Notes: ordinals as universes, universes as ordinals

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Contents

1 Some basic sets

Empty set.

$$\begin{array}{l} \emptyset \;:\; \mathbf{Set} \\ S\,A \;=\; \mathbf{data}\ \ - \end{array}$$

Successor of a set.

$$\begin{array}{lll} S & : & \mathbf{Set} \to \mathbf{Set} \\ S\,A & = & \mathbf{data}\ \mathsf{0} \\ & & \qquad \mathsf{S}\,a\ \ (a : A) \end{array}$$

So we can define finite sets of all cardinalities.

$$\begin{array}{l} 1, 2, 3, \ldots \;:\; \mathbf{Set} \to \mathbf{Set} \\ 1 \;=\; S\,\emptyset \\ 2 \;=\; S\,1 \\ 3 \;=\; S\,2 \\ \ldots \end{array}$$

We can define the set of natural numbers as the first fixedpoint of S.

$$\begin{array}{lll} \mathbf{rec}\ \ \mathit{Nat} & : & \mathbf{Set} \\ \qquad \mathit{Nat} & = & S\ \mathit{Nat} \end{array}$$

2 Families

A family of As is a A-valued function whose domain is a set.

$$\begin{array}{lll} \mathit{Fam} & : & \mathbf{Type} \to \mathbf{Type} \\ \mathit{Fam}\ A & = & (\,\Sigma\, I : \mathbf{Set}\,)\, I \to A \end{array}$$

A family has the form (I, f), where $A : \mathbf{Set}$ and $f : I \to A$.

We very often deal with families of sets, so have a short name for this.

$$\begin{array}{l} \mathcal{F}\!\mathit{am} : \mathbf{Type} \\ \mathcal{F}\!\mathit{am} = \mathit{Fam}\ \mathbf{Set} \end{array}$$

A family of sets (an $\mathcal{F}\!\mathit{am}$) has the form (A, B), where $A : \mathbf{Set}$ and $B : A \to \mathbf{Set}$.

For example, we have the empty family

$$\begin{array}{l} 0_{\mathcal{F}} : \mathcal{F}\!\mathit{am} \\ 0_{\mathcal{F}} = (\emptyset, \lambda x.\, \mathbf{case}\ x\ \mathbf{of}\ \ -\ ) \end{array}$$

Successor of a family of sets. We ensure that this contains a representative of the empty set, and of the successor of each of the original sets. (It may or may not be that $(A,B)_\perp$ extends (A,B).)

$$
\begin{array}{lcl}
(\cdot_\perp) & : & \mathcal{F}am \to \mathcal{F}am \\
(A,B)_\perp & = & \textbf{let}\ A' \ :\ \textbf{Set} \\
 & & \quad\ A' = S\ A \\
 & & \quad\ B' \ :\ S\ A \to \textbf{Set} \\
 & & \quad\ B'\ a' = \textbf{case}\ a'\ \textbf{of} \\
 & & \qquad\qquad \mathsf{0} \quad \to \emptyset \\
 & & \qquad\qquad \mathsf{S}\,a \to S\ (B\ a) \\
 & & \textbf{in}\ (A', B')
\end{array}
$$

We have also a family of finite sets, defined by iterating S.

$$
\begin{array}{lcl}
Fin & : & \mathcal{F}am \\
Fin & = & (Nat, \lambda n.\, S^n\ \emptyset)
\end{array}
$$

This can be thought of as a representation of the ordinal ω. It is 'distinguished' in the sense that the representation is minimal, having a unique representative for each finite set. We shall come across another representation which has more redundancy, but is more suitable (perhaps) for generalisation.

3 Internal families

An internal family of a family of sets (A,B) is one in which the index set has the form $B\ a$ for some element a of A, and the set valued function factors through B, so that it has the form $B \circ b$ for some $b : B\ a \to A$.

$$
\begin{array}{lcl}
\mathit{ifam} & : & \mathcal{F}am \to \textbf{Set} \\
\mathit{ifam}\ (A,B) & = & (\,\Sigma\, a : A\,)\, B\ a \to A
\end{array}
$$

An internal family of a family of sets can be interpreted as a family of sets.

$$
\begin{array}{lcl}
(\cdot \uparrow \cdot) & : & (\,\Pi\,(A,B) : \mathcal{F}am\,)\ \mathit{ifam}\ (A,B) \to \mathcal{F}am \\
(A,B) \uparrow (a,b) & = & (B\ a, B \circ b)
\end{array}
$$

We can sum this up by observing that the internal families of a family of sets form a family.

$$
\begin{array}{lcl}
O1 & : & \mathcal{F}am \to Fam\ \mathcal{F}am \\
O1 & = & \lambda(A,B).\,(\mathit{ifam}(A,B), ((A,B) \uparrow \cdot) \\
O1 & = & \lambda(A,B).\,(\mathit{ifam}(A,B), \lambda(a,b).\,(B\,a, B \circ b))
\end{array}
$$

Given a family of sets (A,B), the following operator (which is analogous to the von Neumman successor operation $\lambda x.\, x \cup \{x\}$) extends (A,B) to a new family which contains (A,B) as an internal family, and (transitively) whatever

