity pp and satisfies (cl) the closure property (where X is progressive):

$$f : pt\,X \to X$$

- A *closure lens* for $f : \Omega \to \Omega$ is a predicate transformer pt that preserves progressivity pp and satisfies (cl_1, cl_2) the closure properties (where X is progressive):

$$pt\,X \subseteq X$$
$$f : pt\,X \to pt\,X$$

Lenses: the basic constructions

- If we have a lens for each for $f_i : \Omega \to \Omega$, we can get a lens for (pointwise) $\sqcup_i f_i$. `SupLens` - proof by general nonsense.
- Given a lens for f, we can get a closure lens for f. `MkClens` - Key idea.
- A closure lens for f is also a closure lens for f^ω. `ItClens`.
- If we have a closure lens for f, we can get a lens for ∇f. `NablaLens`

Lenses: putting them together

```
SuccLens (f :: op Ot)
  :: Lens f -> Lens (Nabla (OpItw f))
  = \(lf :: Lens f) ->
    NablaLens (OpItw f)
       (ItClens f (MkClens f lf))

LimLens (fs :: Seq (op Ot))
        (lfs :: (n::Nat)-> Lens (fs n))
  :: Lens (Nabla (OpLim fs))
  = let  f :: op Ot = OpLim fs
    in NablaLens f
         (MkClens f (SupLens fs lfs))

ZeroLens :: Lens (deriv wexp)
  = SuccLens wexp GentzenLens
```


Relativisation

(U', T') is the next universe after (U, T).

$Pred_{(U,T)} : \mathsf{Set}$

$Prog_{(U,T)}\,(p : Pred_{(U,T)}) : \mathsf{Set}$

$Acc_{(U,T)} : Pred_{(U',T')}$,

$Lens_{(U,T)}\,(f : \Omega \to \Omega) : \mathsf{Set}$,

$L_{(U,T)} = \lambda\,\alpha.\, Lens_{(U,T)}\, v_\alpha : Pred_{(U',T')}$,

etc.

Main fact:

$$Prog_{(U',T')}\, L_{(U,T)}$$

So:

$$\frac{Acc_{(U',T')}\alpha \qquad Acc_{(U,T)}\beta}{Acc_{(U,T)}(v_\alpha \beta)}$$