7.2 Bounded Arithmetic

$I\Delta_0$ is the class of (axioms for) discretely ordered rings in which induction is restricted to sets defined using only **polynomially bounded quantifiers.** Such sets are called Δ_0-sets and are defined recursively as follows (for a DOR R with $N = \{x \in R : x \geq 0\}$).

a) if $f \in R[x_1, ...x_n]$, $\{\bar{x} \in N^n : f(\bar{x}) = 0\}$ is Δ_0, and $\{\bar{x} \in N^n : f(\bar{x}) \geq 0\}$ is Δ_0;

b) closure under Boolean operations;

c) $[|x| = x$ if $x \geq 0$, $= -x$ if $x \leq 0]$.
If $X \subseteq R^{n+1}$ is Δ_0, and $g \in R[x_2, ...x_{n+1}]$, then $\{(x_2, ...x_{n+1}) \in N^n : (\exists x_1)[\ |x| \leq |g(x_2, ...x_{n+1})| \wedge (x_1, ...x_{n+1}) \in X\ ]$ is in Δ_0. ["proper direct image"].

Obviously if M is a model of PA then M is a model of $I\Delta_0$. The converse is not true, as we shall see.

Parikh's Construction

Let M be a model of DOR, and let M_1 be an **initial segment** of M in the sense that

(i) M_1 is an ordered subring of M;

(ii) if $x \in M_1,\ y \in M,\ |y| \leq |x|$, then $y \in M_1$.

Example: $\mathbb{Z}$ in any M.

In the above situation we write: $M_1 \subseteq_e M$, and say M is **an end extension** of M_1.

The most basic fact here is:

Lemma 8 If $M_1 \subseteq_e M$, and M is a model of $I\Delta_0$, so is M_1.

Proof By induction on the recursive definition of Δ_0-set, we show first:

Lemma 9 If $A(x_1, ...x_n)$ is Δ_0 over M_1, and $b_1, ...b_n \in M_1$ then

$$M_1 \models A(b_1, ...b_n) \Leftrightarrow M \models A(b_1, ...b_n)$$

(clearly A is Δ_0-over M).

If $A(x_1, ...x_n)$ is either

$$\{\bar{x} \in N_1^n : f(\bar{x}) > 0\} \text{ or } \{\bar{x} \in N_1^n : f(\bar{x}) = 0$$

for $f \in M_1[x_1, ...x_n]$, result clearly true.

Result clearly propagates through Boolean combinations.

Crucial case is: $X \subseteq M_1^{n+1}$, Δ_0, and $g \in M_1[x_2, ...x_{n+1}]$, and we want

$$\begin{array}{lll} M_1 & \models & (b_2...b_{n+1}) \in \{(x_2, ...x_{n+1}) \in N_1^n(\exists x_1)[\ |x_1| \leq |g(x_2, ...x_{n+1})| \wedge (x_1, ...x_{n+1}) \in X\ ]\} \\ & \Leftrightarrow & \\ M & \models & (b_1, ...b_{n+1}) \in \{(x_2, ...x_{x+1}) \in N^n : (\exists x_1)[\ |x_1| \leq |g(x_2, ...x_{n+1})| \wedge (x_1, ...x_{n+1}) \in X)\ ]\}. \end{array}$$

$\Rightarrow$: Suppose $\exists x_1 \in N$,

$$|x_1| \leq |g(b_2, ...b_{n+1})| \wedge (x_1, b_2, ...b_{n+1}) \in X.$$

By induction $(x_1, ...b_{n+1}) \in X$ in sense of M, so

$$M \models |x_1| \leq |g(b_2, ...b_{n+1})| \wedge (x_1, b_2, ...b_{n+1}) \in X.$$

$\Leftarrow$: Suppose $\exists_1' \in N$

$$|x_1'| \leq |g(b_2, ...b_{n+1})| \wedge (x_1' b_2, ...b_{n+1}) \in X \text{ (in sense of } M).$$

Since $M_1 \subseteq_e M$, $x_1' \in N_1$, and then by induction $(x_1', b_2, ...b_{n+1}) \in X$ (in sense of M_1), whence result. □

Now we prove Lemma 8. Let X be a Δ_0-subset of N defined over M_1, and suppose that $0 \in X$ and $(\forall t)(t \in X \rightarrow t + 1 \in X)$ in sense of M_1. Then $0 \in X$ in sense of M by Lemma 9. Now suppose $X \neq N_1$, and pick $a \in N_1$, so $M_1 \models \neg(a \in X)$. So by Lemma 9, $M \models \neg(a \in X)$. So $M \models (\exists x)(|x| \leq a \wedge a \notin X)$, so since $M \models I\Delta_0$,

$$M \models (\exists y)(|y| \leq a \wedge y \in X \wedge (y + 1) \notin X).$$

By Lemma 9,

$$M_1 \models (\exists y)(|y| \leq a \wedge y \in X \wedge (y + 1) \notin X),$$

contradicting

$$M_1 \models \forall t(t \in X \rightarrow t + 1 \in X).$$

□

Corollary If $M \models PA$, and $M_1 \subseteq_e M$ then $M_1 \models IA_0$. This leads to Parikh's construction, inside any $M_1 \models I\Delta_0$. Let $a \in N$, and let

$$M_1 = \{x \in M : \text{ for some } k \in \mathbb{N}, \ |x| \leq (a+1)^k\}.$$

Then easily M_1 is a DOR and $M_1 \subseteq M$. We define $BP[a]$ (rel. to M) as the above M_1. By the above, $BP[a] \models I\Delta_0$.

This can be used to show that $I\Delta_0$ is a very restricted system.