(A, B) contains internally.

$$
\begin{array}{lcl}
(\cdot^{\top}) & : & \mathcal{F}am \to \mathcal{F}am \\
(A,B)^{\top} & = & \textbf{let } A' : \textbf{Set} \\
& & \qquad A' = S\ A \\
& & \qquad B' : S\ A \to \textbf{Set} \\
& & \qquad B'\ a' = \textbf{case } a' \textbf{ of} \\
& & \qquad\qquad \mathsf{0} \ \to A \\
& & \qquad\qquad \mathsf{S}\, a \to B\ a \\
& & \textbf{in } (A', B')
\end{array}
$$

Then the family (A, B) is representable as the internal family

$$(\mathsf{0}, \lambda a.\, \mathsf{S}\, a) : \mathit{ifam}\ (A, B)^{\top}$$

Moreover, there is a kind of transitivity. If (a, b) is an internal family of (A, B), then $(\mathsf{S}\, a, \mathsf{S} \circ b)$ is an internal family of $(A, B)^{\top}$ which represents the same family of sets.

In general, if we have a family of sets $(A, B) : \mathcal{F}am$ and an internal family of sets $(a, b) : \mathit{ifam}(A, B)$, then for any element of $\mathit{ifam}((A, B) \uparrow (a, b))$ there is an element of $\mathit{ifam}(A, B)$ which represents the same family of sets.

$$
\begin{array}{l}
\mathit{trans}\ :\ (\,\Pi\,(A,B) : \mathcal{F}am, (a,b) : \mathit{ifam}(A,B)\,) \\
\qquad\qquad\qquad \mathit{ifam}((A,B) \uparrow (a,b)) \to \mathit{ifam}(A,B) \\
\mathit{trans}(A,B)(a,b)(c,d) = (b\, c, b \circ d)
\end{array}
$$

One can sum this up by saying that the internal families of a family of sets form a transition system.

$$
\begin{array}{lcl}
O2 & : & \mathcal{F}am \to (\,\Sigma\, S : \textbf{Set}\,)\, S \to \mathit{Fam}\ S \\
O2 & = & \lambda(A,B).\,(\mathit{ifam}(A,B) \\
& & \qquad , \lambda(a,b).\,(\mathit{ifam}((A,B) \uparrow (a,b)) \\
& & \qquad\qquad , \lambda(c,d).\, \mathit{trans}(A,B)(a,b)(c,d))) \\
O2 & = & \lambda(A,B).\,(\mathit{ifam}(A,B) \\
& & \qquad , \lambda(a,b).\,(\mathit{ifam}(B\, a, B \circ b) \\
& & \qquad\qquad , \lambda(c,d).\,(b\, c, b \circ d)
\end{array}
$$

4 Closure operator of lowest type

We call a function from families of sets to families of sets a family operator. Note that being given a family operator

$$\Theta \;:\; \mathcal{F}\!am \to \mathcal{F}\!am$$

is equivalent to being given a pair of functions

$$\begin{array}{l}
\Phi \;:\; \mathcal{F}\!am \to \mathbf{Set} \\
\Psi \;:\; (\,\Pi\,(A,B) : \mathcal{F}\!am\,)\, \Phi(A,B) \to \mathbf{Set} \\
(\Phi(A,B), \Psi(A,B)) = \Theta(A,B)
\end{array}$$

We define a 'closure operator' $\_^*$ which acts on a family operator $\Theta : \mathcal{F}\!am \to \mathcal{F}\!am$ to give a new family operator Θ^* that we call the *closure* of Θ. The closure Θ^* acts on a given family (A,B) to return the least family extending (A,B) which is also closed under Θ.

$$\begin{array}{l}
\Theta^* \;:\; \mathcal{F}\!am \to \mathcal{F}\!am \\
\Theta^*(A,B) = \mathbf{let\ rec}\ (A',B') : \mathcal{F}\!am \\
\qquad\qquad\quad A' = \mathbf{data}\ \begin{array}[t]{ll}
\mathsf{p}\,a & (a : A) \\
\mathsf{q}\,a\,b & (a : A', b : B'\,a \to A') \\
\mathsf{r}\,a\,b\,c & (a : A', b : B'\,a \to A' \\
 & ,c : \Phi(B'\,a, B' \circ b))
\end{array} \\
\qquad\qquad\quad B'\ a' = \mathbf{case}\ a'\ \mathbf{of} \\
\qquad\qquad\qquad\qquad \begin{array}{ll}
\mathsf{p}\,a & \to B\ a \\
\mathsf{q}\,a\,b & \to \Phi(B'\,a, B' \circ b) \\
\mathsf{r}\,a\,b\,c & \to \Psi(B'\,a, B' \circ b)\ c
\end{array} \\
\qquad\qquad \mathbf{in}\ (A',B')
\end{array}$$

