Lenses and the Veblen hierarchy

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Ordinals, notation systems.

Functor $\lambda X : \mathsf{Set} \triangleright 1 + X + X^{\mathbb{N}}$.

Limit−system *constructors* $: F\,X \to X$

$$X : \mathsf{Set}$$
$$0 : X$$
$$(+1) : X \to X$$
$$\sqcup : X^{\mathbb{N}} \to X$$

Notation−system *predecessors* $: X \to F\,X$

$$X : \mathsf{Set}$$
$$cf : X \to \{\{\}, \{*\}, \mathbb{N}\}$$
$$\_[\_] : (\, x : X\,) \to cf\ x \to X$$

$\_^{\Box}$, Progressivity, Accessibility

For any predicate P on a notation system, $P^{\Box}$ is the predicate which holds of α exactly when P holds of all predecessors of α.

$$P^{\Box}\,\alpha = (\, t : cf(\alpha)\,) \to P(\alpha[t])$$

X **is progressive** if $X^{\Box} \subseteq X$.

α **is accessible** if $X(\alpha)$ for all progressive X.

An ordinal notation system is a notation system in which all notations are accessible.

"Brouwer" ordinals Ω

Initial algebra (limit system, and notation system).

$$\begin{array}{lcl} 0 & : & \Omega \\ (+1) & : & \Omega \to \Omega \\ \sqcup & : & (\mathbb{N} \to \Omega) \to \Omega \end{array}$$

λ is a limit $\sqcup_n(\lambda[n])$.

If $(X, 0, (+1), \sqcup)$ is a limit system, we have an 'iteration' function

$$([X, 0, (+1), \sqcup]) : \Omega \to X$$

$$It_\alpha\, f = \lambda\, x \triangleright ([X, x, f, \sqcup])(\alpha)$$

Cantor hierarchy

$$w_\alpha \beta = \beta + \omega^\alpha$$

$$\begin{array}{lcl} w_0 & = & (+1) \\ w_{\alpha+1} & = & (w_\alpha)^\omega \\ w_\lambda & = & \sqcup_n w_{\lambda[n]} \end{array}$$

$$w_\alpha \beta = It_\alpha \, It_\omega \, (+1) \, \beta$$

Veblen hierarchy: $v_\alpha \, \beta$

$(= \phi_\alpha \beta = \chi_\alpha \beta = \chi^\alpha_\beta = \ldots)$

$$\begin{array}{lcl} v_0 \, \beta & = & \nabla \overbrace{(\lambda \, \alpha \triangleright w_\alpha \, 0)^\omega}^{(\lambda \, \alpha \triangleright \omega^\alpha)^\omega} \beta = \epsilon_\beta \\ v_{\alpha+1} & = & \nabla (v_\alpha)^\omega \\ v_\lambda & = & \nabla (\sqcup_n v_{\lambda[n]}) \end{array}$$

$$\nabla : (\Omega \to \Omega) \to \Omega \to \Omega$$

Enumerates range of a closure operator.

$$\nabla f \, \alpha = (f \cdot (+1))^\alpha (f \, 0)$$

$$\begin{array}{lcl} \nabla f \, 0 & = & f \, 0 \\ \nabla f \, (\alpha + 1) & = & f \, ((\nabla f \, \alpha) + 1) \\ (\nabla f \, \lambda)[n] & = & \nabla f (\lambda[n]) \end{array}$$

Gentzen transform

$$X_G \, \alpha = w_\alpha : X \to X$$

- progressive $X \Rightarrow$ progressive X_G.
- $(\lambda \, \alpha \triangleright w_\alpha 0) : X_G \to X$ for progressive X.

  ${-}_G : \mathbb{P}\Omega \to \mathbb{P}\Omega$ is a **lens** for $\omega^{-} : \Omega \to \Omega$

Gentzen transformer and accessibility

$$\frac{Acc \, \alpha \qquad\qquad Acc \, \beta}{Acc \, (w_\alpha \beta)}$$

$$\frac{X_G \, \alpha \qquad\qquad X \beta}{X (w_\alpha \beta)}$$

$$\frac{\dfrac{Prog \, X \vdash Prog X_G \quad Prog X_G \vdash X_G \, \alpha}{Prog \, X \vdash X_G \, \alpha} \qquad Prog \, X \vdash X \, \beta}{Prog \, X \vdash X \, (w_\alpha \beta)}$$

General nonsense

predicates $X : \mathbb{P}\,\Omega$:
partial ordered by $\subseteq$,
infs $(\cap_i X_i)$

predicate transformers $F : \mathbb{P}\,\Omega \to \mathbb{P}\,\Omega$:
(pointwise)
partially ordered by $\sqsubseteq$,
infs $(\sqcap_i F_i)$.