This can also be written

$$\begin{array}{l}
\Theta^* \;:\; \mathcal{F}\!am \to \mathcal{F}\!am \\
\Theta^*(A,B) = \mathbf{let\ rec}\ (A',B') : \mathcal{F}\!am \\
\qquad\qquad\quad A' = \mathbf{data}\ \begin{array}[t]{ll}
\mathsf{p}\,a & (a : A) \\
\mathsf{q}\,x & (x : \mathit{ifam}\ (A',B')) \\
\mathsf{r}\,x\,c & (x : \mathit{ifam}\ (A',B'), c : B'(\mathsf{q}\,x))
\end{array} \\
\qquad\qquad\quad B'\ a' = \mathbf{case}\ a'\ \mathbf{of} \\
\qquad\qquad\qquad\qquad \begin{array}{ll}
\mathsf{p}\,a & \to B\ a \\
\mathsf{q}\,x & \to \Phi\ ((A',B') \uparrow x) \\
\mathsf{r}\,x\,c & \to \Psi\ ((A',B') \uparrow x)\ c
\end{array} \\
\qquad\qquad \mathbf{in}\ (A',B')
\end{array}$$

We can consider $(\cdot_\perp)^*\ 0_{\mathcal{F}}$ as a representation of the ordinal ω. (It contains masses of redundancy.) Another highly redundant representation is as $(\cdot^\top)^*\ 0_{\mathcal{F}}$.

Perhaps this construction can be lifted to higher types.

5 Derivative

Suppose we have introduced a type *Ord* of ordinals, and using ordinal recursion have defined iteration of family operators by taking the Σ union at limit ordinals. Then any operator has a 'derivative'. (To be reconsidered.)

$$\begin{array}{l} \mathit{Deriv}\ :\ (\mathcal{F}\mathit{am} \to \mathcal{F}\mathit{am}) \to \mathit{Ord} \to \mathcal{F}\mathit{am} \\ \mathit{Deriv}\ \Theta\ \alpha = (\Theta^* \circ (\cdot^{\top}))^{\alpha}(\Theta^* 0_{\mathcal{F}}) \end{array}$$

6 Mahlo operators

In section 4 on the preceding page we considered family operators, from general families of sets to general families of sets. Now we restrict attention to families of sets representable as internal families of a given family

$$(A, B) : \mathcal{F}\mathit{am}$$

Let an *internal family operator* for (A, B) be a function of the following type.

$$\mathit{ifam}\ (A, B) \to \mathit{ifam}\ (A, B)$$

It is a very strong closure property on (A, B) that any internal family operator has a closure, representable by another internal family operator.

$$\begin{array}{l} \mathit{Super} : (\mathit{ifam}\ (A, B) \to \mathit{ifam}(A, B)) \to \mathit{ifam}(A, B) \to \mathit{Fam}\, A \\ \mathit{Super}\ \ (\lambda(a, b).\,(\phi(a, b), \psi(a, b)) : \mathit{ifam}(A, B) \to \mathit{ifam}(A, B) \\ \qquad\quad , (a, b) : \mathit{ifam}(A, B)) \\ \qquad\quad = \textbf{let rec}\ \ (A', b') : \mathit{Fam}\, A \\ \qquad\qquad\qquad A' = \textbf{data}\ \begin{array}[t]{ll} \mathsf{p}\, y & (y : B\, a) \\ \mathsf{q}\, x & (x : \mathit{ifam}(A', B \circ b')) \\ \mathsf{r}\, x\, c & (x : \mathit{ifam}(A', B \circ b'), c : (B \circ b')\, (\mathsf{q}\, x)) \end{array} \\ \qquad\qquad\qquad b'\, a' = \textbf{case}\ a'\ \textbf{of} \\ \qquad\qquad\qquad\qquad \begin{array}{ll} \mathsf{p}\, y & \to b\, y \\ \mathsf{q}\, (i, f) & \to \phi(b'\, i, b' \circ f) \\ \mathsf{r}\, (i, f)\, c & \to \psi(b'\, i, b' \circ f)\, c \end{array} \\ \qquad\qquad \textbf{in}\ \ (A', b') \end{array}$$

This should be written in some notation appropriate to internal subfamilies. (This should be introduced in section 3.)

$$\begin{array}{l} \mathit{Mahlo} = \textbf{let}\ \ \textbf{rec}\ (A, B) : \mathcal{F}\mathit{am} \\ \qquad\qquad A = \textbf{data}\ \ \mathsf{m}\, \theta\, \xi\ \ \begin{array}[t]{l} (\theta : \mathit{ifam}(A, B) \to \mathit{ifam}(A, B) \\ , \xi : \mathit{ifam}(A, B)) \end{array} \\ \qquad\qquad B\, a = \textbf{case}\ \ a\ \ \textbf{of} \\ \qquad\qquad\qquad \mathsf{m}\, \theta\, \xi \to \textbf{let}\ \ (A', b') = \mathit{Super}\, \theta\, \xi \\ \qquad\qquad\qquad\qquad\quad\ \ \textbf{in}\ \ A' \\ \qquad\quad \textbf{in}\ \ (A, B) \end{array}$$

The superuniverse for a given internal family operator is actually an internal family of the Mahlo universe.