$B : \mathbb{P}\,\Omega \to \mathbb{P}\,\Omega$ **is monotone** if $B\,X \subseteq B\,Y$ for $X \subseteq Y$. *e.g.* $\_^{\Box}$, $\lambda\,P \triangleright P \cdot f$.

For monotone B, progressive predicates closed under intersections. (So progressive predicates, and transformers of them, have all mod cons.)

Lenses: the basic kinds

lens for $f : \Omega \to \Omega$ a predicate transformer pt that preserves progressivity and satisfies (where X is progressive):

$$f : pt\,X \to X$$

closure lens for $f : \Omega \to \Omega$ is a predicate transformer pt that preserves progressivity and satisfies (where X is progressive):

$$pt\,X \subseteq X$$
$$f : pt\,X \to pt\,X$$

Lenses: the basic constructions

- If we have a lens for each for $f_i : \Omega \to \Omega$, we can get a lens for (pointwise) $\sqcup_i f_i$. `SupLens` – proof by general nonsense.
- Given a lens for f, we can get a closure lens for f. `MkClens` – Key idea.
- A closure lens for f is also a closure lens for f^ω. `ItClens`.
- If we have a closure lens for f, we can get a lens for ∇f. `NablaLens`

Lenses: putting them together

```
SuccLens (f :: op Ot)
  :: Lens f -> Lens (Nabla (OpItw f))
  = \(lf :: Lens f) ->
    NablaLens (OpItw f)
       (ItClens f (MkClens f lf))

LimLens (fs :: Seq (op Ot))
        (lfs :: (n::Nat)-> Lens (fs n))
  :: Lens (Nabla (OpLim fs))
  = let  f :: op Ot = OpLim fs
    in NablaLens f
         (MkClens f (SupLens fs lfs))

ZeroLens :: Lens (deriv wexp)
  = SuccLens wexp GentzenLens
```

Relativisation

(U', T') is the next universe after (U, T).

$Pred_{(U,T)} : \mathsf{Set}$,

$Prog_{(U,T)}(p : Pred_{(U,T)}) : \mathsf{Set}$,

$Acc_{(U,T)} : Pred_{(U',T')}$,

$L_{(U,T)} = \lambda\,\alpha \triangleright Lens_{(U,T)}\, v_\alpha : Pred_{(U',T')}$,

etc.

Key fact.

$$Prog_{(U',T')}\, L_{(U,T)}$$

So:

$$\frac{Acc_{(U',T')}\alpha \qquad\qquad Acc_{(U,T)}\beta}{Acc_{(U,T)}(v_\alpha \beta)}$$

Simple lens

A simple lens is a function L from limit structures $(X, 0, (+1), \sqcup)$ to limit structures $(X_L, 0_L, (+1)_L, \sqcup_L)$ together with a 'projection' (between base sets).

$$p : X_L \to X$$

We have a lens for f if

$$\begin{aligned} & p \circ ([X_L, 0_L, (+1)_L, \sqcup_L]) \\ & = ([X, 0, (+1), \sqcup]) \circ f \\ & : \Omega \to X \end{aligned}$$

Simple lens for $1 + \alpha$

$$\begin{array}{ccc} \begin{pmatrix} X \\ 0 \\ (+1) \\ \sqcup \end{pmatrix} & \mapsto & \begin{pmatrix} X \\ 1 \\ (+1) \\ \sqcup \end{pmatrix} \\ x & \xleftarrow{p} & x \end{array}$$

Simple lens for $1 + \alpha$

$$\begin{array}{ccc} \begin{pmatrix} X \\ 0 \\ (+1) \\ \sqcup \end{pmatrix} & \mapsto & \begin{pmatrix} X \to X \\ id \\ \lambda\, f \triangleright (+1) \cdot f \\ \sqcup \end{pmatrix} \\ f\,1 & \xleftarrow{p} & f \end{array}$$

Simple lens for ω^α

$$
\begin{array}{ccc}
\begin{pmatrix} X \\ 0 \\ (+1) \\ \sqcup \end{pmatrix} & \mapsto & \begin{pmatrix} X \to X \\ (+1) \\ It_\omega \\ \sqcup \end{pmatrix} \\
f\,0 & \xleftarrow{p} & f
\end{array}
$$